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PREFACE

The following pages contain a list of the additions to the Society's Library uring the year 1948.

Users of this third Supplement are referred to the Catalogue published in 1946 Bulletin, vol. 52, No. 5, part 2) and to the preface to Supplement No. 1 (Bulletin, ol. 53, No. 1, part 2) for an explanation of the abbreviations and for a statement of he plan followed in reporting additions to the serial publications.

ARNOLD DRESDEN
Librarian

January 1949

SECTION I

SERIAL PUBLICATIONS

- Academia brasileira de ciencias, Rio de Janeiro
  - Anais
    - 1-6, 11-14, 1929-42
- Académie des sciences, Paris
  - Comptes rendus hebdomadaires des séances
    - Index: 152-181, 182-211
- Akademie der wissenschaften, Berlin
  - Physikalisch-mathematische klasse
    - Abhandlungen
      - 1940 +
- K. Akademie der wissenschaften, Vienna
  - Mathematisch-naturwissenschaftliche klasse
    - Sitzungsberichte
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- K. Akademie van wetenschappen, Amsterdam
  - Afdeeling natuurkunde
    - Proceedings of the section of sciences
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- Akademii͡a nauk S.S.S.R., Leningrad
  - Doklady sA
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  - Izvestii͡a
    - Otdelenie matematicheskikh i estestvennykh nauk
      - Serii͡a fizicheskai͡a
        - 1945 +
- Cahiers scientifiques, Paris
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- Časopis pro pěstování matematiky a fysiky
  - 68 +, 1939 +
- Česká akademie věd a umění v Praze
  - Bulletin international
    - 39 +, 1938 +
  - Třída II: Mathematicko-přírodnická
    - Rozpravy
      - 47 +, 1937 +
- Hellēnikē mathematikē hetaireia, Athens
  - Deltion
    - 20 +, 1940 +
- Jugoslavenska akademija znanosti i umjetnosti, Zagreb
  - Ljetopis
    - 51 +, 1937 +
  - Rad
    - 82 +, 1939 +
- Lyons
  - Université
    - Annales
      - 32-33, 1912
- Matematikai és physikai lapok
  - 46 +, 1939 +
- Matematikai és természettudományi értesítö
  - 60 +, 1941 +
- Nova Scotian institute of science, Halifax
  - Proceedings and transactions
    - 10-13, 17-20, 1898-1942
- Padua
  - Università
    - Seminario matematico
      - Rendiconti
        - 11 +, 1940 +

- Provinciaal Utrechtsch genootschap van kunsten en wetenschappen
    - Verslag van het verhandelde in de algemeene vergadering
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- Revue scientifique. (Revue rose) Paris
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- Royal society of Canada
    - Proceedings and transactions
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        - Index: 1882-1906
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    - Boletin
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- Societatea română de ştiinţe. Bucharest
    - Secţia matematică
        - Bulletin mathématique
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- Société mathématique de France, Paris
    - Bulletin
        - 64 +, 1936 +
- ockholm
    - Astronomiska observatoriet
        - Annalen
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- hoku, Formosa
    - University
        - Faculty of science and agriculture
            - Memoirs (mathematical series)
                - 48 +, 1940 +
- Wiskundige opgaven met de oplossingen
    - 17 +, 1938 +

SECTION II

NONSERIAL PUBLICATIONS

d'Ailly, G. H. *See* D'Ailly, G. H.

Aharoni, J. Antennae, an introduction to their theory. Oxford, 1946.

Almech, J. M. I. *See* Iniguez Almech, J. M.

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Berwald, F. R. Quelques études sur une généralisation de certaines identités dues à Euler. Diss., Stockholm, 1917.

Bjerre, F. F. *See* Fabricius-Bjerre, F. G.

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Coxeter, H. S. M. *See* Ball, W. W. R.

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REPORT OF COMMITTEE ON AID TO DEVASTATED LIBRARIES

A Committee on Aid to Devastated Libraries was appointed by the Council of the American Mathematical Society in May, 1945 (Garrett Birkhoff, H. S. Bray, William Feller, Gabor Szegö, A. W. Tucker, Arnold Dresden), for the purpose of gathering information as to the needs of foreign libraries and formulating plans for meeting their needs. In August, 1946, the Committee was reorganized (P. A. Smith, Leo Zippin, Arnold Dresden), for the task of distributing the material that had been received.

The Trustees of the Society put at the disposal of the Committee available publications of the Society to a value not exceeding $1500, as well as an amount not exceeding $300, for operating expenses. This generous aid was supplemented by many gifts in kind and in cash, from the Mathematical Association of America, American Philosophical Society, National Bureau of Standards, the Annals of Mathematics, the Duke Journal, and a considerable number of individuals including groups of students at Mount Holyoke College, Long Island University, and Swarthmore College. The Committee takes this opportunity to express their appreciation to all these contributors.

The distribution of the material thus obtained would not have been possible without the aid of the American Book Center. Thanks are due to former and present officers of this organization, in particular to Mr. Kenneth R. Shaffner, Mr. Lawrence J. Kipp, and to Miss Alice D. Ball. Through their cooperation it has been possible to send packages of periodicals and of books to thirty-five institutions in fifteen countries in Europe and Asia.

That this material has been gratefully received is evidenced by the letters of appreciation which have been received by the Committee. But what has been sent does not begin to fill the great needs which exist in many places, and which are emphasized by the remarkable upsurge of mathematical activity in many parts of the world. A great deal of aid has been given by other groups, such as the University of California (especially to the University of the Philippines), the Institute for Advanced Study, Princeton University, and a group for Aid to Polish mathematicians, and so on. But legitimate requests continue to reach the Committee.

A report presented to the Council and approved at the Annual Meeting of the Society at Columbus, Ohio on December 28, 1948.

Since this Committee has now completed the task given it by the Council of the American Mathematical Society, such requests have to be referred to other agencies. Among them the most active is the United States Book Exchange, Incorporated (Miss Alice D. Ball, Executive Director), Library of Congress Building, Washington 25, D. C., which carries forward and extends the work begun by the American Book Center. The Committee directs therefore the attention of American mathematicians to this organization, in the hope that they will keep in mind the tremendous need for books and periodicals which continues to exist in many countries, and that they will communicate with the United States Book Exchange in all cases in which it is not feasible for them to forward their gifts abroad directly.

It is hoped that it will be possible to publish in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY from time to time the specific requests which are made known to the undersigned. Only a short time ago a request came from Professor Threlfall in Heidelberg for the following books:

Lectures on topology, University of Michigan, 1941
Duke Mathematical Journal, vol. 5, except Nos. 1, 4; vol. 6, except No. 3; vol. 7 and following
Mathematical Reviews, vol. 1, except No. 1; vol. 2, except No. 11; vol. 7 and following
Wintner, *The analytical foundations of celestical mechanics*
von Kármán, *Mathematical methods in engineering*
Courant, *What is mathematics?*
Courant, *Differential and integral calculus*

For the Committee
ARNOLD DRESDEN

TOPOLOGICAL METHODS IN ABSTRACT ALGEBRA. COHOMOLOGY THEORY OF GROUPS[1]

SAMUEL EILENBERG

1. **Introduction.** The title of this article requires some explanation. The term "abstract algebra" was used to indicate that we shall deal with purely algebraic objects like groups, algebras and Lie algebras rather than topological groups, topological algebras, and so on. The method of study is also purely algebraic but is the replica of an algebraic process which has been widely used in topology, thus the words "toplogical methods" could be replaced by "algebraic methods suggested by algebraic topology." These purely algebraic theories do, however, have several applications in topology.

The algebraic process borrowed from topology is the following. Consider a sequence of abelian groups $\{C^q\}$ and homomorphisms δ,

$$C^0 \xrightarrow{\delta} C^1 \rightarrow \cdots \rightarrow C^{q-1} \xrightarrow{\delta} C^q \xrightarrow{\delta} C^{q+1} \rightarrow \cdots \tag{1.1}$$

such that $\delta\delta = 0$. In each group C^q two subgroups are distinguished

$$Z^q = \text{kernel of } \delta: C^q \rightarrow C^{q+1},$$

$$B^q = \text{image of } \delta: C^{q-1} \rightarrow C^q$$

with the second definition completed by setting $B^0 = 0$. The condition $\delta\delta = 0$ is then equivalent with $B^q \subset Z^q$. The group

$$H^q = Z^q/B^q$$

is called the qth cohomology group of the sequence (1.1). The elements of the groups C^q, Z^q, B^q and H^q are called q-dimensional cochains, cocycles, coboundaries and cohomology classes respectively. Each cocycle $z \in Z^q$ determines a cohomology class as the coset $\{z\} = z + B^q$. Two cocycles in the same coset are called cohomologous.

The process just described can be applied in the following three instances.

I. *Groups.* Let Q be a multiplicative group and G an abelian group written additively. Assume further that Q operates on G, that is, that

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An address delivered before the Summer Meeting of the Society on September 15, 1945, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors May 29, 1948.

[1] The presentation has been brought up to date and includes many results obtained after 1945.

for each $x \in Q$ and $g \in G$ an element $xg \in G$ is given such that

$$x(g_1 + g_2) = xg_1 + xg_2, \qquad x_2(x_1 g) = (x_2 x_1)g, \qquad 1g = g$$

where 1 is the unit element in Q. The group $C^q(Q, G)$ of the q-dimensional cochains of Q in G is defined as the group of all functions f of q variables in Q with values in G. Addition in $C^q(Q, G)$ is defined by

$$(f_1 + f_2)(x_1, \cdots, x_q) = f_1(x_1, \cdots, x_q) + f_2(x_1, \cdots, x_q).$$

The coboundary homomorphism $\delta: C^q(Q, G) \to C^{q+1}(Q, G)$ is defined by the formula

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{q+1}) = {} & x_1 f(x_2, \cdots, x_{q+1}) \\ & + \sum_{i=1}^{q} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{q+1}) \\ & + (-1)^{q+1} f(x_1, \cdots, x_q).\end{aligned}$$

The relation $\delta\delta = 0$ is verified by computation. The resulting cohomology group $H^q(Q, G)$ is called the qth cohomology group of Q over G. Note that the operators of Q on G are part of the definition and a change in these operators will generally alter the group $H^q(Q, G)$.

II. *Associative algebras.* Let A be an associative algebra over a field F, and V a vector space over the same field. Assume further that V is a two sided A-module, that is, that for every $a \in A$, $v \in V$ elements av and va of V are defined which are bilinear functions of a and v, and satisfy

$$a_2(a_1 v) = (a_2 a_1)v, \qquad a_2(va_1) = (a_2 v)a_1, \qquad (va_2)a_1 = (va_2 a_1).$$

The group $C^q(A, V)$ of the q-dimensional cochains of A in V is defined as the group of all functions f of q variables in A with values in V, linear with respect to each variable. Addition in $C^q(A, V)$ is defined as before. The coboundary formula is

$$\begin{aligned}(\delta f)(a_1, \cdots, a_{q-1}) = {} & a_1 f(a_2, \cdots, a_{q-1}) \\ & + \sum_{i=1}^{q} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{q+1}) \\ & + f(a_1, \cdots, a_q) a_{q+1}.\end{aligned}$$

The resulting cohomology group $H^q(A, V)$ is the qth cohomology group of A over V. The groups $C^q, \cdots, H^q$ in this case are clearly vector spaces over the field F.

III. *Lie algebras.* Let L be a Lie algebra over a field F and let V

be a vector space over F. Assume further that a representation P of L in V is given, that is, that for every $x\in L$ a linear transformation $P_x\colon V\to V$ is given such that P_x is a linear function of $x\in L$ and that

$$P_yP_x - P_xP_y = P_{[x,y]}$$

for $x, y\in L$. The group $C^q(L, P)$ is defined as the group of all functions f of q variables in L with values in V, linear with respect to each variable and alternating with respect to permutations of the variables. Addition in $C^q(L, P)$ is defined as before. The coboundary formula is

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{q+1}) &= \sum_{k<l} (-1)^{k+l+1} f([x_k, x_l], x_1, \cdots, \hat{x}_k, \cdots, \hat{x}_l, \cdots, x_{q+1}) \\ &\quad + \sum_{i=0}^{q+1} (-1)^{i+1} P_{x_i} f(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1}),\end{aligned}$$

where the symbol under the circumflex is to be omitted. The resulting cohomology group $H^q(L, P)$ is called the qth cohomology group of L over P and clearly is a vector space over F. In the special case when $V=F$ and the representation P is trivial (that is, when $P_x=0$ for all $x\in L$), the second part of the coboundary formula is zero. We then write $H^q(L)$ instead of $H^q(L, P)$.

In all the three cases discussed above the convention prevails that a function of zero variables with values in G is an element of G. Thus $C^0(Q, G)=G$, $C^0(A, V)=V$, $C^0(L, P)=V$.

A certain common feature can be observed in all the points at which the cohomology theories defined above come into contact with established parts of algebra. The cocycles almost invariably arise as "deviations" from a certain simple behavior. This point is not sufficiently formalized to be discussed here, but will be emphasized as the various instances come up. A strong analogy with the "obstructions" considered in topology in problems on extension and classification of mappings and in the theory of fiber bundles is suggested. In some instances the "deviations" and the "obstructions" actually come into contact.

The definition of the cohomology groups of groups was given by the author and S. MacLane in 1943 ([12], [13], [15], see also Hopf [26] and Eckmann [6]).[2] The remainder of this paper is devoted to the discussion of these groups. After some preliminaries (§§2–3) we describe (§§4–8) the connections that have been established between this cohomology theory and other parts of algebra. This is followed (§§9–13) by the description of some intrinsic properties of the cohomology groups; these are often helpful in actual computations. Some contacts with topology become apparent. Finally (§§14–17) we give an account of the topological applications of the cohomology groups.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

The cohomology theory of associative algebras was studied by Hochschild [22], [23], [24]. As yet, there are no applications to topology. Some of the algebraic results are analogous to the corresponding theorems in cohomology theory of groups.

The cohomology groups of Lie algebras were studied by Chevalley and the author [5]. The contact with topology is established by the fact that the cohomology groups (real coefficients) of the space of a compact Lie group are isomorphic to the cohomology groups of its Lie algebra. The theory is still in a very rudimentary stage and needs further development.

2. **Alternative definitions.** If we compare the definition of the coboundary δf for the cases of groups and associative algebras we observe that they are formally identical except for the right operator on the last term. The two formulae can be brought to coincidence by assuming that the group Q acts on G as a group of *two-sided operators*, that is, that in addition to the left operators of Q on G, the group Q operates on G also on the right in such a way that

$$(g_1 + g_2)x = g_1x + g_2x, \qquad (gx_1)x_2 = g(x_1x_2), \qquad g1 = g,$$
$$(x_1g)x_2 = x_1(gx_2).$$

The formula for δf is then modified by replacing the last term $(fx_1, \cdots, x_q)$ by $f(x_1, \cdots, x_q)x_{q+1}$. The cohomology group $H^q(Q, G)$ thus obtained includes the earlier groups as special cases, namely when the right operators of Q on G are simple (that is, when $gx = g$ for all $x \in Q$, $g \in G$). It turns out that this generalization is not essential. In fact given that Q operates on G two-sidedly, define the group $\circ G$ which coincides with G as a group but has Q as a group of left operators as follows

$$x \circ g = xgx^{-1}.$$

Define a map

$$\phi : C^q(Q, \circ G) \to C^q(Q, G)$$

by setting

$$(\phi f)(x_1, \cdots, x_q) = f(x_1, \cdots, x_q)x_1 \cdots x_q.$$

The maps ϕ are isomorphisms and commute with the coboundary operators, thus inducing isomorphisms of the cohomology groups $H(Q, G)$ (with two-sided operators) and $H(Q, \circ G)$ (with left operators). From now on, unless the contrary is specified, we shall adhere to the original definition involving left operators only.

A cochain $f \in C^q(Q, G)$ is called *normalized* provided $f(x_1, \cdots, x_q) = 0$ whenever $x_i = 1$ for at least one index $i = 1, \cdots, q$. The normalized cochains form a subgroup C^q_n of C^q. Since the coboundary of a normalized cochain is again normalized, the groups $\{C^q_n\}$ define cohomology groups H^q_n the same way as the sequence $\{C^q\}$ defines H^q. Since every normalized cocycle also is a cocycle in the ordinary sense, a "natural" homomorphism $H^q_n \rightarrow H^q$ is induced. We have (Eilenberg-MacLane [15])

(2.1) *The natural homomorphism* $H^q_n(Q, G) \rightarrow H^q(Q, G)$ *maps the cohomology group of* Q *defined using normalized cochains isomorphically onto the cohomology group of* Q.

The same is true for two-sided operators.

3. **The groups** H^0, H^1. A 0-cochain $f \in C^0(Q, G)$ is by definition an element $g \in G$. Since $(\delta f)(x) = xg - g$ it follows that f is a cocycle if and only if $xg = g$. Since by definition $B^0(Q, G) = 0$ we have

(3.1) *The 0th cohomology group* $H^0(Q, G)$ *is the subgroup of those elements of* G *on which* Q *operates simply.*

A 1-cochain $f \in C^1(Q, G)$ is a function $f: Q \rightarrow G$ while its coboundary is

$$(\delta f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1),$$

thence f is a 1-cocycle if and only if

$$f(x_1 x_2) = f(x_1) + x_1 f(x_2).$$

Such functions f are known as *crossed homomorphisms* of Q onto G. In order that $f \in B^1(Q, G)$ we must have $f(x) = xg - x$ for some constant $g \in G$. Such crossed homomorphisms are called *principal homomorphisms.* Hence

(3.2) *The first cohomology group* $H^1(Q, G)$ *is the group of crossed homomorphisms of* Q *into* G *reduced modulo the principal homomorphisms. If the operators of* Q *on* G *are simple* (*that is, if* $xg = g$ *for all* x *and* g) *then* $H^1(Q, G)$ *is the group* Hom (Q, G) *of all homomorphisms of* Q *into* G.

The first cohomology group is useful in connection with the following concepts. Let G and P be groups with the group Q as a group of left operators. It is not assumed that P is abelian and it will be written multiplicatively. A pair (E, ϕ) will be called a Q-enlargement of P by G provided: (1°) E is a group with Q as left operators, (2°) G is a subgroup of E and a direct summand of E, (3°) the operators of Q on G agree with the operators of Q on E, (4°) ϕ is an operator homomorphism of E onto P with G as kernel. Two Q-enlargements (E_1, ϕ_1), (E_2, ϕ_2) of P by G are called equivalent if there is an operator isomorphism $\tau: E_1 \approx E_2$ leaving G pointwise fixed and such that $\phi_2\tau = \phi_1$.

A Q-enlargement (E, ϕ) of P by G is said to be ***inessential*** if there is an *operator* homomorphism $\psi: P \to E$ such that $\phi\psi$ is the identity. The inessential enlargements form an equivalence class.

Multiplication of equivalence classes of enlargements is defined as follows. Given Q-enlargements (E_1, ϕ_1), (E_2, ϕ_2) of P by G, consider the group $E_1 \times E_2$ and its subgroup F consisting of pairs (e_1, e_2) with $\phi_1(e_1) = \phi_2(e_2)$. Clearly F contains the group $G \times G$. Let $\overline{G}$ be the subgroup of $G \times G$ defined by all the pairs of the form (g, g^{-1}). The group $G \times G/\overline{G}$ can be identified with the group G by identifying each coset $(g_1, g_2)\overline{G}$ with the element $g_1 + g_2$. Now let $E = F/\overline{G}$ and define $\phi: E \to P$ by $\phi[(e_1, e_2)\overline{G}] = \phi_1(e_1) = \phi_2(e_2)$, also define the operators of Q on E by setting $x[(e_1, e_2)\overline{G}] = (xe_1, xe_2)\overline{G}$. With these definitions (E, ϕ) is a Q-enlargement of P by G which we define as the product $(E_1, \phi_1) \otimes (E_2, \phi_2)$.

(3.3) *The equivalence classes of enlargements of P by G over Q form a commutative group* Enl (Q, P, G). *The equivalence class of inessential enlargements is the zero element of this group.*

To exhibit the connection of the group Enl (P, G, Q) with cohomology theory we consider homomorphisms $\theta: P \to G$. With addition and operators defined by

$$(\theta_1 + \theta_2)(p) = \theta_1(p) + \theta_2(p), \qquad (x\theta)(p) = x[\theta(x^{-1}p)]$$

the homomorphisms θ form an abelian group Hom (P, G) with Q as left operators.

Let (E, ϕ) be a Q-enlargement of P by G. Since G is a direct summand of E and ϕ maps E onto P with G as kernel, there exists a homomorphism $\psi: P \to E$ such that $\phi\psi = $identity. Unless the enlargement is inessential ψ cannot be chosen to be an operator homomorphism. The expression

$$(*) \qquad x\psi(x^{-1}p) - \psi(p), \qquad x \in Q,\ p \in P,$$

measures the deviation of ψ from being an operator homomorphism. An application of ϕ shows that (*) is in G, and for a fixed $x \in Q$, (*) gives a homomorphism $\psi^*(x)$ of P into G. Thus $\psi^*(x) \in$Hom (P, G) and $\psi^* \in C^1(Q,$ Hom $(P, G))$. A computation yields

$$\begin{aligned}\psi^*(xy)(p) &= xy\psi(y^{-1}x^{-1}p) - \psi(p)\\ &= x[y\psi(y^{-1}x^{-1}p) - \psi(x^{-1}p)] + [x\psi(x^{-1}p) - \psi(p)]\\ &= x[\psi^*(y)(x^{-1}p)] + \psi^*(x)(p).\end{aligned}$$

Thus

$$\psi^*(xy) = x\psi^*(y) + \psi^*(x),$$

that is, $\delta\psi^* = 0$ and $\psi^* \in Z^1(Q, \text{Hom}\,(P, G))$. If ψ is replaced by another homomorphism $\psi_1 : P \to E$ such that $\phi\psi_1 = \text{identity}$ then $\psi_1 - \psi = \theta \in \text{Hom}\,(P, G) = C^0(Q, \text{Hom}\,(P, G))$, and a computation shows that $\psi_1^* - \psi^* = \delta\theta$. Thus the cohomology class $\{\psi^*\}$ of the cocycle ψ^* is independent of the choice of ψ and is completely determined by the enlargement (E, ϕ). This yields a map Enl $(Q, P, G) \to H^1(Q, \text{Hom}\,(P, G))$. It is easy to see that this correspondence yields an isomorphism onto.

(3.4) *The group* Enl (Q, P, G) *of equivalence classes of Q-enlargements of P by G is isomorphic with the cohomology group* $H^1(Q, \text{Hom}\,(P, G))$.

The previous argument becomes more transparent if instead of defining Hom (P, G) as a group with Q as left operators, we define Hom (P, G) with Q as two-sided operators as follows

$$(x\theta)(p) = x\theta(p), \qquad (\theta x)(p) = \theta(xp).$$

The "deviation" $\psi^*(x)$ is then defined as

$$\psi^*(x)(p) = x\psi(p) - \psi(xp).$$

Then

$$\psi^*(xy) = x\psi^*(y) + \phi^*(x)y$$

and ψ^* is a cocycle following the formula with two-sided operators. The discussion with left-operators corresponds then to the discussion with two-sided operators by means of the mapping ϕ of §2.

This discussion concerning enlargements is capable of the following generalization. Suppose that in addition to the group Q another group R is given that also operates on G and P on the right. We shall assume in addition that the operators of Q and R commute, that is, that

$$(xg)r = x(gr), \qquad (xp)r = x(pr)$$

for $x \in Q$, $r \in R$, $g \in G$, $p \in P$. We may then consider pairs (E, ϕ) which are simultaneously Q and R-enlargements and such that the operators of Q and R on E also commute. We shall then say that (E, ϕ) is a Q-R-enlargement. The equivalence classes of Q-R-

enlargements which are inessential as R-enlargements form a commutative group $\mathrm{Enl}_R\ (Q, P, G)$. The zero element of this group consists of the equivalence class of Q-R-enlargements (E, ϕ) which are Q-R-inessential, that is, of those for which there exists a homomorphism $\psi:P\rightarrow E$ which is both a Q- and R-operator homomorphism such that $\phi\psi=$identity. Replacing in the previous discussion all the homomorphisms $\theta:P\rightarrow G$ by R-operator homomorphisms, we find that the group $\mathrm{Enl}_R\ (Q, P, G)$ is isomorphic with the cohomology group $H^1(Q, \mathrm{Hom}_R\ (P, G))$ where $\mathrm{Hom}_R\ (P, G)$ is the group of R-operator homomorphisms of P into G with the operators of Q defined as before.

The above discussion of enlargements is new, but is a close analogue of a similar discussion for associative algebras (Hochschild [24]).

4. **Group extensions.** The discussion of the 2nd and 3rd cohomology groups is closedly connected with the problem of group extensions (Eilenberg-MacLane [16]).

Consider a group E, an invariant subgroup K of E and a homomorphism $\phi:E\rightarrow Q$ mapping E onto Q with kernel K. Let G denote the center of the group K. For every $e\in E$ the mapping

$$k \rightarrow eke^{-1}$$

is an automorphism of K; thus a homomorphism $E\rightarrow A(K)$ of E into the group $A(K)$ of automorphisms of K is defined. The subgroup K maps into the subgroup $I(K)$ of inner automorphisms of K. There results a homomorphism

$$\theta:Q \rightarrow A(K)/I(K).$$

Replacing $k\in K$ by $g\in G$ we also find a homomorphism $Q\rightarrow A(G)$. This means that Q operates on G. From now on we shall assume that *Q, G and the operators of Q on G are fixed.* The pair (E, ϕ) is then called *an extension of Q belonging to the center* G, the pair (K, θ) is then called the *kernel* of the extension (E, ϕ).

Two extensions (E_1, ϕ_1) and (E_2, ϕ_2) are called equivalent if there is an isomorphism τ of E_1 onto E_2 such that $\phi_2\tau=\phi_1$ and $\tau(g)=g$ for each $g\in G$.

Multiplication of equivalence classes of extensions is defined as follows. Given extensions (E_1, ϕ_1), (E_2, ϕ_2) of Q belonging to the center G, consider the group $E_1\times E_2$ and its subgroup F consisting of pairs (e_1, e_2) with $\phi_1(e_1)=\phi_2(e_2)$. Clearly F contains the group $G\times G$. Let $\overline{G}$ be the subgroup of $G\times G$ defined by all pairs of the form (g, g^{-1}). The group $G\times G/\overline{G}$ can be identified with the group G by identifying each coset $(g_1, g_2)\overline{G}$ with the element g_1+g_2. Now let $E=F/\overline{G}$ and define $\phi: E \rightarrow Q$ by $\phi[(e_1, e_2)\bar{G}] = \phi_1(e_1) = \phi_2(e_2)$. The extension (E, ϕ) is defined to be the product $(E_1, \phi_1) \otimes (E_2, \phi_2)$.

(4.1) *The equivalence classes of extensions of Q belonging to the center G (with prescribed operators of Q on G) form a commutative and associative system $\mathcal{E}(Q, G)$.*

Extensions (E, ϕ) in which $K = G$ (that is, the kernel is abelian) form a multiplicative subsystem $\mathcal{A}(Q, G)$ of $\mathcal{E}(Q, G)$. The extension (E, ϕ) with $K = G$ is said to *split* provided there exists a homomorphism $\psi: Q \rightarrow E$ such $\phi\psi = \text{identity}$. The splitting extensions constitute an equivalence class.

(4.2) *The extensions with $K = G$ form an abelian subgroup $\mathcal{A}(Q, G)$ of $\mathcal{E}(Q, G)$. The zero element of $\mathcal{A}(Q, G)$ is also a zero element for $\mathcal{E}(Q, G)$ and is the equivalence class of the splitting extensions.*

The extensions (E, ϕ) with $K = G$ are called the group extensions of Q by G, and $\mathcal{A}(Q, G)$ is called the group of group extensions of Q by G (with the prescribed operators of Q on G). The first result connecting group extension and cohomology is the following.

(4.3) *The group $\mathcal{A}(Q, G)$ of extensions of Q by G is isomorphic with the second cohomology group $H^2(Q, G)$.*

The nature of this isomorphism will be explained in detail since it is typical of the method by which the cohomology theory enters into a problem of algebra.

Let (E, ϕ) be an element of an equivalence class of $\mathcal{A}(Q, G)$. Then $\phi: E \rightarrow Q$ maps E onto Q with G as kernel and with prescribed operators. In an effort to split the extension we consider an arbitrary map $u: Q \rightarrow E$ such that $\phi u = \text{identity}$. If the map u is a homomorphism the extension splits. In general u will not be a homomorphism and the *deviation* of u from a homomorphism is a function $f(x, y)$ defined by

$$u(x)u(y) = f(x, y)u(xy)$$

for $x, y \in Q$. Clearly $f(x, y) \in G$ so that $f \in C^2(Q, G)$. A computation yields

$$\begin{aligned} [u(x)u(y)]u(z) &= f(x, y)f(xy, z)u(xyz), \\ u(x)[u(y)u(z)] &= [xf(y, z)]f(x, yz)u(xyz). \end{aligned}$$

Hence the associativity of E implies

$$f(x, y)f(xy, z) = [xf(y, z)]f(x, yz)$$

and since all the elements are in G we can pass to additive notation:

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0$$

that is,

$$\delta f = 0$$

and $f \in Z^2(Q, G)$. Thus the deviation f (called *factor set*) is a cocycle; the cohomology class $\{f\}$ of this cocycle is independent of the choice of the representative function u and also of the choice of the extension (E, ϕ) within its equivalence class. This yields a map $\mathcal{A}(Q, G) \rightarrow H^2(Q, G)$ which is easily verified to be an isomorphism onto. The cohomology class $\{f\} = F^2(E, \phi)$ attached to the extension (E, ϕ) may be regarded as the obstruction against splitting (E, ϕ); the extension splits if and only if $F^2(E, \phi) = 0$.

The study of extensions of Q by G also leads to an interpretation of the first cohomology group $H^1(Q, G)$ (Baer [2]). Let (E, ϕ) be an extension of Q by G. In the group $A(E)$ of automorphisms of E consider the subgroup A_1 consisting of automorphisms α which satisfy $\phi\alpha = \phi$. Then setting $m(e) = \alpha(e)e^{-1}$ yields a cocycle $m \in Z^1(E, G)$ with E operating on G as inner automorphisms. In A_1 consider the subgroup A_2 of all automorphisms which leave G pointwise fixed. If $\alpha \in A_2$ then there is a unique cocycle $n \in Z^1(Q, G)$ such that $n\phi = m$. It is easy to see that A_2 is abelian and that the correspondence $\alpha \rightarrow n$ yields an isomorphism $A_2 \approx Z^1(Q, G)$. Under this isomorphism $B^1(Q, G)$ corresponds to the subgroup A_3 of A_2 consisting of all automorphisms $e \rightarrow geg^{-1}$, $g \in G$. Thus $A_2/A_3 \approx H^1(Q, G)$.

Before we proceed with the discussion of extensions whose kernel is not abelian, we must devote some attention to kernels.

5. **Kernels.** A Q-kernel is a pair (K, θ) where K is a group and

$$\theta: Q \rightarrow A(K)/I(K)$$

is a homomorphism. If G is the center of K, then θ induces a homomorphism

$$Q \rightarrow A(G),$$

that is, Q operates on G. We shall only consider Q-kernels with a fixed center G and fixed operators of Q on G. Equivalence of kernels is defined in the obvious way. Multiplication of equivalence classes of Q-kernels is defined as follows. Given kernels (K_1, θ_1), (K_2, θ_2) consider the group $K = K_1 \times K_2/\overline{G}$ where G is the subgroup of $G \times G$ considered earlier. The group $G \times G/\overline{G}$, that we have identified earlier with G, is then the center of K. Let $x \in Q$ and let $\theta_i(x) = \alpha_i I(K_i)$ where $\alpha_i \in A(K_i)$, $i = 1, 2$. Since α_1 and α_2 coincide on G it follows that (α_1, α_2) is an automorphism of $K_1 \times K_2$ which leaves $\overline{G}$ invariant. Thus (α_1, α_2) defines an automorphism α of K. We define $\theta(x) = \alpha I(K)$. The resulting kernel (K, θ) is the product $(K_1, \theta_1) \otimes (K_2, \theta_2)$.

(5.1) *The multiplication of equivalence classes of Q-kernels is commutative and associative. The kernel (K, θ) with $K = G$ acts as a unit.*

We have observed in §4 that each extension (E, ϕ) of Q belonging to the center G determines a Q-kernel (K, θ). However not every kernel is the kernel of an extension. This defines a subclass of extendible kernels, that is, of those Q-kernels which are kernels of extensions. Two Q-kernels (K_1, θ_1) and (K_2, θ_2) are called ***similar*** provided there exist extendible Q-kernels (L_1, γ_1), (L_2, γ_2) such that $(K_1, \theta_1) \otimes (L_1, \gamma_1)$ is equivalent with $(K_2, \theta_2) \otimes (L_2, \gamma_2)$. Since the product of extendible kernels is extendible, similarity classes of kernels can be multiplied.

(5.2) *The similarity classes of Q-kernels with center G (and prescribed operators of Q on G) form an abelian group $\mathcal{K}(Q, G)$. The zero element of this group consists of the class of extendible Q-kernels.*

The main result concerning kernels is the following (Eilenberg-MacLane [16]).

(5.3) *The group $\mathcal{K}(Q, G)$ is isomorphic with the 3rd cohomology group $H^3(Q, G)$.*

This isomorphism is established by a method similar to that used for (4.3). Let (K, θ) be a Q-kernel. For each $x \in Q$ select $\alpha(x) \in A(Q)$ such that $\theta(x) = \alpha(x) I(K)$. Then

$$\alpha(x)\alpha(y) = \beta(x, y)\alpha(xy)$$

where $\beta(xy) \in I(K)$. Select $h(x, y) \in K$ so that

$$h(x, y) k h(x, y)^{-1} = \beta(xy)(k).$$

Finally define $f(x, y, z)$ by

$$[\alpha(x), h(y, z)] h(x, yz) = f(x, y, z) h(x, y) h(xy, z).$$

In the noncommutative sense f is then the coboundary of h. A computation shows that f is in the center of K, that is, in G, and that f is a cocycle $f \in Z^3(Q, G)$. The characteristic cohomology class $\{f\} = F^3(K, \theta)$ is independent of the choices made in the definition of f, and of the choice of (K, θ) within its equivalence class.

(5.4) *A Q-kernel (K, θ) with center G is extendible if and only if its characteristic cohomology class $F^3(K, \theta) \in H^3(Q, G)$ is zero.*

Since the characteristic cohomology classes add when kernels are multiplied it follows that $(K, \theta) \to F^3(K, 0)$ is a homomorphism $\mathcal{K}(Q, G) \to H^3(Q, G)$. Proposition (5.3) is proved by showing that this map is an isomorphism onto.

A remark should be made concerning the normalization of the cocycles considered in §4 and this section. The cocycle of §4 will be normalized if and only if we select the representative function u so that $u(1)=1$. In order to insure that the 3-dimensional cocycle of a kernel be normalized we select $\alpha(1)=1$ so that $\beta(1,y)=\beta(x,1)=1$. Then select h so that $h(1,y)=h(x,1)=1$.

For further results concerning kernels see MacLane [32].

6. **Extensions with a non-abelian kernel.** Let (K,θ) be a Q-kernel with center G. Assume further that $F^3(K,\theta)=0$ so that there is at least one extension (E_0,ϕ_0) of Q with (K,θ) as kernel. Let $[E_0,\phi_0]$ be the element of $\mathcal{E}(Q,G)$ determined by (E_0,ϕ_0). The main results then are (Eilenberg-MacLane [16]):

(6.1) *The map* $(E,\phi)\to(E_0,\phi_0)\otimes(E,\phi)$, *where* (E,ϕ) *is any extension of* Q *with* G *as kernel, establishes a* 1-1 *map of the group* $\mathcal{A}(Q,G)$ *onto the totality of equivalence classes of extensions of* Q *with kernel equivalent with* (K,ϕ). *The latter form then a coset* $[E_0,\phi_0]\otimes\mathcal{A}(Q,G)$.

Combining (6.1) with (4.3) yields:

(6.2) *If the* Q-*kernel* (K,θ) *is extendible then the equivalence classes of extensions* (E,ϕ) *of* Q *with kernel* (K,θ) *are in a* 1-1 *correspondence with the elements of the group* $H^2(Q,G)$ *where* G *is the center of* K *and the operators of* Q *on* G *are induced by* θ.

7. **Non-associative systems.** The fact that the deviation f defined in §4 by the equation $u(x)u(y)=f(x,y)u(xy)$ is a 2-cocycle is equivalent with the associativity law in the group E. This suggests that higher-dimensional cocycles may be obtained as deviations from the associativity law in suitable multiplicative systems. This point of view has been explored (Eilenberg-MacLane [17]) and leads to theorems generalizing (4.3) and employing the higher cohomology groups.

Let L be a loop (that is, a non-associative group). Given $a,b,c\in L$ define the *associator* $A(a,b,c)\in L$ by the equation

$$a(bc) = A(a,b,c)[(ab)c].$$

The associator is then a measure of the deviation from associativity. To illustrate the tendency of the associator to behave like a 3-cocycle assume that L is commutative and that the associativity law holds whenever one of the three elements is itself an associator. Then

$$\begin{aligned} a[b(cd)] &= A(a,b,cd)(ab)(cd) = A(a,b,cd)A(ab,c,d)[(ab)c]d,\\ a[b(cd)] &= A(b,c,d)a[(bc)d] = A(b,c,d)A(a,bc,d)[a(bc)]d\\ &= A(b,c,d)A(a,bc,d)A(a,b,c)[(ab)c]d. \end{aligned}$$

Thus (using additive notation)

$$A(b, c, d) - A(ab, c, d) + A(a, bc, d) - A(a, b, cd) + A(a, b, c) = 0$$

which shows that A is a suitable generalization of a 3-cocycle.

Higher associators are defined inductively as

$$A(a_1, \cdots, a_{2n+1}) = A(a_1, \cdots, a_{2n-2}, A(a_{2n-1}, a_{2n}, a_{2n+1}))$$

and also have a tendency to behave like cocycles.

The generalization of (4.3) is achieved by the study of prolongations. A *prolongation* of Q by G (with given operators of Q on G) is a pair (L, ϕ) satisfying the following five conditions:

(i) L is a loop containing G as a subgroup,

(ii) ϕ is a homomorphism of L onto Q with a kernel K containing G,

(iii) $A(k, a, b) = 1 = A(a, k, b)$ for $a, b \in L$, $k \in K$,

(iv) $kA = Ak$ for any $k \in K$ and any associator $A = A(a, b, c)$,

(v) $ag = (\phi(a)g)a$ for $a \in L$, $g \in G$.

The equivalence and multiplication of prolongations is defined as in §4. The equivalence classes of prolongations of Q by G form then a commutative and associative system $\mathcal{P}(Q, G)$. The splitting extension of Q by G acts as a unit for $\mathcal{P}(Q, G)$.

The non-associativity of L in a prolongation (L, ϕ) is further controlled by the consideration of the following four classes defined by the following conditions

O_n^L: $\quad A(a_1, \cdots, a_{2n}, a) = 1,$
O_n^K: $\quad A(a_1, \cdots, a_{2n}, k) = 1,$
G_n^L: $\quad A(a_1, \cdots, a_{2n}, a) = G,$
G_n^K: $\quad A(a_1, \cdots, a_{2n}, k) = G,$

where $n > 0$, $a_1, \cdots, a_{2n}, a \in L$ and $k \in K$. In addition we define the class R_n consisting of those prolongations (L, ϕ) of Q by G which contain a prolongation (L', ϕ') of Q by G (that is, $L' \subset L$ and ϕ' agrees with ϕ on L') of class O_n^K. We also define the class G_0^K by the condition $K = G$.

Next we consider the class $O_n^K \cap G_n^L$. This class contains both O_n^L and G_{n-1}^K and therefore contains the class $Q_n^K \otimes G_{n-1}^K$ of products of prolongation in O_n^L and G_{n-1}^K. We then define as in §5 the similarity classes of $O_n^K \cap G_n^L$ modulo $Q_n^L \otimes G_{n-1}^K$.

(7.1) *The similarity classes of* $O_n^K \cap G_n^L$ *modulo* $O_n^L \otimes G_{n-1}^K$ *form a group* $O_n^K \cap G_n^L / O_n^L \otimes G_{n-1}^K$ *which is isomorphic with the cohomology group* $H^{2n-1}(Q, G)$, $n > 0$.

Similarly

(7.2) *The similarity classes of* $O^L_{n-1}\cap G^K_n$ *modulo* $R_n\otimes G^L_n$ *form a group* $O^L_{n-1}\cap G^K_n/R_n\otimes G^L_n$ *isomorphic with the cohomology group* $H^{2n+2}(Q, G)$, $n>0$.

In the case $n=0$ the various classes can be interpreted so that this result reduces to the group extension case (4.3).

Let us consider the case $n=1$ of (7.1) in more detail. A prolongation (L, ϕ) is in the class $O^K_1\cap G^L_1$ if $A(a, b, k)=1$ and if $A(a, b, c)\in G$. Thus we obtain the class $O^K_1\cap G^L_1$ by replacing conditions (iii) and (iv) by

(iii)′ $A(k, a, b)=A(a, k, b)=A(a, b, k)=1$ for $a, b\in L$, $k\in K$,

(iv)′ all associators of L are in G.

The class O^L_1 consists of those prolongations (L, ϕ) for which L is associative, that is, a group, while G^K_0 is defined by the condition $K=G$. The similarity classes of $O^K_1\cap G^L_1$ modulo $O^L_1\otimes G^K_0$ form then a group isomorphic with $H^3(Q, G)$. This characterization of $H^3(Q, G)$ is quite different from the characterization of $H^3(Q, G)$ as the group of Q-kernels $\mathcal{K}(Q, G)$ asserted in (5.3). It is very likely that some sort of nonassociative generalization of the concept of a kernel and of the group $\mathcal{K}(Q, G)$ exists, that will give a characterization of the cohomology groups $H^q(Q, G)$ for $q>3$.

8. Connections with Galois theory.

Let P be a field, N a finite, separable, and normal extension of P, and Q the Galois group of N over P. Each element $\lambda\in Q$ is then an automorphism $\lambda: N\rightarrow N$ leaving P pointwise fixed. Thus Q acts as a group of operators on both the additive group N^+ of N and the multiplicative group $N^\times$ of the elements of N different from zero. Thus we are in a position to study the cohomology groups $H^q(Q, N^+)$ and $H^q(Q, N^\times)$. It is easy to verify that $H^q(Q, N^+)=0$ for all dimensions q. Thus our attention will be centered on the group $H^q(Q, N^\times)$, from now on written as $H^q(Q, N)$. The base field P is not indicated in this notation as it simply is the field of all fixed points of N under the automorphisms of the group Q.

The "principal genus theorem in the small" asserts that

$$H^1(Q, N) = 0. \tag{8.1}$$

The cohomology groups $H^2(Q, N)$ and $H^3(Q, N)$ enter the picture in connection with the theory of simple algebras and crossed products.

Consider the totality of all central simple algebras A over P with the Kronecker product over P as multiplication. The similarity classes modulo algebras which are full matrix algebras over P form a group $\mathcal{A}(P)$ called the Brauer group of algebra classes over P.

Let N be a finite separable and normal extension of P. For each algebra A over P construct the scalar extension $A_N = A \times_P N$ which is an algebra over N. If A is central simple over P then A_N is central simple over N. The correspondence $A \rightarrow A_N$ yields a homomorphism

$$(*) \qquad \mathcal{A}(P) \rightarrow \mathcal{A}(N).$$

The kernel of this homomorphism will be denoted by $\mathcal{A}^N(P)$ and consists of the algebra classes over P for which N is a splitting field.

(8.2) *The second cohomology group* $H^2(Q, N)$ *is isomorphic with the group* $\mathcal{A}^N(P)$ *of algebra classes over* P *split by* N.

This is a reformulation of a classical result and is obtained by constructing for each $f \in Z^2(Q, N)$ a crossed product algebra $A = (f, Q, N)$ as follows. A is a vector space over N generated by symbols $u(x)$ corresponding to elements $x \in Q$. The multiplication table is given by

$$[au(x)][bu(y)] = a(xb)f(x, y)u(x, y),$$

or equivalently

$$u(x)b = (xb)u(x), \qquad u(x)u(y) = f(x, y)u(xy).$$

With f assumed normalized, $u(1)$ is the unit element of A. The algebra A is central simple over P and has N as splitting field. Further every central simple algebra over P which is split by N is similar to a crossed product (f, Q, N) with f determined uniquely modulo coboundaries. This leads to (8.2) (see for instance Artin-Nesbitt-Thrall [1]).

The image of the homomorphism (*) will be denoted by $\mathcal{A}_N(P)$ and will be used presently.

A central simple algebra A over N will be called Q-normal provided every automorphism in the group Q can be extended to an automorphism of the algebra A. Teichmüller [33] has studied such algebras and has shown that if A is Q-normal then any algebra similar to A is Q-normal. Thus the Q-normal algebras determine a subgroup $\mathcal{N}_Q(N)$ of $\mathcal{A}(N)$. It is easy to see that

$$\mathcal{A}_N(P) \subset \mathcal{N}_Q(N).$$

The connection between Q-normal algebras and the cohomology group $H^3(Q, N)$ was introduced by Teichmüller who has defined a homomorphism

$$T^3: \mathcal{N}_Q(N) \rightarrow H^3(Q, N).$$

This correspondence could be described in the language of §5 as follows.

Let A be Q-normal and let K be the group of regular elements of A. Then $N^\times$ is the center of K. Since each automorphism $x \in Q$ can be extended to an automorphism of A each $x \in Q$ gives rise to subset $\theta(x)$ of the automorphism group $A(K)$. Since A is central simple over N, any two extensions of x differ by an inner automorphism and $\theta(x)$ is an element of $A(K)/I(K)$. Thus $\theta: Q \to A(K)/I(K)$ and the pair (K, θ) is a Q-kernel with $N^\times$ as center. The cohomology class $T^3(A)$ is then the characteristic cohomology class of the kernel (K, θ)

$$T^3(A) = F^3(K, \theta) \in H^3(Q, N).$$

Teichmüller [33] has proved that $\mathcal{A}_N(P)$ is the kernel of the homomorphism T^3, and thus that the group $\mathcal{N}_Q(N)/\mathcal{A}_N(P)$ is isomorphic with a subgroup of $H^3(Q, N)$. This subgroup was subsequently determined by Eilenberg-MacLane [18] as follows.

Let K be a finite normal and separable extension of P such that $P \subset N \subset K$ and let G be the Galois group of K over P. Each $\alpha \in G$ defines then an element $\phi(\alpha) \in Q$. Given a cochain $f \in C^q(Q, N)$ define the "lifted" cochain $\Lambda_K f \in C^q(G, K)$ by setting

$$\Lambda_K f(\alpha_1, \cdots, \alpha_q) = f(\phi(\alpha_1), \cdots, \phi(\alpha_q)).$$

This lifting operation defines a homomorphism

$$\Lambda_K : H^q(Q, N) \to H^q(G, K).$$

A cohomology class in the kernel of Λ_K is said to be *split* by K or to have *ceiling* K. The cohomology classes in $H^q(Q, N)$ which have a ceiling form a subgroup $H_0^q(Q, N)$. For $q=2$ this group can be shown to be trivial.

(8.3) *The homomorphism T^3 maps the group of Q-normal algebra classes over P onto the subgroup $H_0^q(Q, N)$ of $H^q(Q, N)$. The kernel of T^3 is the group $\mathcal{A}_N(P)$. Thus*

$$\mathcal{N}_Q(N)/\mathcal{A}_N(P) \approx H_0^3(Q, N).$$

The proof of (8.3) requires a detailed analysis of the Q-normality of crossed products. The computations are considerable and the normalization conditions discussed in §2 are used extensively. See also MacLane [31] for an alternative proof of a main lemma.

In the case when P is an algebraic number field, MacLane [30] has proved (without appeal to cohomology theory) that the group $\mathcal{N}_Q(N)/\mathcal{A}_N(P)$ is cyclic of order s where s is determined by the arithmetic properties of the extension N of P. Examples can be found with $s>1$; therefore $H_0^3(Q, N) \neq 0$ in this case.

9. **The first reduction theorem.** We now turn to the intrinsic prop-

erties of the cohomology groups of groups. The main problem consists in devising a method for computing the cohomology groups. In this connection the so-called "reduction theorems" are of great assistance. We call a "reduction theorem" any theorem that asserts that the cohomology group $H^q(Q, G)$ is isomorphic with a lower dimensional cohomology group of Q with a suitably altered coefficient group G.

In the reduction theorem that we are about to state it is convenient to use two-sided operators and normalized cochains. We first define Q as a group of two-sided operators on the group of cochains $C^n(Q, G)$ as follows

$$\begin{aligned}(xf)(x_1, \cdots, x_n) &= x[f(x_1, \cdots, x_n)],\\ (fx)(x_1, \cdots, x_n) &= x[f(x_1, \cdots, x_n)] - (\delta f)(x, x_1, \cdots, x_n).\end{aligned}$$

Next we establish an isomorphism

$$\tau_{q,n}: C^q(Q, C^n(Q, G)) \approx C^{q+n}(Q, G)$$

by setting

$$(\tau f)(x_1, \cdots, x_q, y_1, \cdots, y_n) = [f(x_1, \cdots, x_q)](y_1, \cdots, y_n)$$

for $f \in C^q(Q, C^n(Q, G))$. A computation shows that $\tau_{q,n}$ is an operator isomorphism and that it commutes with the coboundary operator

$$\delta\tau_{q,n} = \tau_{q-1,n}\delta.$$

Consequently we obtain isomorphisms

$$\tau_{q,n}: H^q(Q, C^n(Q, G)) \approx H^{q+n}(Q, G), \qquad q > 0. \tag{9.1}$$

This is the first reduction theorem (Eilenberg-MacLane [15]); it was originally discovered by Hochschild [22] for cohomology groups of associative algebras.

If Q operates on G only on the left (that is, the right operators of Q on G are simple) we may combine the map $\tau_{q,n}$ with the map ϕ of §2 and express the reduction theorem in terms of left operators only. Explicitly the formulae take the following form:

$$\begin{aligned}(xf)(x_1, \cdots, x_n) &= f(x_1, \cdots, x_n) - x(\delta f)(x^{-1}, x_1, \cdots, x_n),\\ (\bar{\tau}_{q,n}f)(x_1, \cdots, x_q, y_1, \cdots, y_n) &= x_1 \cdots x_q[f(x_1, \cdots, x_q)(y_1, \cdots, y_n)]\\ &\quad - \delta[f(x_1, \cdots, x_q)](x_1 \cdots x_q, y_1, \cdots, y_n).\end{aligned}$$

For $n = 1$ the above formulae give

$$(xf)(y) = xf(x^{-1}y) - xf(x^{-1}),$$

$$(\bar{\tau}_q f)(x_1, \cdots, x_q, y) = f(x_1, \cdots, x_q)(x_1, \cdots, x_q y)$$
$$- f(x_1 \cdots, x_q)(x_1 \cdots x_q)$$

and yield an isomorphism

$$\text{(9.2)} \qquad \bar{\tau}_q: H^q(Q, C^1(Q, G)) \approx H^{q-1}(Q, G) \qquad q > 0,$$

with left operators only.

As an application of the reduction theorem we prove:

(9.3) *If Q is a free group then $H^q(Q, G) = 0$ for $q \geqq 2$.*

In view of the reduction theorem it suffices to prove that $H^2(Q, G) = 0$. This amounts to proving that every extension (E, ϕ) of Q by G splits. Let $\{x_\alpha\}$ be a base for Q; select $\psi(x_\alpha) \in E$ so that $\phi\psi(x_\alpha) = x_\alpha$. The map ψ can be extended to a homomorphism $\psi: Q \rightarrow E$. Since $\phi\psi =$ identity, the extension splits.

10. **The cup product reduction theorem.** Let G_1, G_2, G be additive abelian groups. We shall say that G_1 and G_2 are *paired* to G, if for each $g_1 \in G_1$ and $g_2 \in G_2$ an element $g_1 \cup g_2 \in G$ is given such that

$$(g_1 + g_1') \cup g_2 = g_1 \cup g_2 + g_1' \cup g_2,$$
$$g_1 \cup (g_2 + g_2') = g_1 \cup g_2 + g_1 \cup g_2'.$$

If in addition Q acts as a group of left operators on G_1, G_2, G then we also require that for all $x \in Q$

$$x(g_1 \cup g_2) = xg_1 \cup xg_2.$$

Given cochains $f_1 \in C^p(Q, G_1)$, $f_2 \in C^q(Q, G_2)$ we define a cochain $f_1 \cup f_2 \in C^{p+q}(Q, G)$ as follows:

$$(f_1 \cup f_2)(x_1, \cdots, x_{p+q})$$
$$= f_1(x_1, \cdots, x_p) \cup x_1 \cdots x_p f_2(x_{p+1}, \cdots, x_{p+q}).$$

This yields a pairing of the groups $C^p(Q, G_1)$ and $C^q(Q, G_2)$ to the group $C^{p+q}(Q, G)$ with the following property

$$\delta(f_1 \cup f_2) = (\delta f_1) \cup f_2 + (-1)^p f_1 \cup \delta f_2.$$

This yields

$$\text{cocycle} \cup \text{cocycle} = \text{cocycle},$$
$$\text{cocycle} \cup \text{cobound.} = \text{cobound.},$$
$$\text{cobound.} \cup \text{cocycle} = \text{cobound.},$$

and thus leads to a pairing of the cohomology groups $H^p(Q, G_1)$ and $H^q(Q, G_2)$ to the group $H^{p+q}(Q, G)$.

In the case of two-sided operators we also require that $(g_1 \cup g_2)x = g_1 x \cup g_2 x$ and define $f_1 \cup f_2$ by

$$(f_1 \cup f_2)(x_1, \cdots, x_{p+q}) \\ = f_1(x_1, \cdots, x_p)x_{p+1} \cdots x_{p+q} \cup x_1 \cdots x_p f_2(x_{p+1}, \cdots, x_{p+q}).$$

Suppose now that Q is given as a factor group F/R where F is a free group (this amounts to describing Q by generators and relations). More precisely, consider an extension (F, ψ) where $\psi: F \rightarrow Q$ maps F onto Q and has kernel R. Let $[R, R]$ denote the commutator group of R; then $[R, R]$ is an invariant subgroup of both R and F. Define

$$F_0 = F/[R, R], \qquad R_0 = R/[R, R].$$

The extension (F, ψ) then defines an extension (F_0, ψ_0) with $\psi_0: F_0 \rightarrow Q$ mapping F_0 onto G with the abelian kernel R_0. According to §4 this extension leads to left operators of Q on R_0 and yields an element $f_0 \in H^2(Q, R_0)$ which describes the extension.

Now consider the group Hom (R_0, G) of all homomorphisms $\phi: R_0 \rightarrow G$; since $R_0 = R/[R, R]$ this group may be identified with the group Hom (R, G). Define Q as a group of left operators on Hom (R_0, G) by setting for $x \in Q$, $\phi: R_0 \rightarrow G$

$$(x\phi)(r) = x[\phi(x^{-1}r)], \qquad r \in R_0.$$

With these operators, the formula

$$r \cup \phi = \phi(r), \qquad r \in R_0,\ \phi \in \text{Hom}\ (R_0, G),$$

yields a pairing of the groups R_0 and Hom (R_0, G) to the group G. This in turn leads to a pairing of the cohomology groups $H^2(Q, R_0)$ and $H^q(Q, \text{Hom}\ (R_0, G))$ to the group $H^{q+2}(Q, G)$. The cup product reduction theorem (Eilenberg-MacLane [15]) then asserts

(10.1) *The homomorphism*

$$\lambda_q: H^q(Q, \text{Hom}\ (R_0, G)) \rightarrow H^{q+2}(Q, G)$$

defined by

$$\lambda_q(f) = f_0 \cup f$$

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for $f \in H^q(Q, \text{Hom}\ (R_0, G))$ *is an isomorphism onto for* $q > 0$ *and a homomorphism onto for* $q = 0$.

The kernel of λ_0 can be described as follows. The group $H^0(Q, \text{Hom}\ (R_0, G))$ consists of the elements of Hom (R_0, G) which are invariant under the operators of Q. This is precisely the group Ophom (R_0, G) of all operator homomorphisms $\phi: R_0 \rightarrow G$ (that is, homomorphisms satisfying $x(\phi(r)) = \phi(xr)$ for all $x \in Q$). Next define

F_0 as a group of left operators on G by setting $ag=\psi_0(a)g$ for $a\in F$, $g\in G$. Then every cocycle $f\in Z^1(F_0, G)$ when considered on R_0 yields an element Ophom (R_0, G) while every coboundary $f\in B^1(F, G)$ is zero on R_0. This yields a homomorphism

$$\eta: H^1(F_0, G) \to \text{Ophom}\,(R_0, G) = H^0(Q, \text{Hom}\,(R_0, G)).$$

(10.2) *The kernel of* λ_0 *is the image of* η.

For an alternative proof of (10.1) see Lyndon [29].

11. Cohomology groups of cyclic groups.

The cup product reduction theorem is particularly adaptable to the computation of the cohomology groups $H^q(Q, G)$ in the case of a cyclic group Q. Indeed if Q is cyclic of order h then $Q\approx J/hJ$ and the group Hom (R_0, G) in the reduction theorem is isomorphic to G. This method of computation is adopted in [15]. In view of the frequency with which cyclic groups occur in the applications it may be of interest to have a more direct method for computing the cohomology groups of cyclic groups. The following, hitherto unpublished, discussion has been obtained by the author jointly with S. MacLane.

Let Q be a cyclic group of order h and γ a generator of Q. Let G be an abelian group with Q as left operators. We set

$$Dg = \gamma g - g, \qquad Ng = \sum_{i=0}^{h-1} \gamma^i g, \qquad \text{for } g \in G,$$

$$DG = \{\text{all } Dg,\ g \in G\}, \qquad G_D = \{\text{all } g \text{ with } Dg = 0\},$$
$$NG = \{\text{all } Ng,\ g \in G\}, \qquad G_N = \{\text{all } g \text{ with } Ng = 0\}.$$

We verify readily that

$$NG \subset G_D, \qquad DG \subset G_N,$$

and that these subgroups are independent of the choice of γ. Clearly $H^0(Q, G) = G_D$.

For each cochain $f\in C^{q+2}(Q, G)$ define the cochain $\sigma f\in C^q(Q, G)$ by setting

$$(\sigma f)(x_1, \cdots, x_q) = \sum_{i=0}^{h-1} \gamma^{-(i+1)} f(\gamma, \gamma^i, x_1, \cdots, x_q).$$

A direct computation shows that

$$\delta(\sigma f) = \sigma(\delta f)$$

so that σ defines homomorphisms

$$\sigma: H^{q+2}(Q, G) \to H^q(Q, G), \qquad q > 0.$$

By iterating σ we obtain homomorphisms ($n\geqq 0$)

$$\sigma^n : H^{q+2n}(Q, G) \to H^q(Q, G), \qquad q > 0.$$

(11.1) *Let Q be a cyclic group of order h with generator γ. Then*

$$H^{2n+2}(Q, G) \approx G_D/NG, \qquad n \geqq 0,$$

where the isomorphism is obtained by assigning to each cocycle $f \in Z^{2n+2}$ the element $\sigma^{n+1}f$ of G. Further

$$H^{2n+1}(Q, G) \approx G_N/DG, \qquad n \geqq 0,$$

where the isomorphism is obtained by assigning to each cocycle $f \in Z^{2n+1}$ the element $(\sigma^n f)(\gamma)$ of G.

In the argument the following lemma is useful.

(i) If $f \in Z^q(Q, G)$ and $f(\gamma, x_2, \cdots, x_q) = 0$ then $f = 0$.

Indeed, developing $(\delta f)(\gamma, \gamma^i, x_2, \cdots, x_q)$ yields

$$f(\gamma^{i+1}, x_2, \cdots, x_q) = f(\gamma^i, x_2, \cdots, x_q)$$

and the proposition follows by induction in i.

We shall consider the group of integers J (with Q operating trivially) as paired with G to the group G by setting

$$m \cup g = mg, \qquad m \in J, \; g \in G.$$

Further, define a cochain $s_\gamma \in C^2(Q, J)$ by

$$\begin{aligned} s_\gamma(\gamma^i, \gamma^j) &= 0 \quad \text{if } i + j < h, \; 0 \leqq i, j < h, \\ s_\gamma(\gamma^i, \gamma^j) &= 1 \quad \text{if } i + j \geqq h, \; 0 \leqq i, j < h. \end{aligned}$$

(ii) s_γ is a cocycle.

This may be shown by a direct computation, using (i) to simplify the argument. Alternatively s_γ may be realized as the deviation (=factor set) of a group extension (E, ϕ) with E the additive group of rational numbers with denominator h, and $\phi : E \to Q$ the homomorphism $\phi(m/h) = \gamma^m$. The kernel of this extension is the group J. For each $\gamma^i \in Q$, $0 \leqq i < h$, choose $u(\gamma^i) = i/h$ as representative. Then $u(\gamma^i) + u(\gamma^j) = s_\gamma(\gamma^i, \gamma^j) + u(\gamma^{i+j})$, so that s is indeed the deviation and thus a cocycle. Since E is free this argument also shows that the cohomology class of s_γ is a cohomology class of the type used in the cup product reduction theorem.

A direct computation shows that

(iii) $$\sigma(s_\gamma \cup f) = f, \qquad f \in C^q(Q, G).$$

Setting for $f \in C^{q+2}(Q, G)$ and $0 \leqq i < h$

$$g(\gamma^i, x_2, \cdots, x_{q-1}) = \sum_{j=0}^{i-1} \gamma^{i-j-1} f(\gamma, \gamma\ , x_2, \cdots, x_{q-1}),$$

we also find by direct computation (using (i)) that

$$\text{(iv)} \qquad f + \delta g = s_\gamma \cup \sigma f, \qquad f \in Z^{q+2}(Q, G).$$

Formulae (iii) and (iv) show that for $q>0$, σ maps $H^{q+2}(Q, G)$ isomorphically onto $H^q(Q, G)$. The inverse isomorphism is given by the correspondence $f \rightarrow s_\gamma \cup f$. This reduces the proof of (11.1) to the cases when $n=0$.

For each $g \in G$ define $g_\gamma \in C^1(Q, G)$ by

$$g_\gamma(\gamma^i) = \sum_{j=0}^{i-1} \gamma^j g, \qquad 0 \leqq i < h.$$

Then, by direct computation (using (i)),

$$\text{(v)} \qquad \delta(g_\gamma) = s_\gamma \cup Ng.$$

For each $f \in Z^1(Q, G)$ define $g = f(\gamma)$. Since $f(\gamma^{i+1}) = \gamma f(\gamma^i) + f(\gamma)$ it follows that $g_\gamma = f$. Further by (v)

$$0 = \delta f = \delta g_\gamma = s_\gamma \cup Ng$$

so that $Ng=0$ and $g \in G_N$. Conversely, $g_\gamma(\gamma) = g$ and (v) shows that if $Ng=0$ then g_γ is a cocycle. Thus the correspondence $f \rightarrow f(\gamma)$ yields an isomorphism $Z^1(Q, G) \approx G_N$. If $f = \delta h$ then $f(\gamma) = \gamma h - h = Dh \in DG$ and vice versa. Thus $B^1(Q, G)$ corresponds to DG. Thus $H^1(Q, G)$ is shown to be mapped by $f \rightarrow f(\gamma)$ isomorphically onto G_N/DG.

For each $f \in Z^2(Q, G)$ we have $\sigma f \in Z^0(Q, G) = G_D$. If $g \in G_D$ then $s_\gamma \cup g \in Z^2(Q, G)$ and by (iii) $\sigma(s_\gamma \cup g) = g$. Thus σ maps Z^2 onto G_D. If $f = \delta h$ for $h \in C^1(Q, G)$ then $\sigma f = \sigma \delta h$ which computes to be $Nh(\gamma)$, thus $\sigma f \in NG$. Conversely if $\sigma f \in Ng$ then by (v)

$$s_\gamma \cup \sigma f = s_\gamma \cup Ng = \delta(g_\gamma).$$

Since by (iv) f is cohomologous to $s_\gamma \cup \sigma f$, it follows that f is a coboundary. Thus $f \in B^2(Q, G)$ if and only if $\sigma f \in NG$. Thus $H^2(Q, G)$ is shown to be mapped by $f \rightarrow \sigma f$ isomorphically onto G_D/NG.

This concludes the proof of (11.1). Incidentally we have shown that every cocycle $f \in Z^{2n+2}(Q, G)$ is cohomologous to a cocycle

$$s_\gamma \cup \cdots \cup s_\gamma \cup g$$

where $g \in G_D$ is uniquely determined modulo NG; and that every cocycle $f \in Z^{2n+1}(Q, G)$ is cohomologous to a cocycle

$$s_\gamma \cup \cdots \cup s_\gamma \cup g_\gamma$$

where $g \in G_N$ is uniquely determined modulo DG.

If $G = J$ is the group of integers with Q as trivial operators the result takes the following form:

(11.2) $H^{2n+1}(Q, J) = 0$. *The group* $H^{2n}(Q, J)$ *is cyclic of order* h *and is generated by the cohomology class of the* n*-fold cup product* $s^{2n} = s_\gamma \cup \cdots \cup s_\gamma$. *Every cocycle* $f \in Z^{2n}(Q, J)$ *is cohomologous to* $\sigma^n(f)s^{2n}$. *If the generator* γ *of* Q *is replaced by* α *where* $\gamma = \alpha^k$, $(k, h) = 1$, *then* s_α^{2n} *is cohomologous to* $k^n s_\gamma^{2n}$.

The last proposition requires verification. Observe that

$$\begin{aligned} \sigma(s_\alpha) &= \sum_{i=0}^{h-1} s_\alpha(\gamma, \gamma^i) = \sum_{i=0}^{h-1} s_\alpha(\alpha^k, \alpha^{ik}) \\ &= \sum_{i=0}^{h-1} s_\alpha(\alpha^k, \alpha^i) = \sum_{i=h-k}^{h-1} s(\alpha^k, \alpha^i) = k. \end{aligned}$$

Thus, by (iv), s_α is cohomologous $s_\gamma \cup k = ks_\gamma$.

12. **Abelian extensions.** We return to the interpretation of the cohomology group $H^2(Q, G)$, given in §4 as the group $\mathcal{A}(Q, G)$ of equivalence classes of group extensions (E, ϕ) with Q as quotient and G as kernel and with prescribed operators of Q on G. Assume now that Q is abelian and that the operators of Q on G (which is also abelian) are simple. This does not in general imply that E is abelian. In order that E be abelian it is necessary and sufficient that the deviation (or factor set) $f \in Z^2(Q, G)$ associated with the extension satisfy the additional symmetry condition

$$f(x, y) = f(y, x).$$

This leads to the definition of the symmetric subgroup $Z_s^2(Q, G)$ of $Z^2(Q, G)$. Since every coboundary $\delta h \in B^2(Q, G)$ is symmetric we obtain a subgroup $H_s^2(Q, G)$ of $H^2(Q, G)$. This subgroup is isomorphic with the subgroup of $\mathcal{A}(Q, G)$ given by extensions (E, ϕ) with E abelian. Thus $H_s^2(Q, G)$ may be called the group of abelian group extensions of Q by G. It is usually written as Ext (Q, G).

Suppose now that $Q = F/R$ where F is a free abelian group. Consider the homomorphism

$$\text{Hom}\,(F, G) \to \text{Hom}\,(R, G)$$

obtained by restricting each homomorphism of F to the subgroup R. Let Hom $(F, G) \mid R$ denote the image subgroup in Hom (R, G). Then

$$\text{Ext}\,(Q, G) \approx \text{Hom}\,(R, G)/\text{Hom}\,(F, G) \mid R. \tag{12.1}$$

This isomorphism is a consequence of the cup product reduction theorem, but can also easily be proved directly. It was proved in Eilenberg-MacLane [11] in connection with the following application to homology theory:

(12.2) *In any closure finite abstract complex K the cohomology group* $H^q(K, G)$ *can be expressed by means of the coefficient group G and the integral homology groups* $H_{q-1}(K)$ *and* $H_q(K)$ *as the direct sum*

$$H^q(K, G) \approx \operatorname{Hom}(H_q(K), G) + \operatorname{Ext}(H_{q-1}(K), G).$$

A similar result for homology groups has the form

$$H_q(K, G) \approx H_q(K) \otimes G + H_{q-1}(K) * G$$

where $\otimes$ *stands for the tensor product while* $*$ *denotes a product called the "dual product" that will be discussed in* §13 (H. Cartan and Eilenberg [4]).

13. **Homology groups of groups.** So far our discussion was concerned with the cohomology groups $H^q(Q, G)$ and their relations with other concepts of algebra. We now pass to the connections of these groups with topology. For the purpose of this discussion it is useful to have the *homology groups* $H_q(Q, G)$, G discrete abelian. We shall assume throughout this section that Q operates simply on the coefficient groups.

There are at least two methods of introducing these groups. One is based on Pontrjagin's duality theory of abelian groups. Consider a compact abelian group G. The group $C^q(Q, G)$ of cochains, which is isomorphic to a product of many copies of G, can then be topologized in a natural fashion so as to become a compact abelian group. In this way the cohomology group $H^q(Q, G)$ becomes a compact abelian group, if G is compact. Given now a discrete abelian group G consider the character group ChG which is compact and define

$$H_q(Q, G) = ChH^q(Q, ChG).$$

A more direct method of obtaining the homology groups proceeds through the construction of an abstract complex $K(Q)$ associated with the group Q. A q-dimensional "simplex" of $K(Q)$ is an ordered q-tuple $\sigma = (x_1, \cdots, x_q)$ of elements of Q. For each $0 \leqq i \leqq q$ we define the ith "face" $\sigma^{(i)}$ of σ as follows

$$\begin{aligned} \sigma^{(0)} &= (x_2, \cdots, x_q), \\ \sigma^{(i)} &= (x_1, \cdots, x_i x_{i+1}, \cdots, x_q), \qquad 0 < i < q, \\ \sigma^{(q)} &= (x_1, \cdots, x_{q-1}). \end{aligned}$$

This complex $K(Q)$ has sufficiently many properties of a simplicial complex to allow for the usual definition of chains, cochains, homology and cohomology groups. For instance a q-cochain with coefficients in G is a function f which to each q-simplex σ assigns an element $f(\sigma) \in G$. The coboundary δf is defined by

$$(\delta f)(\sigma) = \sum_{i=0}^{q+1} (-1)^i f(\sigma^{(i)})$$

for every $(q+1)$-simplex σ. It thus becomes apparent that the cochains in $K(Q)$ are simply the cochains of Q in G. With Q operating simply on G, the coboundary operators also coincide so that $H^q(K(Q), G) = H^q(Q, G)$. Now we can define the homology group $H_q(Q, G)$ to be the group $H_q(K(Q), G)$ of $K(Q)$.

The second method has the added advantage in that it shows that the various theorems concerning the relations of the various homology and cohomology groups of groups in an abstract complex also hold for the homology and cohomology groups of groups. In particular the isomorphisms

$$(13.1) \qquad H^q(Q, G) \approx \operatorname{Hom}(H_q(Q), G) + \operatorname{Ext}(H_{q-1}(Q), G),$$

$$(13.2) \qquad H_q(Q, G) \approx H_q(Q) \otimes G + H_{q-1}(Q) * G$$

hold, where $H_q(Q) = H_q(Q, J)$ is the integral homology group.

The "Künneth relations" in the cartesian product of two complexes are also applicable to the direct product of two groups and yield an isomorphism (H. Cartan and Eilenberg [4])

$$(13.3) \quad H_r(Q \times R) \approx \sum_{p+q=r} H_p(Q) \otimes H_q(R) + \sum_{p+q+1=r} H_p(Q) * H_q(R).$$

Formulae (13.1)–(13.3) allow for a complete computation of the groups $H_q(Q, G)$ and $H^q(Q, G)$ for Q an abelian group with a finite number of generators and with Q operating trivially on G. This was done using a more explicit method by Lyndon [28].

The integral homology group $H_2(Q)$ can be given an alternative description. Let $Q = F/R$ where F is a free group. In addition to the commutator group $[F, F]$, consider the group $[F, R]$ spanned by the elements $ara^{-1}r^{-1}$, $a \in F$, $r \in R$. Then

$$(13.4) \qquad H_2(Q) \approx (R \cap [F, F])/[F, R].$$

This isomorphism is an easy consequence of the cup product reduction theorem. Hopf [25] used this isomorphism as a definition of $H_2(Q)$.

An alternative description of the groups $H_q(Q, G)$ in the case G is a ring with a unit element (and Q operates simply on G) was given by Hopf [26]; the proof that Hopf's definition agrees with ours can be found in [9].

Assume that Q is abelian. The homology group $H_1(Q, G)$ is then symmetric in Q and G and is actually the tensor product $Q \otimes G$. The dual product $Q*G$ may be obtained as follows. Adjoin to the complex $K(Q)$ the symmetry relations

$$(x_1, x_2) = (x_2, x_1)$$

(in dimension two only) and denote the thus modified complex by $K'(Q)$. Then $H_2(K'(Q); G)$ is symmetric in Q and G and may serve as one of several alternative definitions of the dual product $Q*G$ (H. Cartan and Eilenberg [4]).

14. Relations between the fundamental group and homology groups. Let X be an arcwise connected topological space and $x^* \in X$. Relative to x^* as base point we define the fundamental group $\pi_1 = \pi_1(X)$ and the higher (abelian) homotopy groups $\pi_i = \pi_i(X)$, $i > 1$. The homology and cohomology groups $H_q(X, G)$ and $H^q(X, G)$ are defined using singular simplexes, that is, maps $T: s \to X$ where s is a euclidean simplex with ordered vertices. Since X is arcwise connected, we may without loss of generality assume that T maps all the vertices of s into the point x^*.

The homotopy group $\pi_q(X)$ has a natural homomorphism into the integral homology group $H_q(X)$, the image of this homomorphism is denoted by $\Sigma_q(X)$ and is called the *spherical subgroup* of $H_q(X)$. The spherical subgroups $\Sigma_q(X, G)$ of $H_q(X, G)$ can also be defined for an arbitrary coefficient group G. In the cohomology group $H^q(X, G)$ we distinguish the subgroup $\Lambda^q(X, G)$, called the group of *spherical annihilators*, which is the annihilator of $\Sigma_q(X)$ under the natural pairing of $H^q(X, G)$ and $H_q(X)$ to the group G.

Let now $T: s \to X$ be a singular q-simplex where s is a simplex with ordered vertices $v_0 < \cdots < v_q$ all of which are mapped by T into the base point x^*. The map of the edge $v_{i-1}v_i$ into X is then a closed path about x^* in X and determines an element x_i of the fundamental group π_1. We assign to the singular simplex T the q-tuple $(x_1, \cdots, x_q)$ of elements of π_1. If $T^{(i)}$ is the ith face of T, that is, the face obtained by omitting the vertex v_i of s, $i = 0, \cdots, q$, then $T^{(i)}$ corresponds respectively to the $(q-1)$-tuples

$$\begin{array}{cr} (x_2, \cdots, x_q) & \text{if } i = 0, \\ (x_1, \cdots, x_i x_{i+1}, \cdots, x_q) & \text{if } 0 < i < q, \\ (x_1, \cdots, x_{q-1}) & \text{if } i = q. \end{array}$$

If we compare this with the definition of the complex $K(\pi_1)$ in §13 we notice that the correspondence $\kappa: T \rightarrow (x_1, \cdots, x_q)$ maps the singular chains of X into chains of $K(\pi_1)$ and that this correspondence commutes with the boundary operator. This leads to homomorphisms

$$\kappa: H_q(X, G) \rightarrow H_q(\pi_1, G).$$

Similarly we obtain a map for cochains $\kappa: C^q(\pi_1, G) \rightarrow C^q(X, G)$. Indeed, if $f \in C^q(\pi_1, G)$, define

$$(\kappa f)(T) = f(x_1, \cdots, x_q)$$

where $T \rightarrow (x_1, \cdots, x_q)$. This map yields homomorphisms

$$\kappa: H^q(\pi_1, G) \rightarrow H^q(X, G).$$

The main result may now be stated

(14.1) *In an arcwise connected space X with vanishing homotopy groups π_i for $1 < i < q$ the following isomorphisms hold*:

$$H_i(X, G) \approx H_i(\pi_1, G), \qquad H^i(X, G) \approx H^i(\pi_1, G) \qquad \textit{for } i < q,$$
$$H_q(X, G)/\Sigma_q(X, G) \approx H_q(\pi_1, G), \qquad \Lambda^q(X, G) \approx H^q(\pi_1, G).$$

This theorem shows the extent to which the group π_1 determines the homology structure of X. All the isomorphisms are achieved by the correspondences κ described above. The correspondence κ for cochains commutes properly with the operation of constructing cup products (in π_1 and in X). More precisely if G_1 and G_2 are paired to the group [illegible] $H^{i_1}(\pi_1, G_1)$, $h_2 \in H^{i_2}(\pi_1, G_2)$ then

$$\kappa(h_1 \cup h_2) = \kappa(h_1) \cup \kappa(h_2).$$

Thus for $i_1 + i_2 \leqq q$, the group π_1 also determines the cup products in X.

In (14.1) the groups $H_i(\pi_1, G)$ and $H^i(\pi_1, G)$ are understood with π_1 operating trivially on G. If we consider homology and cohomology groups with so-called "local coefficients," that is, coefficients G admitting π_1 as a group of operators, the isomorphisms (14.1) remain valid, provided the various groups are suitably interpreted.

As a very special case of (14.1) we obtain the relation

$$H_2(X)/\Sigma_2(X) \approx H_2(\pi_1).$$

This result (with X a connected simplicial complex and $H_2(\pi_1)$ defined by (13.4)) was proved by Hopf [25] in 1942 and was the starting point of this whole investigation. Theorem (14.1) was first published by Eilenberg-MacLane [12; 13] and independently and in slightly different forms by Hopf [26], Freudenthal [21], and Eckmann [6].

The isomorphisms (14.1) have been combined with the cup-prod-

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uct reduction theorem to yield some, not yet fully understood, relations between the cohomology groups (with local coefficients) in a simplicial complex (Eilenberg [10]).

15. **Generalizations of** (14.1). Theorem (14.1) has been extended in several directions.

In the situation of (14.1), the group $\pi_1(X)$ determines the factor group $H_q(X)/\Sigma_q(X)$ of the integral homology group, while the subgroup $\Sigma_q(X)$ is a factor group of the qth homotopy group. This leads one to conjecture that perhaps π_1, π_q and the operators of π_1 on π_q jointly determine $H_q(X)$ (assuming $\pi_i=0$ for $1<i<q$). This is not the case as can be shown by examples [19; 34]. In order to obtain a description of $H_q(X)$ in terms of homotopy invariants, a new invariant seems to be necessary. This new invariant (Eilenberg-MacLane [14; 19]) is an element $k^{q+1}(X)$ of the (algebraic) cohomology group $H^{q+1}(\pi_1, \pi_q)$, with π_1 operating on π_q, and is defined whenever $\pi_i=0$ for $1<i<q$. The invariant $k^{q+1}(X)$ suffices to describe the groups $H_q(X, G)$ and $H^q(X, G)$ in a purely algebraic manner. The theorem to this effect yields (14.1) as a special case. The definition of $k^{q+1}(X)$ may be roughly described as follows. Consider the function κ which maps the singular chains of X into those of $K(\pi_1)$. In order to prove (14.1), an "inverse" function $\bar{\kappa}$ mapping the chains of $K(\pi_1)$ is defined such that $\bar{\kappa}$ commutes with ∂ and $\kappa\bar{\kappa}$ is the identity. The definition of $\bar{\kappa}$ utilizes the fact that $\pi_i=0$ for $1<i<q$ and can therefore be carried out only up to the dimension q inclusive. The invariant $k^{q+1}(X)$ is obtained by measuring the obstruction against the continuation of $\bar{\kappa}$ to the dimension $q+1$.

The invariant $k^3(X)\in H^3(\pi_1, \pi_2)$ is defined for every (arcwise connected) topological space. Suppose now that X is a simplicial complex containing at least two 2-simplices. Let S be the 1-dimensional skeleton of X and assume that the base point x^* has been chosen in S. The group $\pi_2(X)$ then maps isomorphically onto the center of the relative homotopy group $\pi_2(X, S)$ and there also is a natural homomorphism $\pi_1(X)\rightarrow A(\pi_2(X, S))/I(\pi_2(X, S))$ (see §5). Thus $\pi_2(X, S)$ may be regarded as a $\pi_1(X)$-kernel. The characteristic cohomology class of this kernel is precisely $k^3(X)$ (MacLane [32]).

Theorem (14.1) has also been extended by replacing the fundamental group by some higher homotopy group. In this order an abstract complex $K(Q, n)$ is defined for an *abelian* group Q and an integer $n>1$. The homology and cohomology groups of $K(Q, n)$ over G are denoted by $H_q(Q, n, G)$ and $H^q(Q, n, G)$. Then (14.1) generalizes as follows:

(15.1) *In an arcwise connected topological space X with vanishing* *homotopy groups* π_i *for* $i<n$ *and* $n<i<q$ *the following isomorphisms hold*:

$$H_i(X, G) \approx H_i(\pi_n, n, G), \qquad H^i(X, G) \approx H^i(\pi_n, n, G) \qquad \text{for } i<q,$$

$$H_q(X, G)/\Sigma_q(X, G) \approx H_q(\pi_n, n, G), \qquad \Lambda^q(X, G) \approx H^q(\pi_n, n, G).$$

Again in this case an invariant $k_n^{q+1}(X)$ belonging to the group $H^{q+1}(\pi_n, n, \pi_q)$ is defined and this invariant completely determines the groups $H_q(X, G)$ and $H^q(X, G)$ (Eilenberg and MacLane [19; 20]).

There are many indications that the study of the groups $H^q(Q, n, G)$ will cast light on some of the unsolved problems of homotopy theory. Interesting results have been obtained by G. W. Whitehead [35] for the case $q\leqq n+3$, by constructing a space X satisfying the conditions of (15.1) with $\pi_n(X)\approx Q$. A purely algebraic study of the groups $H^q(Q, n, G)$, falling back on the algebraic complex $K(Q, n)$, has been undertaken by the author and S. MacLane, and is in progress. So far, complete results were obtained only for $q\leqq n-2$.

Extensions of (14.1) and (15.1) to the case of relative homotopy and homology groups have been made by Blakers [3].

16. **Equivariant groups of complexes with operators.** The constructions leading to the proof of (14.1) were carried out in the space X. Instead one might carry out these constructions in the universal covering space $\tilde{X}$ of X and treat $\pi_1(X)$ as a group of homeomorphisms operating on $\tilde{X}$. One is thus lead to the study of complexes and spaces with a group of operators (Eckmann [6], Eilenberg [9]) which furnishes generalizations of (14.1) in a different direction.

Let K be an abstract complex. We shall say that Q operates on K if, for every q, the group Q operates (on the left) on the group of q-chains $C_q(K)$ in such fashion that the boundary operator $\partial: C_q(K) \to C_{q-1}(K)$ is an operator homomorphism. Let G be an abelian coefficient group with Q as a group of (left) operators. A q-cochain f of K over G, that is, a homomorphism $f: C_q(K)\to G$, will be called *equivariant* provided

$$f(xc) = xf(c), \qquad x\in Q,\ c\in C_q(K).$$

The coboundary of an equivariant is equivariant and therefore using equivariant cochains throughout we arrive at an equivariant cohomology group $H_e^q(K, G)$. This group admits a natural homomorphism

$$H_e^q(K, G) \to H^q(K, G)$$

into the ordinary cohomology group of K.

The complex K is called acyclic in dimensions less than q if the integer homology groups $H_i(K)$ vanish for $i<q$, with $H_0(K)$ interpreted as the reduced homology group. We shall say that Q operates on K freely provided for each $q\geqq 0$, the group $C_q(K)$ contains a set Ω_q such that the elements $x\omega$ ($x\in Q$, $\omega\in\Omega_q$) form a free base for the abelian group $C_q(K)$. It can be shown quite easily that if K is acyclic in dimensions less than q and Q operates freely on K, then the equivariant cohomology groups $H^i_e(K, G)$, $i<q$, are completely determined by Q, G and the operators of Q on G. In order to find precisely what these groups are, associate with the group Q an abstract complex K_Q as follows:

A q-dimensional "simplex" of K_Q is an ordered $(q+1)$-tuple $\sigma=(x_0, \cdots, x_q)$ of elements of Q. The ith face of σ $(0\leqq i\leqq q)$ is $\sigma^{(i)}=(x_0, \cdots, \hat{x}_i, \cdots, x_q)$, where the circumflex indicates that x_i is omitted. The operators of Q on K_Q are given by $x(x_0, \cdots, x_q)=(xx_0, \cdots, xx_q)$. The complex K is acyclic in all dimensions and Q operates freely on K. A q-dimensional cochain of K over G is simply a function f of $q+1$ variables in Q with values in G. The coboundary δf is defined by

$$\delta f(x_0, \cdots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(x_0, \cdots, \hat{x}_i, \cdots, x_{q+1})$$

while the equivariance condition is

$$f(xx_0, \cdots, xx_q) = xf(x_0, \cdots, x_q).$$

With each such equivariant cochain f we associate a cochain $\bar{f}\in C^q(Q, G)$ by setting

$$\bar{f}(x_1, \cdots, x_q) = f(1, x_1, x_1x_2, \cdots, x_1x_2\cdots x_q).$$

Conversely $\bar{f}$ defines f by

$$f(x_0, \cdots, x_q) = x_0\bar{f}(x_0^{-1}x_1, \cdots, x_{q-1}^{-1}x_q).$$

Moreover $\delta\bar{f}=\overline{\delta f}$. Thus the equivariant cohomology groups $H^q_e(K_Q, G)$ simply yield an alternative definition of the cohomology groups $H^q(Q, G)$. We refer to this definition as the "homogenous definition" in contrast with the original "nonhomogenous definition". If in the complex K_Q we introduce the relations $x(x_0, \cdots, x_q)=(xx_0, \cdots, xx_q)$ we obtain a complex isomorphic with $K(Q)$ as defined in §13. Thus K_Q may be regarded as a "univeral covering" of $K(Q)$.

The precise theorem is then the following.

(16.1) *Let K be a complex acyclic in dimensions less than q with the* *group Q operating freely on K. For every abelian coefficient group G with Q as (left) operators, the following isomorphisms hold:*

$$H^i(Q, G) \approx H^i_e(K, G) \qquad \textit{for } i < q,$$
$$H^q(Q, G) \approx \textit{kernel of } H^q_e(K, G) \to H^q(K, G).$$

In particular if Q operates without fixed points on a topological space X, acyclic in dimensions less than q, then Q operates freely on the singular complex $S(X)$ of X and (16.1) can be applied.

In order to derive (14.1) (for cohomology only) from (16.1) the following facts are needed: (1°) $\pi_1(X)$ operates on the universal covering space $\tilde{X}$ without fixed points, (2°) the condition $\pi_i(X)=0$ for $1<i<q$ is equivalent with the acyclicity of $\tilde{X}$ in dimensions less than q, (3°) under the natural projection $\tilde{X}\to X$, the group $H^i_e(\tilde{X}, G)$ maps isomorphically onto the group $H^i(X, G)$, to be taken with local coefficients if $\pi_1(X)$ operates on G, (4°) if $\pi_i(X)=0$ for $1<i<q$ then the kernel of $H^q_e(\tilde{X}, G)\to H^q(\tilde{X}, G)$ is mapped isomorphically onto the subgroup $\Lambda^q(X, G)$ of $H^q(X, G)$.

In addition to the ordinary and equivariant cohomology group $H^q(K, G)$ and $H^q_e(K, G)$ we may introduce a *residual* cohomology group $H^q_r(K, G)$ which is obtained by calculating modulo equivariant cochains. More precisely $H^q_r=Z^q_r/B^q_r$ where $f\in Z^q_r$ if and only if δf is equivariant, while $f\in B^q_r$ if and only if $f=\delta g+h$ with h equivariant. The three kinds of cohomogy groups are connected by homomorphisms

$$\cdots \to H^q_e(K, G) \to H^q(K, G) \to H^q_r(K, G) \to H^{q+1}_e(K, G) \to \cdots$$

and form an *exact sequence* in the sense that the image of each homomorphism is the kernel of the next one.

In the residual group $H^q_r(K, G)$ a subgroup $D^q_r(K, G)$ is distinguished by considering cochains $f\in C^q(K, G)$ such that $f(xz)=xf(z)$ for any $x\in Q$ and $z\in C_q(K)$ with $\partial z=0$.

Proposition (16.1) may then be supplemented as follows (Eilenberg [9]):

(16.2) *Under the conditions of* (16.1) *the following isomorphisms hold:*

$$H^{i+1}(Q, G) \approx H^i_r(K, G) \qquad \text{for } i < q,$$
$$H^{q+1}(Q, G) \approx D^q_r(K, G).$$

This last isomorphism, expressed in the dual language of homology, leads to the following result due to Hopf [27].

(16.3) *Let X be a q-dimensional connected polyhedron with $\pi_i(X)=0$*

for $1 < i < q$. *Denote by* γ_q *the kernel of the natural homomorphism* $\pi_q(X) \rightarrow H_q(X)$, *and by* $\pi_{q,0}$ *the subgroup of* π_q *spanned by all elements* $\alpha - x\alpha$ *with* $\alpha \in \pi_q$, $x \in \pi_1$. *Then* $\pi_{q,0} \subset \gamma$ *and*

$$H_{q+1}(\pi_1) \approx \gamma_q/\pi_{q,0}.$$

An analogous discussion of equivariant and residual homology groups is possible without interesting variations.

17. Generalizations. In analogy to the invariant $k^{q+1}(X) \in H^{q+1}(\pi_1, \pi_q)$ indicated in §15, an invariant $l^{q+1}(K) \in H^{q+1}(Q, H_q(K))$ may be defined for every complex K acyclic in dimensions less than q with Q as a group of operators. It was shown by the author and S. MacLane [**19**] that this invariant $l^{q+1}(K)$ determines the group $H^q_e(K, G)$ in its entirety as well as the homomorphism $H^q_e(K, G) \rightarrow H^q(K, G)$. Further it was proved by the author (unpublished) that $l^{q+1}(K)$ determines also the group $H^q_r(K, G)$ and the homomorphism $H^q(K, G) \rightarrow H^q_r(K, G)$. This implies that $l^{q+1}(K)$ also determines the kernel of $H^{q+1}_e(K, G) \rightarrow H^{q+1}(K, G)$.

Applying these results to the universal covering $\tilde{X}$ of a space X leads to the following result, supplementing (14.1).

(17.1) *Let X be an arcwise connected space with vanishing homotopy groups* π_i *for* $1 < i < q$. *The invariant* $k^{q+1}(X) \in H^{q+1}(\pi_1, \pi_q)$ *then determines the homology groups* $H_q(X, G)$ *as well as the factor group of* $H_{q+1}(X, G)$ *by the image of* $H_{q+1}(\tilde{X}, G)$ *under the natural projection of the universal convering* $\tilde{X}$ *into* X. *For integer coefficients* $k^{q+1}(X)$ *determines* $H_{q+1}(X)/\Sigma_{q+1}(X)$. *For cohomology* $k^{q+1}(X)$ *determines* $H^q(X, G)$ *and the kernel of* $H^{q+1}(X, G) \rightarrow H^{q+1}(\tilde{X}, G)$.

An interesting generalization of the equivariant theory has been recently introduced by Eckmann [**7**; **8**]. Let K be a complex with Q as a group of operators and let G be any abelian coefficient group (not necessarily with Q as operators). Let ψ be any subgroup of the group of all functions $Q \rightarrow G$. A cochain $f \in C^q(K, G)$ is called ψ-cochain if for every $c \in C_q(K)$ the expression $f(xc)$ treated as function of $x \in Q$ belongs to ψ. Replacing the equivariant cochains by ψ-cochains the ψ-groups $H^q_\psi(K, G)$ and the corresponding residual groups are defined.

In order to obtain the equivariant theory assume that Q operates on G and take ψ to be the group of all functions $x \rightarrow xg$, $g \in G$.

The next important case is the "almost zero" theory where ψ is taken to be the group of all functions $Q \rightarrow G$ which are zero except for a finite number of elements of Q. This choice is especially important if K is a locally finite simplicial (or cellular) complex which has a finite fundamental domain relative to the operators of Q. In this case the ψ-cochains are the *finite* cochains of Q so that $H^q_\psi(Q, G)$ is the cohomology group of K based on finite cochains. It is independent of the group Q.

The group $H^q_\psi(K_Q, G)$ in the "almost zero" theory is denoted by $\Pi^q(Q, G)$. More directly, consider as q-cochains, functions $f(x_0, \cdots, x_q) \in G$, defined for $x_0, \cdots, x_q \in Q$ such that for each system $x_0, \cdots, x_q$

$$f(xx_0, \cdots, xx_q) = 0$$

for all, except for a finite number, of $x \in Q$. Define the coboundary δf as

$$(\delta f)(x_0, \cdots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(x_0, \cdots, \hat{x}_i, \cdots, x_{q+1}).$$

Then $\Pi^q(Q, G)$ may be defined using the usual procedure for defining homology and cohomology groups.

All the results discussed in §16 remain valid for any choice of ψ. The same holds for the results involving an invariant $l^{q+1}(K)$. Curiously enough, this invariant remains in the group $H^{q+1}(Q, G)$, which is the equivariant group of K_Q. This shows that the equivariant theory has some features of universality.

Using the "almost zero" theory Eckmann has proved the following theorem due to Specker (unpublished).

(17.2) *In an orientable 3-dimensional manifold the fundamental group* π_1 *determines the homotopy group* π_2:

$$\pi_2 \approx \Pi^1(\pi_1, J)$$

where J *is the additive group of integers.*

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Columbia University

THE OCTOBER MEETING IN NEW YORK

The four hundred thirty-ninth meeting of the American Mathematical Society was held at New York University on Saturday, October 30. The attendance was approximately three hundred, including the following two hundred fifty-seven members of the Society:

Milton Abramowitz, C. R. Adams, C. F. Adler, E. J. Akutowicz, R. G. Albert, E. B. Allen, C. B. Allendoerfer, R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, Helmut Aulbach, Frederick Bagemihl, F. E. Baker, Joshua Barlaz, A. F. Bartholomay, F. D. Bateman, P. T. Bateman, F. P. Beer, Stefan Bergman, R. R. Bernard, D. L. Bernstein, Lipman Bers, Garrett Birkhoff, H. W. Bode, A. M. F. Borgers, D. G. Bourgin, H. W. Brinkmann, Paul Brock, Azelle Brown, A. B. Brown, J. D. Campbell, P. G. Carlson, P. W. Carruth, K. Chandrasekharan, Y. W. Chen, Alonzo Church, Randolph Church, L. W. Cohen, I. S. Cohn, H. R. Cooley, T. F. Cope, Natalie Coplan, Richard Courant, L. M. Court, V. F. Cowling, J. S. Cronin, D. A. Darling, Martin Davis, M. M. Day, B. V. Dean, J. B. Diaz, R. L. Dietzold, M. P. Dolciani, J. L. Doob, Avron Douglis, C. H. Dowker, Y. N. Dowker, R. H. Downing, Arnold Dresden, Nelson Dunford, Aryeh Dvoretzky, J. J. Eachus, Samuel Eilenberg, H. M. Elliott, Bernard Epstein, R. M. Exner, W. H. Fagerstrom, A. B. Farnell, J. M. Feld, C. D. Firestone, D. A. Flanders, Edward Fleisher, W. W. Flexner, R. M. Foster, R. H. Fox, D. A. S. Fraser, Bernard Friedman, G. N. Garrison, B. P. Gill, Sidney Glusman, V. D. Gokhale, Adolph Goodman, R. E. Goodman, M. J. Gottlieb, G. E. Gourrich, P. H. Graham, Harriet Griffin, George Grossman, Emil Grosswald, E. J. Gumbel, C. B. Hailperin, Carl Hammer, R. W. Hamming, E. E. Hammond, L. B. Hedge, G. A. Hedlund, M. H. Heins, Alex Heller, L. H. Herbach, H. L. Herrick, Einar Hille, A. J. Hoffman, Banesh Hoffmann, T. R. Hollcroft, G. B. Huff, E. M. Hull, T. R. Humphreys, Witold Hurewicz, L. C. Hutchinson, Eugene Isaacson, Nathan Jacobson, Fritz John, L. G. F. Jones, Irving Kaplansky, M. E. Kellar, D. E. Kibbey, S. A. Kiss, M. S. Klamkin, J. R. Kline, Morris Kline, E. G. Kogbetliantz, Horace Komm, B. O. Koopman, Saul Kravetz, Wouter van der Kulk, H. K. Kutman, H. G. Landau, A. W. Landers, J. R. Lee, Solomon Lefschetz, Joseph Lehner, R. A. Leibler, Max LeLeiko, D. C. Lewis, Samuel Linial, M. A. Lipschutz, Charles Loewner, A. J. Lohwater, E. R. Lorch, Lee Lorch, L. A. MacColl, H. M. MacNeille, A. J. Maria, M. H. Maria, W. T. Martin, Imanuel Marx, F. I. Mautner, F. M. Mautner, D. G. Mead, Paul Meier, K. S. Miller, G. C. Miloslavsky, Deane Montgomery, C. R. Morris, F. J. Murray, D. S. Nathan, C. A. Nelson, David Nelson, J. D. Newburgh, Morris Newman, Louis Nirenberg, P. B. Norman, I. L. Novak, T. E. Oberbeck, L. F. Ollmann, Paul Olum, Morris Ostrofsky, E. R. Ott, J. C. Oxtoby, J. S. Oxtoby, T. E. Peacock, A. M. Peiser, C. R. Phelps, Everett Pitcher, E. L. Post, Walter Prenowitz, M. H. Protter, W. T. Puckett, A. L. Putnam, Hans Rademacher, Arthur Radin, J. F. Randolph, G. N. Raney, Eric Reissner, Daniel Resch, Helene Reschovsky, Moses Richardson, C. E. Rickart, J. F. Ritt, J. H. Roberts, S. L. Robinson, V. N. Robinson, R. E. Root, I. H. Rose, P. C. Rosenbloom, Bernard Sachs, Raphael Salem, H. E. Salzer, L. A. Santaló, Arthur Sard, Samuel Schecter, Henry Scheffé, I. J. Schoenberg, Pincus Schub, Abraham Schwartz, H. M. Schwartz, J. T. Schwartz, G. E. Schweigert, Abraham Seidenberg, D. B. Shaffer, C. E. Shannon, H. N. Shapiro, I. M. Sheffer, Seymour Sherman, R. A. Siegel, D. N. Silver, R. C.

Simpson, James Singer, M. M. Slotnick, N. E. Steenrod, S. K. B. Stein, Fritz Steinhardt, Wolfgang Sternberg, J. J. Stoker, R. C. Strodt, Walter Strodt, Fred Supnick, Otto Szasz, Feodor Theilheimer, J. M. Thomas, D. L. Thomsen, G. L. Tiller, A. W. Tucker, J. W. Tukey, P. M. Treuenfels, Annita Tuller, J. L. Ullman, A. H. Van Tuyl, Oswald Veblen, J. L. Walsh, W. G. Warnock, Wolfgang Wasow, G. C. Webber, Alexander Weinstein, Louis Weisner, J. G. Wendel, M. E. White, P. M. Whitman, Hassler Whitney, Albert Wilansky, Jacob Wolfowitz, Arthur Zeichner, J. J. Zeig, J. A. Zilber, H. J. Zimmerberg, Leo Zippin.

On Saturday morning there were two sections for contributed papers: one in Analysis, in which Professor J. F. Randolph presided, and one in Algebra, Topology, Logic, and Applied Mathematics, in which Professor Eric Reissner presided.

At 2:00 P.M. Professor Raphael Salem of Massachusetts Institute of Technology gave an address on *Convexity theorems*. President Einar Hille presided.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 582 was read by Dr. Elliott, paper number 591 by Mr. Linial, and paper number 596 by Professor Cohen. Mr. Leutert was introduced by Dr. Stefan Bergman and Dr. M. M. Schiffer, and Mrs. Macintyre was introduced by Professor I. E. Segal.

ALGEBRA AND THEORY OF NUMBERS

1. Mary P. Dolciani: *On a problem in the additive theory of numbers.*

Using the Hardy-Littlewood circle method, the author has proved that an indefinite diagonal, ternary quadratic form $f(x)$ represents almost all integers consistent with the generic character of the form. More precisely, if $E(n)$ denotes the number of integers c less than n such that the equation $f(x)=c$ is solvable in the ring of p-adic integers for all primes p, but is not solvable in rational integers, then $E(n)=o(n)$. Similarly, the corresponding result can be proved for the case in which the variables are prime. (Received September 18, 1948.)

2*t*. A. L. Foster: *On the permutational representation of general sets of operations by partition lattices.*

Let $U=\{ \cdots, x, \cdots \}$ be a "ground" class (with or without operational structure), and $\Omega=\{ \cdots, \omega, \cdots \}$ an arbitrary set of unary mappings (= "monotations") of U onto $U: x\rightarrow_{\epsilon} U$. An "$\Omega$-partition" of U is a partition whose "cells" remain intact under all the cellular monotations induced by the various ground monotations $\omega \in \Omega$ (and also by the structure operations, if any). This paper studies the partition lattice structure of the class of all (a) "permutational-," respectively (b) "univoque"-Ω-partitions, in which the above cellular "factor monotations" are exclusively (a) permutations, respectively (b) univoque cellular mappings. This theory considerably generalizes the classical imprimitivity set theory of groups of transformations. As one typical result a simple lattice condition is found (which is always satisfied if, for instance, U is finite) insuring that all permutational Ω-partitions be "derivable" from a unique "atomic" one. One is led to a representative theory, a theory of "permutations-in-the-large," and, by use of previously introduced notions [Alfred L. Foster, *Maximal idempotent sets in a ring with unit*, Duke Math. J. vol. 13 (1946)] to an invariant theory of such general sets of transformations, similar to the homomorphism theory of rings and groups. (Received August 30, 1948.)

3*t*. Joseph Lehner: *Proof of Ramanujan's partition congruence for the modulus* 11^3.

The author proves Ramanujan's congruence: $p(11^3n+721)\equiv 0 \pmod{11^3}$, $n=0, 1, 2, \cdots$, where $p(n)$ is the number of partitions of n. The proof, in which he makes use of the theory of modular functions, is an extension of the method employed in a previous paper (*Ramanujan identities involving the partition function for the moduli* 11^α, Amer. J. Math. vol. 65 (1943)), in which the analogous congruence was established for 11 and 11^2. (Received September 7, 1948.)

4. F. I. Mautner: *On the principal irreducible representations of a discrete group*. Preliminary report.

The author studies in more detail the direct integral decomposition of the regular representation $U:g\rightarrow U(g)$ of a countably infinite discrete group G into irreducible representations. Let G_0 be the union of the finite classes of conjugate elements of G. It is shown that if G_0 is of infinite index in G then almost all the factors (in the sense of Murray and von Neumann, Ann. of Math. vol. 37 (1936) pp. 116–229) into which the weakly closed operator algebra generated by the $U(g)$ decomposes are of class II_1. From this it follows that almost all the irreducible (unitary) representations which occur in any direct integral decomposition of U into irreducible representations must be infinite-dimensional. In the proofs are used von Neumann's theory of generalized direct sums of Hilbert spaces and also results announced by the author in the Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 52–54. (Received September 13, 1948.)

5*t*. T. S. Motzkin: *The Euclidean algorithm*.

A constructive criterion for the existence of a Euclidean algorithm within a given integral domain is derived, and from among the different possible Euclidean algorithms in an integral domain one is singled out. The same is done for "transfinite" Euclidean algorithms. The criterion obtained is applied to some special rings, in particular rings of quadratic integers. Different sets of axioms for the Euclidean algorithm and related notions are compared, and the possible implications for the classification of principal ideal rings, and other integral domains, indicated. (Received October 11, 1948.)

6*t*. Leopoldo Nachbin: *On a characterization of the lattice of all ideals of a Boolean ring*.

A nonempty lattice L is isomorphic to the lattice of all ideals of a Boolean ring R, that is, a ring in which every element is idempotent, if and only if L satisfies the following conditions: (1) L is complete; (2) every element in L is the supremum of all smaller compact elements (x being *compact* if $x\leqq\bigvee_\lambda x_\lambda$ implies $x\leqq\bigvee_i x_{\lambda_i}$ for some finite subfamily); (3) the infimum of two compact elements is compact; (4) L is distributive; and (5) every inf-irreducible element x in L (that is, an element having no representation $x=x_1\wedge x_2$ with $x_1\neq x$ and $x_2\neq x$) distinct from its last element I is a dual atom (that is, $x<y$ implies $y=I$). Then R is essentially unique. R has a unity if and only if L satisfies the following equivalent conditions: (6) the last element of L is compact; (6′) every nonempty chain of elements of L distinct from I has a supremum distinct from I. The lattice of all open sets of a Hausdorff space satisfies (1), · · · , (5) if and only if the space is locally compact and totally disconnected. Therefore, by Stone's topological representation of Boolean rings, these conditions characterize the lattice of all open sets of such a space. The compact case corresponds to adding (6), or (6′). (Received September 9, 1948.)

7*t*. Sam Perlis and G. L. Walker: *Finite abelian group algebras. II.*

Earlier results of the authors (see Bull. Amer. Math. Soc. Abstract 54-7-298) for group algebras G_F of abelian groups G of prime power order n over any nonmodular field F are extended to the case in which n is arbitrary and F is any field of characteristic not dividing n. (Received September 2, 1948.)

8*t*. H. E. Salzer: *Verification of the first twenty thousand cases of an empirical theorem with the aid of a device for mass computation.*

This investigation established that the first 20,000 multiples of 5, that is $5m$ up to $5m=100{,}000$, were a sum of four nonnegative tetrahedral numbers $n(n+1)(n+2)/6$ (incidentally, checking H. W. Richmond's statement about the first 4,000 multiples). Use was made of the author's elementary theorem that every multiple of 5 which is a sum of four tetrahedral numbers is a sum of two numbers of a set S, the members of S consisting of multiples of 5 that are the sum of two tetrahedral numbers. All numbers of $S\leqq 100{,}000$ were indicated by the position of a blacked box on coordinate paper. Simultaneous mass additions of many numbers of S to any particular number of S was performed by superposing translucent coordinate paper bearing a replica of the box-configuration for S, and marking the spaces over black boxes. This computation, being lengthy, was done only once. Although, without checking, this result cannot be absolutely guaranteed, an exception is extremely improbable, since a tiny part of all conceivable combinations of two members of S sufficed for every $5m$. A photographic analogue of this process could handle similar computations of 1000 times the range, and much more rapidly. (Received September 1, 1948.)

ANALYSIS

9*t*. E. F. Beckenbach: *Convexity theorems for Legendre polynomials.*

It is shown that for Legendre polynomials $P_n(x)$ the function $\Delta_{n,1}(x)\equiv P_n^2-P_{n-1}P_{n+1}$ is a concave function of x for all real x. The Turán-Szegö theorem that $\Delta_{n,1}\geqq 0$ for $|x|\leqq 1$ is an immediate consequence, as are other results including the following: for $|x|\leqq 1$ we have $\Delta_{n,1}(0)\geqq\Delta_{n,1}(x)\geqq(1-|x|)\Delta_{n,1}(0)$; for $|x|\geqq 1$ we have $\Delta_{n,k}(x)\equiv P_n^2-P_{n-k}P_{n+k}\leqq 0$ and a fortiori $|P_n|\leqq(|P_{n-k}|+|P_{n+k}|)/2$, $k=1,\ 2,\cdots$. (Received August 30, 1948.)

10*t*. Garrett Birkhoff: *Functionals on spaces of continuous functions.*

Let R be any bounded open set in Euclidean m-space; let S be the boundary of R, and K the compact set $R\cup S$. Let $C(K)$ be the Banach space of continuous functions $f(\vec{x})$ on K, with $\|f\|=\sup|f(\vec{x})|$. It is shown that every linear functional λ on $C(K)$ has the form $\lambda(f)=\int f(\vec{x})d\mu$, where $d\mu=dg(\vec{x})$ is the additive function of intervals corresponding to a Stieltjes integral for a left-continuous function $g(\vec{x})$ of bounded variation. This gives a useful compactness principle for measure functions; the case $m=1$, K an interval, has been treated by F. Riesz. An extension to any compact metrizible compact topological space K is possible. (Received August 10, 1948.)

11. Garrett Birkhoff: *Existence of generalized Poisson integrals.*

Let $U(\vec{x})$ be any continuous function not assuming its maximum on the boundary S of a compact domain $K=R\cup S$ of definition. Then there exists a point $\vec{b}$ in R and an $\epsilon>0$ such that for every mass-function with center of gravity $\vec{b}$, $\int U(\vec{x})d\mu \leqq U(\vec{b})-\epsilon\int|\vec{x}-\vec{b}|^2 d\mu$. Let $\mathfrak{S}$ be any family of continuous functions U, satisfying a fixed integro-differential equation of elliptic type: that is, such that for each point $\vec{a}$ of R there exists a family of measure functions μ_k (with total measure one) such that $\lim_{k\to 0}|\int U(\vec{x})d\mu_k - U(\vec{a})|/\int|\vec{x}-\vec{a}|^2 d\mu_k \to 0$. Then there exists for each $\vec{a}$ in R a measure function μ_a such that $\int(U(\vec{x})d\mu_a = U(\vec{a})$ for all $U\in\mathfrak{S}$, and $\int \vec{x}d\mu_a = \vec{a}$. A maximum principle and uniqueness theorem are corollaries. It also shows that any continuous function whose generalized Laplacian (in the sense of Petrini or Privaloff-Saks) vanishes identically in R is harmonic. (Received August 10, 1948.)

12. P. W. Carruth: *Roots and factors of ordinals.*

An ordinal is termed a *root* of a second ordinal if the latter is a power of the former. Use is made of results of S. Sherman (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 111–116) to determine necessary and sufficient conditions for one ordinal to be a root of another. These conditions lead to a proof of the fact that the set of roots of an ordinal is closed. Results of E. Jacobsthal (Math. Ann. vol. 67 (1909) pp. 130–144) are used to give a short alternative proof of this last fact and of the known fact that the set of left factors of an ordinal is closed. Results also are obtained concerning the decomposition of an ordinal into the product of prime factors. (Received September 17, 1948.)

13. Aryeh Dvoretzky: *On sections of power series.* II.

The results announced in an earlier abstract (Bull. Amer. Math. Soc. Abstract 54-3-130) about the behavior of the partial sums, and so on, of a power series near a point on its circle of convergence can be extended and made more precise. Thus (1) can be strengthened into the statement that for infinitely many n each value of $|w|<\rho_n$ is assumed more than cn times, $c=c(\epsilon)$ being a positive constant. A similar remark applies to (2) and (3). The conclusion of (4) holds under the much weaker assumption $\limsup \log|a_n|/\log n \neq \pm\infty$. Also when this assumption fails, definite results may be given. Thus if $\lim \log|a_n|/\log n = -\infty$ and $\sum_0^\infty a_n z_0^n \neq 0$ we have $\liminf d_n/\gamma_n \leqq 1$ where $\{\gamma_n\}$ is a monotone sequence decreasing to zero defined as follows: Let n_0 be such that $|a_n|<1$ for $n>n_0$, then for $n>n_0$ the points (n, γ_n) are obtained through Newton polygon construction from the sequence $\{(n, -n^{-1}) \log|a_n|)\}$. More precisely: for every $\epsilon>0$ there exist infinitely many n for which the nth partial sum has a root in C_n: $|z-z_0 \exp \gamma_n|<(1+\epsilon)n^{-1}\log n$. Statements are also made about the domains covered by $w=s_n(z)$ when z is in C_n. Similar results hold also in the case $\limsup \log |a_n|/\log n = \infty$. It is sufficient for the validity of these results that $f(z)$ be bounded away from zero and infinity as $z \to z_0$ through some sector containing the origin. When z_0 is a regular point and $f(z_0) \neq 0$, $n^{-1} \log n$ can be strengthened to $O(n^{-1})$. Finally, similar results hold for remainders and general sections. (Received September 20, 1948.)

14*t*. H. Margaret Elliott: *On approximation to functions satisfying a generalized continuity condition.*

Let C be an analytic Jordan curve. Let $u(z)$ be harmonic interior to C, continuous on the corresponding closed region $\overline{C}$, and let $\partial^k u/\partial s^k$, s arc-length measured along C, have modulus of continuity $\omega(\delta)$ on C. It is shown that there exist harmonic polynomials $p_n(z)$ of respective degrees n, $n=1, 2, \cdots$, such that $|u(z)-p_n(z)| \leqq M\omega(1/n)/n^k$, z in $\overline{C}$. Given the existence of a sequence of harmonic polynomials $p_{n_j}(z)$ of respective degree n_j, $j=1, 2, \cdots$, which satisfy $|u(z)-p_{n_j}(z)| \leqq \Omega(n_j)/n_j^k$, z in $\overline{C}$, it is shown under suitable restrictions on C, $\Omega(x)$, and the sequence n_j that $\partial^k u/\partial s^k$ exists on C and has modulus of continuity (1) $\omega(\delta) \leqq L[\delta\int_{n_1}^{1/\delta}\Omega(x)dx + \int_{\eta_0/\delta}^{\infty}\Omega(x)/x)dx]$, $0<\delta \leqq 1/n_1$. Furthermore, if $\partial^k u/\partial s^k$ has modulus of continuity $\omega(\delta)$ on C_R, $R>1$, then there exist harmonic polynomials $p_n(z)$ such that $|u(z)-p_n(z)| \leqq M_1\omega(1/n)/n^k R^n$, z in $\overline{C}$. Conversely if there exist harmonic polynomials $p_{n_j}(z)$, $j=1, 2, \cdots$, such that $|u(z)-p_{n_j}(z)| \leqq \Omega(n_j)/n_j^k R^{n_{j+1}}$, z on $\overline{C}$, $R>1$, then $\partial^k u/\partial s^k$ has modulus of continuity on C_R satisfying a condition of the form (1). Analogous results are obtained for analytic functions. (Received September 17, 1948.)

15*t*. H. Margaret Elliott: *On approximation to functions satisfying a generalized continuity condition, in the sense of least pth powers.*

This paper is a sequel to one on Tchebycheff approximation to functions satisfying a generalized continuity condition. Let $u(z)$ be defined on C. It is shown that if there exist harmonic polynomials $p_{n_j}(z)$ of respective degrees n_j, $j=1, 2, \cdots$, such that $\int_C \Delta(z)|u-p_{n_j}(z)|^p ds \leqq [\Omega(n_j)]^p/n_j^{kp+1}$, $p>0$, then under suitable restrictions on C (it is sufficient for C to be an analytic Jordan curve), $\Delta(z)$, $\Omega(x)$, and the sequence n_j, we have $u(z)=u_1(z)$ almost everywhere on C, where $u_1(z)=\lim_{j\to\infty} p_{n_j}(z)$, z on $\overline{C}$, and $\partial^k u_1/\partial s^k$, s arc-length measured along C, has on C modulus of continuity (1) $\omega(\delta) \leqq L[\delta\int_{n_1}^{1/\delta}\Omega(x)dx + \int_{\eta_0/\delta}^{\infty}(\Omega(x)/x)dx]$, $0<\delta \leqq 1/n_1$. Suppose the rectifiable Jordan curve C is such that $x'(w)$ is continuous and different from zero, $|w|=1$, where $z=x(w)$ maps the interior of $|w|=1$ onto the interior of C; if there exist harmonic polynomials $p_{n_j}(z)$, $j=1, 2, \cdots$, such that $\int_C \Delta(z)|u-p_{n_j}|^p ds \leqq [\Omega(n_j)/n_j^k R^{n_{j+1}}]^p$, $p>1$, $R>1$, then under suitable restrictions on $\Delta(z)$, $\Omega(x)$, and n_j, it is proved that $f_1^{(k)}(z)$ has modulus of continuity in $\overline{C}_R$ which satisfies a condition of the form (1), where $f_1(z)$ is the analytic function such that $\mathfrak{R}[f_1(z)]=u_1(z)$. The corresponding problems are treated for approximation measured by a surface integral. (Received September 17, 1948.)

16*t*. A. S. Galbraith, Wladimir Seidel, and J. L. Walsh: *On the growth of derivatives of functions omitting two values.*

Results of two of the authors (W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle* $\cdots$, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 128–216; in particular Chap. IV) on relations between the radius of p-valence and the rate of growth of the derivatives of functions analytic in the unit circle are extended. The principal result is: If $f(z)$ omits two values, if $\{z_n\}$ is an infinite sequence of points within the unit circle for which $w_n=f(z_n)\rightarrow\infty$, and if the radius of p-valence $D_p(w_n)$, for some integer $p>0$, satisfies $\lim_{n\rightarrow\infty}(|w_n| \log |w_n|)^p D_p(w_n)=0$, then $\lim_{n\rightarrow\infty}(1-|z_n|^2)^k|f^{(k)}(z_n)|=0$, $k=1, 2, \cdots, p$. (Received October 15, 1948.)

17*t*. V. L. Klee. *On a theorem of Mazur.*

A well known theorem of Mazur states that a convex body in a linear normed space is supported at each of its boundary points. The usual proofs lean heavily on the norm, so are not applicable in more general spaces. The author gives a simple proof (using Zorn's lemma) which is valid in an arbitrary linear topological group. (Received August 9, 1948.)

18*t*. V. L. Klee: *The convex hull of a Borel set.*

Suppose that X is a Borel set of type G_α in the Banach space E. Then the convex hull $C(X)$ is also a Borel set, of type $G_{\alpha\sigma}$. If E is finite-dimensional and X is a Borel set of a certain type in E, then $C(X)$ is a Borel set of the same type, unless X is closed and unbounded, in which case $C(X)$ is a G_δ set and F_σ set, but is not necessarily closed. The proof uses Schauder's theorem on the interiority of linear continuous transformations. (Received August 31, 1948.)

19*t*. V. L. Klee: *The Tietze-Matsumura theorem in a linear topological space.*

Suppose that E is a linear topological group in which each line is a continuous image (under the natural mapping) of the real line. Then each closed, connected, locally convex subset of E is convex. It is believed that the proof is simpler, and the restriction on the space is less than those in any preceding paper on this theorem. (Received September 2, 1948.)

20*t*. Werner Leutert: *On the solution of boundary value problems in the theory of elasticity.*

Analogous to the theory of kernel functions for partial differential equations [Bergman, Duke Math. J. vol. 14 (1947) pp. 349–366; Bergman and Schiffer, Duke Math. J. vol. 14 (1947) pp. 609–638; Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1141–1151; Duke Math. J. vol. 15 (1948) pp. 535–566; Aronszajn, Proc. Cambridge Philos. Soc. vol. 39 (1943) pp. 133–153; Work performed under Contract N5ori 76-16 NR 043–046, Technical Report 5, Preliminary Note] a reproducing transformation is introduced for the system of three partial differential equations of second order for the components of the displacement vector in the theory of elasticity. The boundary value problem is stated and solved in a general form of which the three usually considered boundary value problems are limit cases. Formulas are given for the actual computation of the analogs of Green's, Neumann's, and Robin's functions. The results are a generalization of those obtained by Bergman [Math. Ann. vol. 98 (1927) pp. 248–263] for the solution of the first and second boundary value problems. (Received September 16, 1948.)

21*t*. Sheila S. Macintyre: *On the zeros of successive derivatives of integral functions.*

It is shown that if $f(z)$ is an integral function satisfying $\lim\sup_{r\rightarrow\infty}$ $(\log M(r)/r)$

$<.7259$ and if $f^{(n-1)}(z_n)=0$, the sequence $\{z_r\}$ having all its limit points in the unit circle, then $f(z)\equiv 0$. Previous results are thereby improved and extended, see N. Levinson (Duke Math. J. vol. 11 (1944) pp. 729–733; vol. 12 (1944) p. 335). A further theorem of the same type is proved in which the unit circle is replaced by the domain consisting of the points distant less than h from the segment $-1\leqq x\leqq 1$, and the constant .7259 is replaced by $4^{-1}\pi \exp(-\pi^2 h/8)$. This is an extension of a theorem due to I. J. Schoenberg (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 12–23). A shortened proof of Schoenberg's original result is also given. (Received September 7, 1948.)

22*t*. C. N. Moore: *On the functional behavior of a function defined by a certain Dirichlet's series.*

Form a sequence of the odd integers and place below them a sequence obtained by increasing the numbers above by two. Let p_o represent any prime in the second sequence which lies below a number in the first sequence which is a multiple of one of the first λ odd primes, $p_1, p_2, \cdots, p_\lambda$. Consider the function $L_o(s)=\sum p_o(\log p_o)/(p_o)^s$ on the line $s=1+it$. The series on the right converges absolutely for $R(s)>1$, and the analytic function thus defined can be extended to the left of the line $s=1+it$, thus furnishing a means of studying its functional behavior on that line. Such properties of the function have application to the problem of the infinitude of prime pairs. (Received October 11, 1948.)

23*t*. Leopoldo Nachbin: *On strictly minimal topological division rings.*

The author shows that several theorems usually established for real or complex topological vector spaces depend on the scalar domain being strictly minimal or complete. A topological division ring K is *strictly minimal* if, T_K denoting its topology, there exists no Hausdorff topology T on K such that $T<T_K$ and K endowed with T is a topological vector space over K endowed with T_K. *Completeness* is meant in Weil's sense for the additive group of K. The following results are typical and are generalizations of part of the recent work of J. Braconnier and I. Kaplansky: (1) If K is strictly minimal and f is a linear functional defined on a topological vector space over K, then f is continuous if and only if $f^{-1}(0)$ is closed; if K is not strictly minimal this need not be true. (2) In order that every finite-dimensional vector space over K should have only one admissible topology it is necessary and sufficient that K be strictly minimal and complete. (3) In order that every vector space automorphism of any finite-dimensional topological vector space over K should be homeomorphic it is necessary and sufficient that K be strictly minimal and complete. (4) Every topological division ring which admits a nontrivial valuation preserving the topology is strictly minimal. (Received September 9, 1948.)

24. Everett Pitcher: *The index and minimum properties of the characteristic roots of a quadratic function.*

Two approaches to the problem of existence and number of characteristic roots are formulated and the relation between them made clear in the following development. Let (A) $B(\eta, \phi, \lambda)$ be a real valued, symmetric, bilinear function of η, ϕ, differentiable in λ for $0\leqq\lambda<b$. Set $I(\eta, \lambda)=B(\eta, \eta, \lambda)$. Suppose (B) $I(\eta, 0)$ is positive definite; (C) if $I(\eta_0, \lambda_0)\leqq 0$ then $I_\lambda(\eta_0, \lambda)\leqq 0$ for $0\leqq\lambda<b$; (D) if $I(\eta_0, \lambda_0)=0$ and $\eta_0\neq 0$

then $I_\lambda(\eta_0, \lambda_0) < 0$. Of particular interest are the facts that λ need not enter linearly and the sense of variation of I with λ is not specified necessarily for all η. I may possess two properties: the ***index property***, that the index of $I(\eta, \lambda_0)$ be finite and equal to the number of characteristic roots less than λ_0; the ***minimum property***, that the characteristic roots in order are countable and that for any set $\eta_1, \lambda_1; \cdots; \eta_p, \lambda_p$ of characteristic pairs of I, the set of numbers λ which together with an $\eta \neq 0$ satisfies the conditions $I(\eta, \lambda) = 0$, $B(\eta, \eta_i, \lambda) = 0$ $(\lambda \neq \lambda_i)$, $B_\lambda(\eta, \eta_i, \lambda) = 0$ $(\lambda = \lambda_i)$ contains its greatest lower bound. The principal theorem states that if A, B, C, D hold, the index property and the minimum property are equivalent. Connections can be made with a variety of concrete characteristic value problems. (Received September 20, 1948.)

25. P. C. Rosenbloom: *The conformal mapping of nearly circular regions.*

If $f(z)$ maps the unit circle onto a domain D contained in $|w| < 1$ and containing D_1: $|w| < 1$, $|w - c| > \rho$, where $|c| = 1$, $0 < \rho < 1$, and if $f(0) = 0$, $f'(0) = a > 0$, then $f(z) = z - (1-a)z(c+z)/(c-z) + O((1-a)\rho^{1/2})$. This gives a very simple proof of Loewner's differential equation for functions mapping the unit circle on slit domains. (Received October 30, 1948.)

26*t*. I. E. Segal: *A kind of abstract integration pertinent to locally compact groups*. I. Preliminary report.

A *gage* is defined as a self-adjoint linear functional μ on a dense self-adjoint subalgebra A_0 of a C^*-algebra A, such that for $W \in A_0$, $\mu(W^*XW)$ is a continuous function of $X \in A_0$. The concept of gage includes that of state, and also that of a regular measure on a locally compact Hausdorff space. A further example is: A = operator group algebra of the locally compact group G, A_0 = subalgebra of all operators of the form T_f, where $T_f g$ = convolution of f with g, f being a continuous function which vanishes outside a compact set in G and g being an arbitrary element of $L_2(G)$, and $\mu(T_f) = f(e)$, e being the group identity. It is shown that all elements X of A of the form W^*VW, with $W \in A$ and $V \in SA(A_0)$, where $SA(A_0)$ is the collection of self-adjoint elements of A_0, are integrable in the sense that GLB $[Y | Y \geqq X, Y \in SA(A_0)]\mu(Y) =$ LUB $[Y | Y \leqq X, Y \in SA(A_0)]\mu(Y) < \infty$. In case the group G above is abelian, this is a lemma on which a proof of the generalized Plancherel theorem could be based. (Received August 19, 1948.)

27*t*. I. E. Segal: *Extension of representations of subgroups of locally compact groups.*

A continuous unitary irreducible representation (on a Hilbert space) of a compact subgroup of a locally compact group can be extended to the same kind of representation of the full group. The same is true for any subgroup of a discrete group. More generally, a sufficient condition for a continuous unitary irreducible representation of a closed subgroup of a locally compact group to be extendable to the same kind of representation of the full group is obtained, in terms of states of algebras related to the group and subgroup. The author's results naturally include the theorem of A. Weil covering the case in which the full group is compact, but his methods are unrelated to those of Weil, and the proof could be greatly simplified by restricting consideration to this case. (Received September 7, 1948.)

28*t*. Robert Sips: *Asymptotic representations of Mathieu functions and of spheroidal wave functions.*

Asymptotic representations are obtained for the periodic solutions, and for the corresponding separation constants, of the Mathieu equation $y''+(a-2^{-1}\lambda^2c^2\cos^2\eta)y=0$, for large values of the parameter λc. The asymptotic representation of the separation constant a, though obtained by an essentially different method, is identical to the one given earlier by S. Goldstein (Proc. Cambridge Philos. Soc. vol. 23 (1927) pp. 303–336). The Mathieu functions themselves are represented asymptotically, for all real values of the variable η, as a series of functions of the parabolic cylinder $D_n(\alpha)=\exp(-\alpha^2/4)\cdot H_n(\alpha)$, where $\alpha=(2\lambda c)^{1/2}\cdot\cos\eta$ and H_n is the nth Hermite polynomial. The spheroidal wave functions are the periodic solutions for large values of λc of the equation $(\sin\eta)^{-1}\cdot d(\sin\eta dH/d\eta)/d\eta+(A+\lambda^2c^2\sin^2\eta-m^2/\sin^2\eta)H=0$ for prolate wave functions, and the same equation with $\sin\eta$ replaced by $\cos\eta$ and a minus sign before the term $\lambda^2c^2\cos^2\eta$ for oblate wave functions. But, whereas the asymptotic expression of the prolate spheroidal wave functions is a series of parabolic functions, similar to the one representing the Mathieu functions, the corresponding development for the oblate wave functions is a series of functions of the form $\alpha^m\exp(-\alpha^2/4)\cdot L_\nu^{(m)}(\alpha^2/2)$ where $\alpha=(2\lambda c)^{1/2}\cdot\sin\eta$ and $L_\nu^{(m)}$ is the generalized Laguerre polynomial. Numerical results, obtained from the asymptotic expansions, are compared with exact values published by various authors. (Received May 10, 1948.)

29. Otto Szasz: *A generalization of Bernstein's polynomials to the infinite interval.*

A constructive proof of Weierstrass' theorem on the uniform approximation of a continuous function by polynomials over a finite interval is based on Bernstein's polynomials. There the values of the function at equidistant points are employed, with weights corresponding to the Bernoulli probability distribution. In the present paper the values of a function at equidistant points are employed over the infinite interval $(0, \infty)$, but the weights now correspond to the Poisson distribution. Uniform convergence to the function is proved at each point of continuity. A degree of approximation is given on introducing a weighted modulus of continuity. The process is connected with the problem of approximate integration by Cauchy-Riemann sums over the infinite interval. (Received September 6, 1948.)

30. J. L. Walsh, W. E. Sewell, and H. Margaret Elliott: *On the degree of convergence of harmonic polynomials to harmonic functions.*

Let $u(x, y)$ be harmonic inside C, continuous in the corresponding closed region $\overline{C}$; under suitable restrictions on C if the kth directional derivative of $u(x, y)$ satisfies on C a Lipschitz condition of order α $(0<\alpha\leqq 1)$, then a properly chosen sequence of harmonic polynomials $p_n(x, y)$ of respective degrees n converges to $u(x, y)$ in $\overline{C}$ with a degree of convergence $1/n^{k+\alpha}$. Conversely, if a sequence $p_n(x, y)$ converges to $u(x, y)$ in $\overline{C}$ with this degree of convergence, the kth directional derivative of $u(x, y)$ satisfies on C a Lipschitz condition of order α if $\alpha<1$. Furthermore, if $u(x, y)$ is harmonic inside C_R, continuous on $\overline{C}_R$, and if the kth directional derivative of $u(x, y)$ satisfies a Lipschitz condition of order α on C_R $(0<\alpha\leqq 1)$, then a suitable sequence $p_n(x, y)$ converges to $u(x, y)$ on $\overline{C}$ with a degree of convergence $1/n^{k+\alpha}R^n$; and conversely, if a sequence $p_n(x, y)$ converges to $u(x, y)$ on $\overline{C}$ with this degree of convergence, the $(k-1)$th directional derivative of $u(x, y)$ satisfies on C_R a Lipschitz condition of order α if $\alpha<1$. (Received September 7, 1948.)

APPLIED MATHEMATICS

31*t*. Milton Abramowitz: *Asymptotic expansions of Coulomb wave functions.*

The author deals with the problem of determining various expansions for the regular solution of the Coulomb wave equation when $L=0$ for values of the parameters ρ and η where the power series expansion is slowly converging. An asymptotic expansion is derived from the contour integral representations as derived by Sexl. Other approximating expansions are obtained from the differential equations using Sexl's previously determined saddle point approximations as starting point. Also included is a new short table of the zeros of the regular solution. (Received April 9, 1948.)

32*t*. Garrett Birkhoff: *Note on virtual mass.*

Let S be any solid, and Σ any solid of revolution about the x-axis containing S. Then the rotational virtual mass of S in an infinite fluid equals the moment of momentum about the x-axis of the fluid in $\Sigma-S$, when S is rotating about the x-axis with unit angular velocity. The cross-components of the virtual mass tensor satisfy similar identities. For example, that between rotation about the x-axis and translation along the y-axis is the y-momentum of the fluid in $\Sigma-S$, when S is rotated about the x-axis. The above is a generalization of a recent result of Theodorson. Again, if H and K are two infinitesimal rigid motions, and S^* is a sphere of radius r containing S, the K-component of pressure thrust for H-motion of S is $\mathrm{Lim}_{r\to\infty}\{\iint_{S^*}(HUdS_K-KUdS_H)+\iiint_{S^*-S}[(HK-KH)U]dV$, where U is the velocity potential for H-motion, and dS_K the surface flux of the vector field defined by K. Using this identity, it is easy to prove that a rigid body in an infinite fluid behaves dynamically like a Lagrangian system with six degrees of freedom. (Received August 10, 1948.)

33*t*. Harvey Cohn: *Interaction of simple waves.*

The author shows how the interaction of simple waves can be solved in general in the hodograph space by means of constructions such as those of J. Giese (Aberdeen Ballistic Research Laboratories Report, no. 657). For instance, if the flow with velocity $q=(u, v, w)$ involves two waves represented by vectors $q_1(\sigma)$, $q_2(\tau)$ in hodograph space ($\sigma_0\leqq\sigma\leqq\sigma_1$, $\tau_0\leqq\tau\leqq\tau_1$, $q_1(\sigma_0)=q_2(\tau_0)=q_0$ and $q_1(\sigma_1)=q_1$, $q_2(\tau_1)=q_2$), then the resultant velocity q_{12} of the interaction is found by first spanning the two preceding curves by a surface $w(u, v)$ satisfying $Aw_{uu}-2Bw_{uv}+Cw_{vv}=0$, where $A=c^2(1+w_v^2)-(v+w_v)^2$, $B=c^2w_uw_v-(u+ww_u)(v+ww_v)$, and $C=c^2(1+w_u^2)-(u+ww_u)^2$, and next drawing from the points q_1 and q_2 the characteristic curves which will determine q_{12}. The integration of the partial differential equation lends itself to the application of familiar (local) existence and uniqueness theorems. The author integrates the equation by power series to find the approximate interaction of Meyer waves. (Received August 23, 1948.)

34. Eugene Isaacson: *Water waves over a sloping bottom.*

Progressing gravity waves in water bounded by a plane bottom at an angle ω to the water surface can be described with the aid of a potential function $\phi(x, y)$ in the sector $-\omega\leqq\theta\leqq0$ of the (x, y)-plane, satisfying the boundary conditions: $\partial\phi/\partial y-\phi=0$ for $\theta=0$; $\partial\phi/\partial\theta=0$ for $\theta=-\omega$. This potential function had been determined for rational multiplies of 90° and for two limiting cases by Miche, Stoker, Lewy, Friedrichs. For details and references—also to the work of A. E. Heins—see the papers in Communications on Applied Mathematics vol. 1 (1948). In the present paper a simple explicit representation of the solution for arbitrary angle ω, which comprises all previously treated cases, will be given. To obtain this representation, the solution for rational multiples p/q of 90°, first given by Lewy as a sum of many integrals, is expressed in terms of a single integral in such manner that the transition to the case of an arbitrary angle ω can be carried out. (Received June 3, 1948.)

35*t*. C. C. Lin: *On the theory of isotropic turbulence of high Reynolds numbers*. Preliminary report.

Kolmogoroff's theory as applied to isotropic turbulence is discussed in terms of the spectrum without reference to correlation functions. The inverse five-thirds power law for the intermediate range of the spectrum is rederived following the original ideas of Kolmogoroff without making specific assumptions regarding the dependence of the transfer term on the spectrum. A law of decay of turbulence is derived following this idea. It is shown that in general the square of Kolmogoroff's micro-scale η instead of Taylor's micro-scale λ increases linearly in time. The Reynolds number of turbulence R_λ generally decreases linearly in time. The product $u^{-2}R_\lambda$ increases linearly in time, where u is the intensity of turbulence. This law of decay includes the half-power law when R_λ is a constant. The decrease of R_λ seems to be indicated by the experiments of Batchelor and Townsend. Furthermore, other earlier experimental results not fitting the half-power law are very well described by the present theory. (Received June 28, 1948.)

36*t*. C. C. Lin, Eric Reissner, H. S. Tsien: *On two-dimensional non-steady motion of a slender body in a compressible fluid.*

It is well known that in problems of flow of a compressible fluid a perturbation approach often leads to linear differential equations. Recently von Kármán pointed out that for steady motion in the transonic range certain nonlinear terms have to be retained even in a perturbation theory (Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 26 (1947) pp. 182–190). The present paper contains a systematic discussion of *non-steady* perturbation flows with a resultant classification of all possible cases including determination of their similarity parameters. Of particular interest is the result that for non-steady flow a linearized perturbation theory is obtained even as the Mach number of the undisturbed flow approaches unity, provided the reduced frequency parameter k is large compared with the 2/3 power of the amplitude ratio δ. (Received September 16, 1948.)

37*t*. E. J. McShane: *The differentials of certain functionals in exterior ballistics.*

If a projectile is launched with given initial conditions, its range, time of flight, and so on, are functionals of the range-wind, cross-wind, density, temperature (usually regarded as functions of the altitude y alone), and of other functions of position and velocity also. Ranges are ordinarily tabulated for certain simple "standard" choices of wind (identically 0), of density, and so on, and effects of departures from standard conditions are approximated by the differential of the functional. G. A. Bliss proved that the differential exists if the space of (for example) wind functions is normed by

$\|w(\cdot)\| = \max\ [\sup|w(y)|,\ |dw(y)/dy|]$. The presence of dw/dy in this definition is undesirable for both esthetic and physical reasons. In the present paper it is proved that the differential exists even when the norm is the familiar one, $\|w(\cdot)\| = \sup|w(y)|$. (Received August 30, 1948.)

GEOMETRY

38t. Edward Kasner and John DeCicco: *Osculating conics of the integral curves of equations of the type* (G).

The importance of differential equations of the type (G): $y''' = G(x, y, y')y'' + H(x, y, y')y''^2$, in geometry and physics, has been brought out by Kasner in his Princeton Colloquium Lectures, *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 3, 1913, 1934, 1948, and in many later papers. The osculating conics (five-point contact) of the ∞^3 integral curves are studied. The ∞^1 osculating conics, constructed at a lineal element E to the ∞^1 curves of the system passing through E, not only pass through E, but also touch another conic in general position. Their centers describe still another conic passing through the point of E. The foci of these conics describe an algebraic curve of sixth degree with a singular point of fourth order at the point of E. The degenerate cases of these various loci are studied in detail. (Received September 18, 1948.)

LOGIC AND FOUNDATIONS

39. Samuel Linial and E. L. Post: *Recursive unsolvability of the deducibility, Tarski's completeness, and independence of axioms problems of propositional calculus.*

The primitive connectives are $\sim p$ and $(p \vee q)$, corresponding modus ponens and substitution the rules of deduction. Let S be a fixed normal system on a, b with non-null g's and g''s whose deducibility problem is recursively unsolvable (see Post, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 286–287, 292; vol. 52 (1946) footnote 3). To a and b are made correspond $\sim\sim(p \vee \sim p)$ and $\sim\sim\sim\sim(p \vee \sim p)$ respectively. With $\sim(\sim p \vee \sim q)$ as connective, correspondents in propositional calculus result for each non-null string B on a, b, $B(p)$ designating a certain particular correspondent. A normal operation is simulated in propositional calculus by $(\sim, \vee)$ implications which have a corresponding effect for certain correspondents of the strings involved, and implications which transform any one correspondent of a string into any other. A finite set $\{A_i\}$ of tautologies result such that if $\{P_i\}$ is any particular complete finite set of axioms (tautologies) for propositional calculus, $c(p)$ the tautology $\sim\sim(\sim p \vee p)$, then B is asserted in S when and only when: $B(p)$, itself a tautology, is deducible from $\{A_i\}$; $\{A_i\}$, $\{(B(p) \supset T_i)\}$ is a complete set of axioms; $\{A_i\}$, $c(p)$, $(B(p) \supset c(p))$ is not an independent set of axioms. The title results follow for arbitrary finite sets of tautologies as axioms, hence also the first and third without the tautology restriction. (Received May 24, 1948.)

STATISTICS AND PROBABILITY

40t. Aryeh Dvoretzky: *On the strong stability of a sequence of events.*

Let $\{A_n\}$ $(n = 1, 2, 3, \cdots)$ be any sequence of events (which may interdepend in an arbitrary manner) defined in the same set of trials. Put $\alpha_n=\Pr\{A_n\}$, $\alpha_{m,n}=\Pr\{A_mA_n\}$, $n\beta_n=\sum_{i=1}^{n}\alpha_i$, $C_{n,2}\gamma_n=\sum_{1\leqq i<j\leqq n}\alpha_{i,j}$, $\delta_n=\beta_n-\alpha_n^2$ and $f_n=R_n/n$ where R_n is the number of those events among $A_1, A_2, \cdots, A_n$ which are realized. Let $\bar{f}_n=E\{f_n\}$ be the expected value of f_n and $\phi_n=f_n-\bar{f}_n$. The sequence $\{A_n\}$ is called *strongly stable* provided $\{\phi_n\}$ is strongly stable, that is (cf. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933, p. 58), if for every $\epsilon>0$, $\lim_{n\to\infty}\{\text{sup.}_{\nu>n}|\phi_\nu|>\epsilon\}=0$. Extending classical results M. Loève (J. Math. Pures Appl. (9) vol. 24 (1945) pp. 249–318, p. 257) proved that if $n\delta_n$ is bounded then $\{A_n\}$ is strongly stable. Using the lemma that if $a_n\geqq0$ and (*) $\sum_{n=1}^{\infty}a_n/n<\infty$ there exists a sequence of integers $\{n_i\}$ such that $0<n_{i+1}-n_i=o(n_i)$ and $\sum_{i=1}^{\infty}a_{n_i}<\infty$, the author proves that even (**) $\sum_{n=1}^{\infty}\delta_n/n<\infty$ implies the strong stability of $\{A_n\}$. Since the sufficient condition (*) in the lemma becomes also necessary when $\{a_n/n\}$ is monotone it is possible to establish cases where (**) is necessary and sufficient for the strong stability of $\{A_n\}$. (Received September 20, 1948.)

41*t*. C. J. Everett and S. M. Ulam: *Multiplicative systems*. I.

Let $G(x)$: $g_i(x)=\sum p_1(i;j_1,\cdots,j_t)x^j$ (power-series in $x_1,\cdots,x_t$) be a transformation of the unit cube I_t, $p_1(i;j)$ being the probability that a particle of type i produce, upon transmutation, $j_1+\cdots+j_t$ particles, j_s of type s. Then, in the i-component $g_i^k(x)=\sum p_k(i:j)x^j$ of the kth iterate $G^k(x)$, $p_k(i;j)$ gives the probability of the population $(j_1,\cdots,j_t)$ in the kth generation of progeny from one particle of type i. Weak restrictions on nonvanishing of $\partial g_i/\partial x_j$ and certain $\partial^2 g_i/\partial x_j\partial x_k$ imply: (1) if $x_i^0=\lim p_k(i;0)$, x^0 and 1 are the only fix-points of G on I_t; (2) $\lim p_k(i;j)=0$, all $j\neq0$; (3) $\lim G^k(x)=x^0$, all $x\neq1$ on I_t, (4) matrix $M=[m_{ij}]$, $m_{ij}=(\partial g_i/\partial x_j)_1$, has maximal characteristic root $r>0$ and unique positive vectors v, v' of norm 1 such that $vM=rv$, $Mv'=rv'$, a result due to Frobenius; (5) $x^0=1$ if and only if $r\leqq1$; (6)) $\lim jM^k/\|jM^k\|=v$ uniformly on the positive sector of t-space; (7) for every $e>0$, $f>0$, there exists a K such that for all $k\geqq K$, $\sum p_k(i;j)\geqq1-f$, j being summed over all vectors in the e-cone of v, including $j=0$. This is the analogue of the weak law of large numbers in additive probability. (Received September 16, 1948.)

42*t*. C. J. Everett and S. M. Ulam: *Multiplicative systems*. II.

Under the condition $r<1$ (hence $x^0=1$), the following results are obtained: (1) for $x\neq1$ in I_t, vectors $1-G^k(x)$ and $G^{k+1}(x)-G^k(x)$ approach v' in the sense of ray convergence; (2) if $q_k(i)$ is defined by $p_k(i;0)=q_1(i)+\cdots+q_k(i)$, $q_k(i)$ is a probability density on $k=1, 2, \cdots$ with moments of all orders; (3) if a source with generating function $S(x)$ acts alternatively with transmutation, the generating function for the kth population is $H_k(x)=S(G(x))\cdots S(G^k(x))$, $H(x)\equiv\lim H_k(x)$ exists on I_t, $(\partial H/\partial x_i)_1$ exists, is computable from a linear system, and represents the k-limit of expectation of particles of type i in population k; (4) analogous theorems are valid for total progeny in systems with and without source; (5) H_k (and its analogues in (4)) may be realized as a component-function of the kth iterate of a single transformation. (Received September 16, 1948.)

43*t*. C. J. Everett and S. M. Ulam: *Multiplicative system*. III.

The set Z_i of all *possible* genealogies (graphs) z arising from one particle of type i possesses an intrinsic metric $d(z, z')$ under which it is a complete separable zero-dimensional metric space. Simple axioms (A) on intervals and (B) on measure of intervals are given from which completely additive measure theory is derived. Intervals in Z_i are defined intrinsically and shown to satisfy axioms (A). If a generating transformation $G(x)$ is given, the transition probabilities $p_1(i; j)$ serve to define a measure for the intervals of Z_i satisfying axioms (B). The latter is nontrivial due to non-local-compactness of Z_i. Thus $G(x)$ defines a G-measure for the additive class of G-measurable sets of Z_i. Among other results may be mentioned: the set of graphs which terminate has measure x_i^0. Almost all graphs either terminate or have kth generation population $(j_1, \cdots, j_i)$ approaching the ratios $v_1 : \cdots : v_i$ of v. The latter is the analogue for multiplicative systems of the strong law of large numbers in additive probability. (Received September 16, 1948.)

TOPOLOGY

44. L. W. Cohen and Casper Goffman: *The topology of ordered abelian groups.*

The problem is to find complete group extensions of ordered abelian groups G. It is well known that the set of all lower segments will not serve for non-archimedean groups. A subset of the lower segments, called Dedekindean, is defined. These are the elements of the complete group extension G_D of the given group G. The groups G have an ordinal $\xi^*(G)$ in terms of which convergence and topological completeness are defined. G_D is topologically complete. It is shown that groups G complete in the archimedean sense are topologically complete, that the converse is not true but that if a topologically complete group is dense in every archimedean extension it is complete in the archimedean sense. The Hahn groups are examples of topologically complete ordered abelian groups and are of the second ξ^*-category. (Received August 19, 1948).

45. R. H. Fox: *A remarkable simple closed curve.*

In 3-dimensional Euclidean space there exists a simple closed curve whose complementary domain has a non-abelian fundamental group although, in the everyday sense of the word, this curve is not knotted at all. (Received September 18, 1948.)

46*t*. F. I. Mautner: *Completeness relations on locally compact groups.*

The completeness relations which have recently been obtained for the irreducible unitary representations of separable locally compact groups (Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 52–54) are now studied in more detail. One of the results is that two methods which lead to the same Parseval relation for compact or locally compact commutative groups may lead to two essentially different completeness relations for certain classes of locally compact groups. As an illustration, it is shown that the "regular representation" of the group of linear transformations in one variable over any infinite discrete field can be decomposed in an explicit manner into uncountably many inequivalent irreducible unitary representations, despite the fact that a certain algebra of operators associated with the group (the weakly closed ring generated by the regular representation) is a factor of class II_1 in the sense of Murray and von Neumann (Ann. of Math. vol. 37 pp. 116–229) and thus a simple algebra. (Received March 13, 1948.)

47*t*. A. H. Stone: *On the analytic definition of degree of multicoherence.*

It is proved that the "analytic" and "geometric" definitions of the degree of multicoherence (Eilenberg, Fund. Math. vol. 27 (1936) pp. 159, 153) agree for all connected, locally connected, normal spaces. A number of geometrical deductions are made. The principal tool is a modification of the notion of the "nerve" of a system of sets, to take into account the numbers of components of their intersections. (Received August 13, 1948.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN CHICAGO

The four hundred fortieth meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, on Friday and Saturday, November 26–27, 1948. The total attendance was about 200 including the following 136 members of the Society:

A. A. Albert, W. R. Allen, R. V. Andree, M. L. Anthony, Max Astrachan, W. L. Ayres, Reinhold Baer, Walter Bartky, H. R. Beveridge, S. F. Bibb, R. H. Bing, K. E. Bisshopp, W. M. Boothby, A. J. Brandt, J. W. Butler, K. H. Carlson, R. E. Carr, Harold Chatland, Herman Chernoff, E. W. Chittenden, W. G. Clark, E. H. Clarke, H. E. Clarkson, E. G. H. Comfort, J. J. Corliss, A. R. Craw, H. J. Curtis, G. K. DelFranco, Flora Dinkines, J. L. Doob, John Dyer-Bennet, B. J. Eisenstadt, H. M. Esser, H. S. Everett, Chester Feldman, J. V. Finch, C. H. Fischer, Harley Flanders, L. R. Ford, R. S. Fouch, Evelyn Frank, C. G. Fry, R. E. Fullerton, S. H. Gould, L. M. Graves, D. E. Green, L. J. Green, L. W. Griffiths, M. M. Gutterman, P. R. Halmos, Frank Harary, E. D. Hellinger, Vaclav Hlavaty, G. P. Hochschild, Carl Holtom, L. A. Hostinsky, S. P. Hughart, H. K. Hughes, Ralph Hull, Meyer Jerison, Louis Joseph, Samuel Kaplan, William Karush, Leo Katz, Fulton Koehler, J. C. Koken, W. C. Krathwohl, M. Z. Krzywoblocki, E. P. Lane, J. S. Leech, Saunders MacLane, Kenneth May, D. M. Merriell, Clair E. Miller, D. D. Miller, J. M. Mitchell, C. W. Moran, E. J. Moulton, Leopoldo Nachbin, E. D. Nering, August Newlander, O. M. Nikodym, E. P. Northrop, F. S. Nowlan, J. M. H. Olmsted, Daniel Orloff, E. H. Ostrow, Gordon Pall, Sam Perlis, D. H. Potts, D. W. Pounder, G. B. Price, O. W. Rechard, H. W. Reddick, W. P. Reid, W. T. Reid, Haim Reingold, P. R. Rider, L. A. Ringenberg, V. N. Robinson, Arthur Rosenthal, A. E. Ross, Herman Rubin, J. M. Sachs, Hans Samuelson, R. G. Sanger, L. J. Savage, A. C. Schaeffer, O. F. G. Schilling, Lowell Schoenfeld, W. T. Scott, A. S. Shapiro, Harold Shniad, D. N. Silver, G. F. Simmons, H. A. Simmons, Annette Sinclair, D. M. Smiley, M. F. Smiley, G. W. Smith, W. S. Snyder, E. H. Spanier, D. C. Spencer, D. M. Stone, A. F. Svoboda, L. W. Swanson, T. T. Tanimoto, A. H. Taub, E. F. Trombley, H. E. Vaughan, G. L. Walker, S. E. Walkley, D. R. Waterman, André Weil, J. W. T. Youngs, Antoni Zygmund.

The Committee to Select Hour Speakers for Western Sectional Meetings had invited two speakers for the occasion. Professor P. R. Halmos, of the University of Chicago, spoke on *Measurable transformations* at 2:00 P.M., Friday; Professor Saunders MacLane, of the University of Chicago, on *Duality for groups* at 10:00 A.M., Saturday. Presiding officers at these lectures were Professors J. L. Doob and Reinhold Baer, respectively.

Two sessions were held for contributed papers: Friday, 10:30 A.M., Professor Arthur Rosenthal presiding; and Friday, 3:15 P.M., Professor E. W. Chittenden presiding.

At the session on Saturday morning, Professor L. M. Graves moved that the Associate Secretary be instructed to write a letter of thanks to Major Lenox Lohr, Director of the Museum of Science and Industry, for the hospitality which the Museum had extended to the Society. The motion was seconded and passed unanimously. The Associate Secretary has carried out these instructions.

Social life at the meeting was enlivened by a party held by Mrs. Dorothy MacLane for visiting mathematicians on Friday evening from 4:30 to 7:00 P.M.

Abstracts of all papers presented at the meeting are given below. Papers read by title are indicated by the letter "*t*." Paper number 48 was presented by Professor Chatland.

Algebra and Theory of Numbers

48. Harold Chatland and Harold Davenport: *Euclid's algorithm in real quadratic fields.*

The question of the existence of Euclid's algorithm in real quadratic fields, $k(m^{1/2})$ has been settled except for m a prime, of the form $24n+1$. For $m>128^2$ Davenport proved the nonexistence of the algorithm. Chatland then showed that except for $m=193$, 241, 313, 337, 457, 601 the algorithm is invalid for m a prime of the form $24n+1$. In this paper it is proved that the algorithm does not exist in any of these six cases. Hence, as a result of all investigations it may be said that Euclid's algorithm exists in the real field $k(m^{1/2})$ if and only if m has one of the values 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97. (Received August 10, 1948.)

49*t*. V. E. Dietrich and Arthur Rosenthal: *Transcendence of factorial series with periodic coefficients.*

It is well known that every real number α can be represented in the form $\alpha=\sum_{n=1}^{\infty} a_n/n!$ where the a_n are integers and $0\leqq a_n<n$ (for $n=2, 3, \cdots$). In this note it is proved that α is a transcendental number if the a_n are periodic (except for the trivial case where almost all a_n are zero). More generally it is proved: If the power series $\phi(z)=\sum_{n=1}^{\infty}(a_n/n!)z^n$ has algebraic coefficients a_n (not almost all of them being zero) which form a periodic sequence, then $\phi(z)$ is a transcendental number for every algebraic z $(\neq 0)$. (Received October 12, 1948.)

50. Frank Harary: *The structure of finite Boolean-like rings.*

Boolean-like rings (BLR's) were defined by A. L. Foster as the class of rings in which the ring and logical operations are interdefinable through the same equations as for Boolean rings. In this paper, all finite BLR's are classified into isomorphism types by means of a canonical form for their multiplication tables. This canonical form is the basis for all further results on finite BLR's. Indecomposable BLR's are characterized as BLR's whose Boolean subring contains exactly two elements. It is verified that each finite BLR is isomorphic to the direct product of indecomposable BLR's. Not all BLR's have unique prime factorization. A necessary and sufficient condition for a finite BLR to have unique prime factorization is obtained. (Received October 15, 1948.)

Analysis

51*t*. R. P. Boas: *The Charlier B-series.*

The B-series has the form $f(x) \sim \sum_{n=0}^{\infty} a_n \Delta^n \theta(x)$, where Δ denotes a receding difference; when $f(x)$ is defined only for integral values of x, $\theta(x) = e^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \cdots$; $\theta(x) = 0$ for $x = -1, -2, \cdots$; when $f(x)$ is defined for all real x, $\theta(x) = (2\pi)^{-1}e^{-\lambda}\int_{-\pi}^{\pi} e^{-ixu} \exp(\lambda e^{iu})du$. This paper develops an L^2 theory of the series, considering the meansquare approximation to a given $f(x)$ by sequences $\sum_{n=0}^{N} a_n^{(N)} \Delta^n \theta(x)$, where the $a_n^{(N)}$ are not necessarily determined by the usual formulas for the coefficients in the B-series. The resulting expansion theory applies to more general functions $f(x)$ than can be represented by the B-series itself. More general series of similar character are also considered. (Received October 11, 1948.)

52*t*. L. W. Cohen and Casper Goffman: *The completion of ordered fields.*

The problem dealt with is that of defining complete extensions of an ordered field by the methods of Dedekind and Cantor. The equivalence of these methods in the classical completion of the rational field suggests the possibility of their equivalence in general even if the field does not enjoy the property of Archimedes. Let $\mathcal{F}$ be an ordered field. By the method of Dedekind is meant the definition of the field $\mathcal{F}_D$ whose elements are those lower segments L with the property that if $g > \theta$, θ the zero in $\mathcal{F}$, there is an $f \in L$ such that $f + g \notin L$. By the method of Cantor is meant the definition of the field F_C whose elements S are the classes of mutually equivalent fundamental sequences in the topology of $\mathcal{F}$. The principal result obtained is that $\mathcal{F}_D$ and $\mathcal{F}_C$ are isomorphic complete extension of $\mathcal{F}$. Completeness in the sense of Archimedes is considered and the relation between the two concepts of completeness is indicated. The Hahn fields are examples. (Received October 10, 1948.)

53*t*. H. V. Craig and W. T. Guy: *On Jacobian extensors.*

The purpose of this paper is to introduce the concept Jacobian extensor together with certain features of the accompanying theory. The transformation equation involved differs from that of ordinary extensors by the presence of the coefficients X sup α inf ρ, which are equal to the binomial coefficient ${}_\alpha C_\beta$ times the derivative of order $\alpha - \rho$ of the weighted Jacobian, $(x/\bar{x})^w$. Rules are given for constructing Jacobian extensors by differentiation of weighted scalars and tensors, and it is found that the process extensive differentiation carries over to the new theory. A Jacobian connection is developed from the ordinary connection and by extensive differentiation extended to higher orders of M. By means of the extended Jacobian connection, the higher order intrinsic derivatives of weighted scalars and tensors are expressed as contractions. (Received October 15, 1948.)

54. O. W. Rechard: *A note on the representation of real numbers.*

Let p be any integer ≥ 2. For every continuous strictly increasing function $f(x)$ on $[0, p]$ to $[0, 1]$ Everett (*Representations for real numbers*, Bull. Amer. Math. Soc., vol. 52 (1946)) has defined an algorithm which associates with every real number between 0 and 1 a sequence of integers mod p. For the case $f(x) = x/p$ this algorithm yields the decimal representation of a number to the base p. This note contains, among other things, a characterization of the class of one-one correspondences between real numbers and sequences of integers mod p which can be obtained by applying such an algorithm. It is simply the class of those correspondences which can be obtained from the class of all continuous strictly increasing functions $F(x)$ on $[0, 1]$ to $[0, 1]$ by defining the sequence corresponding to the number x to be the sequence of integers in the non-$(p-1)$-terminating decimal expansion of $F(x)$ to the base p. As a result of this characterization, for example, there exists a continuous strictly increasing function $f(x)$ on $[0, 2]$ to $[0, 1]$ which distinguishes between algebraic and transcendental numbers in the sense that a number α between 0 and 1 is algebraic if and only if $\alpha = f(a_1 + f(a_2 + \cdots + f(a_n)) \cdots)$ for some finite sequence of integers $a_1, a_2, \cdots, a_n$, with $a_j = 0$ or 1. (Received October 15, 1948.)

55. W. T. Reid: *Symmetrizable completely continuous linear transformations in Hilbert space.*

This paper treats the spectral theory of a completely continuous linear transformation K in a complete (not necessarily separable) abstract Hilbert space $\mathfrak{H}$, where K is (left) symmetrizable in the sense that there exists a non-negative symmetric bounded transformation S on $\mathfrak{H}$ such that SK is symmetric. It is proved that K has a real nonzero proper value in case $SK \neq 0$; certain expansion theorems are established for transformations K such that $(Su, u) \neq 0$ for each proper element u of K corresponding to a nonzero proper value. Special attention is given to symmetrizable transformations corresponding to the vector integral equations which are equivalent to self-adjoint boundary problems satisfying a condition of definiteness as formulated by Bliss (Trans. Amer. Math. Soc. vol. 44 (1938) pp. 413–428) or by the author (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 381–425). The method of proof is based on extremizing processes. In particular, the paper extends the results of Zaanen (Nieuw Archief voor Wiskunde (2) vol. 22 (1943) pp. 57–80; Nederl. Akademie van Wetenschappen, Proceedings vol. 49 (1946) pp. 194–204; Indagationes Math. vol. 8 (1946) pp. 91-101—see also his further papers in vols. 49, 50 of the same journal on the theory of linear integral equations). (Received October 15, 1948.)

56*t*. I. E. Segal: *Invariant measures on locally compact spaces.*

The existence and uniqueness of Haar measure are generalized by showing that an equicontinuous group G of unimorphisms of a uniformly locally compact space M leaves invariant a nonzero regular measure on M, which is unique (within proportionality) if and only if there is a point in M whose orbit is dense. If G is locally compact and acts continuously on M (that is, if the function on $G \times M$ to M describing the action of G is continuous) and if $f \in L_1(G)$ and $g \in L_\alpha(M)$, then $\int_G f(a) g(a^{-1}p) da$ exists for almost all p, da being the element of Haar measure, and defines a function in $L_\alpha(M)$ $(1 \leqq \alpha \leqq \infty)$. (Received October 1, 1948.)

APPLIED MATHEMATICS

57. M. Z. Krzywoblocki: *On the theory of the limiting lines in compressible, rotational, inviscid, isentropic flow.*

Tollmien and Ringleb created the theory of the so-called "limiting lines" in compressible, inviscid, isentropic irrotational flow by a transformation of the physical plane onto the hodograph plane. In the present paper a similar procedure is applied to a compressible, inviscid, isentropic, rotational flow. By a suitable selection of the "generalized" velocity potential function a simple hodograph transformation is obtained. The condition of the vanishing functional determinant gives the equation of the "limiting lines". (Received September 23, 1948.)

58*t*. H. E. Salzer: *Formulas for numerical integration of first and second order differential equations in the complex plane.*

For numerical integration of a first or second order differential equation in the complex plane, wherever the solution is analytic, it is more natural and advantageous to choose $z=x+iy$ as the variable, rather than to express the equation as a simultaneous real system in x and y. Also, using z as the variable permits a closer choice of the fixed points upon which to base the approximating polynomials for the integrand. Integration formulas are given for a Cartesian grid. They were found by integrating the Lagrangian interpolation polynomial for three to nine points $z_j=z_0+jh$, where h is the length of a square in the grid, and j is a complex integer j_1+ij_2. The configurations of z_j were chosen from considerations of convenience and closeness. A number of formulas for complex extrapolation are given, because they have an essential role in the integration process. All formulas are given with exact coefficients. A simple procedure for systematic integration, with suggested variations, is outlined for general use, suited to large-scale automatic computation devices. For configurations of z_j that are symmetric about the 45° ray, there are proved several useful relations among the coefficients occurring in the integration formulas. (Received October 8, 1948.)

GEOMETRY

59*t*. V. G. Grove: *On the R_λ-associate of a line.*

In this paper the author extends a notion of Popa applied to a plane net to the asymptotic net on a surface. Thereby new geometric characterizations of Bell's R_λ-associate of a line and of the R_λ-derived curves are found. Applications give characterizations of the Darboux and Segre tangents and of the tangents of the Darboux-Segre pencil of conjugate nets. (Received October 15, 1948.)

TOPOLOGY

60*t*. E. G. Begle: *The Vietoris mapping theorem for bicompact spaces.*

The Vietoris mapping theorem is proved for bicompact spaces. For any covering M of a space X, denote by $X(M)$ the complex consisting of all simplexes of X for each of which there is a set of M containing all its vertices. Then the statement of the theorem is: Let f be a mapping of X onto Y such that for each covering M of X, each point y of Y, and each integer k, $0\leq k\leq n$, there is a refinement N of M such that any finite k-cycle on $X(N)\cap f^{-1}(y)$ bounds on $X(M)\cap f^{-1}(y)$. Then the homomorphism of the n-dimensional homology group of X into that of Y which is induced by f is an isomorphism and is onto. The coefficient group is arbitrary. If the coefficient group is compact or a field, then the hypothesis may be replaced by the weaker condition that the k-dimensional homology groups of $f^{-1}(y)$, $0\leq k\leq n$, vanish for every y. For integer coefficients the theorem is not true with this weaker hypothesis, even in the compact metric case. (Received October 6, 1948.)

61. R. H. Bing: *Complements of continuous curves.*

Suppose space is compact, connected, locally connected, and metric. It is shown that each pair of points lies on an arc each of whose complementary domains has property S. If no pair of points separates space, the arc may be so chosen that its complement is also connected. If W is a closed totally disconnected point set, there is a dendron containing W such that each complementary domain of W has property S. (Received October 14, 1948.)

62t. Ky Fan: *Partially ordered additive groups of continuous functions.* I.

Let $C(\Omega)$ be the partially ordered additive group (p.o.a.g.) of all real continuous functions on a compact Hausdorff space Ω. A subgroup G of $C(\Omega)$ is called a characterizing subgroup, if: (1) G contains all constant functions; (2) for $x_0 \in \Omega$ and f, $g \in G$ such that $f(x_0) = g(x_0) = 0$, there is an $h \in G$ with $h \geq f$, $h \geq g$, $h(x_0) = 0$; (3) for any two points $x_1 \neq x_2$ of Ω, there is an $f \in G$ with $f(x_1) \neq f(x_2)$. A subgroup G of $C(\Omega)$ verifying (1), (2) is called a weakly characterizing subgroup. Let Ω, Ω^* be two compact Hausdorff spaces and G^* be a characterizing subgroup of $C(\Omega^*)$. Then Ω and Ω^* are homeomorphic, if and only if G^* is isomorphic (as p.o.a.g.) to a characterizing subgroup of $C(\Omega)$. More generally, Ω^* is a continuous image of Ω, if and only if G^* is isomorphic to a weakly characterizing subgroup of $C(\Omega)$. It is to be noticed that, when $C(\Omega)$ is regarded as a lattice, a characterizing subgroup of $C(\Omega)$ is not necessarily a sublattice of $C(\Omega)$. On the other hand, when $C(\Omega)$ is considered as a ring, a subring G of $C(\Omega)$ verifying (1), (3) fails to determine the topology of Ω. (Received October 14, 1948.)

63t. Ky Fan: *Partially ordered additive groups of continuous functions.* II.

Let G be a partially ordered additive group (p.o.a.g.). An element $f \in G$ is an Archimedean element of G if, for every $g \in G$, there is a natural number n with $nf \geq g$. An element $f \geqq 0$ without this property is a non-Archimedean element of G. A subgroup H of G is singular in G, if every $f \in H$ is of the form $f = g - h$, where $g \in H$, $h \in H$ are non-Archimedean elements of G. A subgroup H of G is convex in G, if $H \neq G$ and if $h_1 \in H$, $h_2 \in H$, $h_1 \leq f \leq h_2$, imply $f \in H$. The following theorem is proved: A p.o.a.g. G is isomorphic to a characterizing subgroup of $C(\Omega)$ (cf. the preceding abstract) for a topologically unique compact Hausdorff space Ω, provided G satisfies these four conditions: (i) G has a subgroup R which is isomorphic to the simply ordered additive group of all real numbers; (ii) at least one element of R is an Archimedean element of G; (iii) if $nf + g \geqq 0$ for all natural numbers n, then $f \geqq 0$; (iv) if a maximal convex subgroup M of G is not maximal singular in G, then there are a finite number of elements $f_1, \cdots, f_n$ such that $f_i \notin M$ $(1 \leq i \leq n)$ and that each maximal singular subgroup of G contains at least one of them. By adding a fifth condition, a characterization of the entire group $C(\Omega)$ is also obtained. (Received October 14, 1948.)

64t. M. K. Fort: *A note on equicontinuity.*

It is proved in this note that the center of every algebraically transitive group of homeomorphisms acting on a compact metric space is equicontinuous. (Received August 12, 1948.)

65t. M. K. Fort: *A proof that the group of all homeomorphisms of the plane onto itself is locally arcwise connected.*

Let P be a plane and let $H(P)$ be the group of all homeomorphisms of P onto itself, topologized by defining convergence to mean uniform convergence on each compact subset of P. It is shown that corresponding to each positive number ϵ there is a positive number δ such that for each element f of $H(P)$ which is in the δ-neighborhood of the identity element I there is an isotopy F such that $F_0 = f$, $F_1 = I$ and F_t is in the ϵ-neighborhood of I for $0 \leq t \leq 1$. It follows that f and I can be joined by an arc inside the ϵ-neighborhood of I. Since $H(P)$ is a topological group, this proves that $H(P)$ is locally arcwise connected. (Received September 16, 1948.)

66*t*. Deane Montgomery: *Theorems on the topological structure of locally compact groups.*

Let G be a locally compact n dimensional group. If U is a neighborhood of the identity there exists a neighborhood V of the identity such that any n or $n-1$ cycle (reals mod one) in $\overline{V}$ bounds in $\overline{U}$. An analogous property is true for homogeneous spaces. If H is a closed subgroup of dimension $n-1$ and if dim G/H is finite, then dim $G/H=1$. If G is locally connected and A is a closed n-dimensional subset then A must contain an inner point; this also holds for homogeneous spaces. (Received October 4, 1948.)

67*t*. G. T. Whyburn: *Equicontinuous collections of mappings.*

If A and B are metric spaces, a collection G of mappings of A into B is called uniformly interior on $A_0 \subset A$ provided that for $e>0$ a $d>0$ exists such that for any $x \in A_0$ and $f \in G$, $f[V_e(x)] \supset V_d[f(x)]$. Similarly, G is uniformly light on A_0 provided that for any $e>0$ a $d>0$ exists such that if V is any connected set in A intersecting A_0 and of diameter $\geqq e$ and $f \in G$, then $\delta[f(V)] \geqq d$. If G is equicontinuous, A is locally compact and B is compact, then G is uniformly light on the compact subsets of A if and only if the limit mapping of every sequence uniformly convergent on a compact subset of A is light. If G is an equicontinuous collection of strongly interior mappings of a locally connected generalized continuum A into a locally connected continuum B and if the limit mapping of every sequence in G uniformly convergent on a compact subset A_0 of A is light, then G is uniformly interior on A_0. A uniformly bounded collection G of analytic functions on a region A is uniformly interior or light on a compact region $\overline{R}$ in A if and only if for some $d>0$, $\max_{z \in \overline{R}} |f'(z)| \geqq d$ for all $f \in G$. (Received October 2, 1948.)

68*t*. G. T. Whyburn: *Normal regions and developability.*

Given locally connected generalized continua A and B and a light interior mapping $f(A)=B$, a region R in A is normal provided $f|\overline{R}$ is interior and f permutes with the boundary of R, that is, $f[F(R)]=F[f(R)]$. (This is similar to but not identical with Stoilow's notion of normal region.) If K is a continuum in B, H is a compact component of $f^{-1}(K)$ and $\epsilon>0$, there exists a normal region R satisfying $H \subset R \subset V_\epsilon(H)$. If A is the union of a strictly monotone increasing sequence of conditionally compact normal regions, f is said to be developable. This property holds if and only if each continuum in A lies in a conditionally compact normal region. Also, f is developable if and only if it is expansive in the sense that A is the union of a strictly monotone increasing sequence of conditionally compact regions $[R_n]$ such that no given compact set in B intersects the images of the boundaries of infinitely many of the R_n. (Received October 2, 1948.)

J. W. T. Youngs,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The four hundred forty-first meeting of the American Mathematical Society was held at the University of California, Los Angeles, on Saturday, November 27, 1948. The attendance was approximately 125, including the following 93 members of the Society:

M. I. Aissen, O. W. Albert, Richard Arens, W. G. Bade, J. L. Barnes, E. F. Beckenbach, H. W. Becker, M. M. Beenken, Clifford Bell, Gertrude Blanch, H. F. Bohnenblust, J. L. Brenner, Herbert Busemann, W. D. Cairns, L. H. Chin, L. M. Coffin, F. E. Cothran, F. G. Creese, E. L. Crow, D. R. Curtiss, J. H. Curtiss, Tobias Dantzig, P. H. Daus, W. W. Denton, C. R. DePrima, R. P. Dilworth, Roy Dubisch, W. D. Duthie, D. T. Finkbeiner, G. E. Forsythe, Arie Gaalswyk, P. R. Garabedian, H. H. Germond, H. E. Glazier, W. H. Glenn, J. W. Green, G. J. Haltiner, H. J. Hamilton, E. M. Henderson, M. R. Hestenes, P. G. Hoel, Alfred Horn, R. E. Horton, D. G. Humm, D. H. Hyers, Rufus Isaacs, Glenn James, P B. Johnson, Samuel Karlin, M. B. Lehman, D. H. Lehmer, Eugene Lukacs, J. C. C. McKinsey, J. E. McLaughlin, A. V. Martin, W. A. Mersman, A. B. Mewborn, A. D. Michal, W. E. Milne, S. F. Neustadter, A. B. J. Novikoff, L. J. Paige, R. S. Phillips, H. H. Price, E. S. Quade, W. C. Randels, L. T. Ratner, Saul Rosen, Wladimir Seidel, Max Shiffman, R. A. Siegel, Ernst Snapper, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, Robert Steinberg, E. G. Straus, L. M. Straus, A. C. Sugar, J. D. Swift, Alfred Tarski, A. E. Taylor, C. J. Thorne, Elmer Tolsted, S. M. Ulam, F. A. Valentine, L. F. Walton, Morgan Ward, P. A. White, A. L. Whiteman, D. V. Widder, V. A. Widder, E. R. Worthington.

The morning session was devoted to research papers and to the invited address, *The geometry of Finsler spaces*, by Professor Herbert Busemann, of the University of Southern California. Professor M. R. Hestenes presided. In the afternoon there were two sections for research papers, at which Professors D. H. Hyers and Morgan Ward, respectively, presided.

Following the meetings, there was a tea at the Institute for Numerical Analysis of the National Bureau of Standards, on the campus.

Abstracts of papers read at the meetings follow. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Mrs. Lehmer was introduced by Professor J. W. Green, Mr. Lackness by Professor G. C. Evans, and Mr. Thompson, Professor Mostowski, and Mrs. Szmielew by Professor Alfred Tarski. Paper number 69 was read by Miss Chin, paper 73 by Mr. McLaughlin, and paper 91 by Professor Phillips.

ALGEBRA AND THEORY OF NUMBERS

69. Louise H. Chin and Alfred Tarski: ***Distributive and modular laws in relation algebras.***

By the left distributive law for two binary operations $\odot$ and $\oplus$ is meant the formula $a\odot(b\oplus c)=(a\odot b)\oplus(a\odot c)$ where a, b, c are elements of a set S. Similarly, by the left modular law is meant the formula $a\odot[b\oplus(a\odot c)]=(a\odot b)\oplus(a\odot c)$. Right distributive and modular laws are analogously defined. Relation algebras (A. Tarski, Journal of Symbolic Logic vol. 6) are systems where four binary operations—absolute $+$ and $\cdot$, relative $\dotplus$ and ;—play a fundamental role. Hence there is a variety of such laws to consider. In some cases, that is, for some couples of binary operations, the laws are satisfied by arbitrary a, b, c of the algebra; for example, distributive laws for ; and $+$, and for $\dotplus$ and $\cdot$. In other cases, elements a are distinguished which are distributive, or modular, for a given couple of operations, that is, which together with arbitrary b, c satisfy a distributive or modular law in question. In most cases the sets of elements thus distinguished prove important even independent of formal reasons. For example, elements a which are left distributive for ; and $\cdot$ (that is, such that $a;\,(b\cdot c)=(a;\,b)\cdot(a;\,c)$ for any b, c) coincide with the functional elements, that is, with functions in the ordinary interpretation of relation algebras. Similarly, elements a which are left and right modular for $\cdot$ and ; are equivalence elements, that is, symmetric and transitive relations in the ordinary interpretation. (Received October 8, 1948.)

70. R. P. Dilworth: *A theorem on the structure of the lattice of continuous functions.*

A prime ideal P of a distributive lattice L is said to be *isolated* if for every prime ideal Q either $Q\supseteq P$, $P\supseteq Q$, or $P\cup Q=L$. The following theorem is proved: let $C(S)$ denote the lattice of continuous, bounded, real functions on a topological space S. Then $C(S)$ has the property that each element (as a principal ideal) is an intersection of isolated prime ideals. This property is, in a sense, a weak form of the requirement that the lattice be a direct union of chains, for it can be shown that a finite-distributive lattice has this property if and only if it is a direct union of chains. (Received October 16, 1948.)

71. D. T. Finkbeiner. *A general imbedding problem for lattices.* Preliminary report.

Consider a property P which holds in a lattice L if and only if P holds in every quotient lattice of L. Such properties are distributivity, modularity, upper and lower semi-modularity, and so on. P is said to hold weakly in L if and only if P holds in every quotient lattice a/b where b is not the null element of L. If L' is a lattice with the property P, and if S is the set sum of principal ideals generated by a set of elements of L', then $L=L'-S+$null element is a lattice in which P holds weakly. In this paper conditions are determined under which a lattice L, in which P holds weakly, can be enlarged to L' in which P holds in general (Received October 18, 1948.)

72. Emma Lehmer: *On the quintic character of* 2.

A criterion for the cubic character of 2 was developed by Gauss. It states that 2 is a cubic residue of p if and only if $x\equiv 1 \pmod 3$ is even in the quadratic partition of $4p=x_3^2+27y^2$. Recently S. Chowla gave an expression for $x_3=1+\phi_3(4)$, where $\phi_k(a)=\sum_{\nu=1}^{p-1}\chi(\nu)\chi(\nu^k+a)$ is a sum discussed by Jacobsthal in a Berlin dissertation (1906). (Here $\chi(n)$ denotes the quadratic character of n with respect to p.) The present paper gives a similar criterion for the quintic character of 2 as follows: Two is a quintic residue of p if and only if the value of $x\equiv 1 \pmod 5$ is *even* in the simultaneous solution of the pair of Diophantine equations $16p=x_5^2+50u^2+50v^2+125w^2$ and $x_5w=v^2-4uv-u^2$. The uniqueness of this value of x_5 has been established by L. E. Dickson (Amer. J. Math. vol. 57 (1935) pp. 391–424). It is also shown that $x_5=1+\phi_5(4)$. From another point of view x_k can be defined in general as $x_k=-\sum_{i=1}^{r-1}a_i$, where a_i are the coefficients in the decomposition of p into reciprocal factors in the field of kth roots of unity for k an odd prime. Then $x_k=1+\phi_k(4)$. Obviously x_k is even if and only if 2 is a kth power residue, but the Diophantine equations involving x_k^2 which are known do not have a unique solution for x_k and therefore cannot serve as criteria for $k>5$. (Received October 15, 1948.)

73. Jack McLaughlin and R. P. Dilworth: *Projectivities in relatively complemented lattices.*

It is known that if two points of a complemented modular lattice are projective, this projectivity can be accomplished by means of two transpositions. It is shown here that although this property does not extend to arbitrary relatively complemented lattices L, every projectivity between points of L can be accomplished by means of at most $2(n-1)$ transpositions, where n is the dimension of L. (Received October 16, 1948.)

74. Alfred Tarski: *Arithmetical classes and types of mathematical systems.* Preliminary report.

Consider the class K of all systems $\mathfrak{A}=\langle A, R\rangle$, A any nonempty set, R any n-termed relation (n fixed). $L\subseteq K$ is an *arithmetical* (or *elementary*) *class*, $L\in \boldsymbol{AC}$, if the condition $\langle A, R\rangle\in L$ is equivalently expressible by a formula containing only elementary logical constants (connectives, quantifiers, identity symbol), bound variables representing elements of A, and the free variable R. (This definition is replaceable by a purely mathematical one. Compare Tarski and Kuratowski-Tarski, Fund. Math. vol. 17.) $\mathfrak{A}, \mathfrak{B}\in K$ are *arithmetically equivalent* (*elementarily indistinguishable*), $\mathfrak{A}\equiv\mathfrak{B}$, if $\mathfrak{A}\in L\leftrightarrow\mathfrak{B}\in L$ whenever $L\in\boldsymbol{AC}$ (Tarski, Fund. Math. vol. 26). $\overline{M}$ denotes the class of all $\mathfrak{A}\in K$ with $\mathfrak{A}\equiv\mathfrak{B}$ for some $\mathfrak{B}\in M$; L is an *arithmetical type*, $L\in\boldsymbol{AT}$, if $L=\overline{M}$ for some one-element $M\subseteq K$. Theorems: I. $\boldsymbol{AC}$ is a Boolean algebra under class addition and complementation, with unit class K. II. $(L_{i+1}\subset L_i\in\boldsymbol{AC}$ for $i=0, 1, \cdots)\rightarrow(\prod_{i<\omega}L_i\notin C)$. III. $(L\in\boldsymbol{AT})\leftrightarrow L=\prod_{M\in F}M$ for some dual prime ideal $\boldsymbol{F}$ in the Boolean algebra $\boldsymbol{AC}$). IV. $\boldsymbol{AC}$ is countably infinite; also $\boldsymbol{AT}$ for the fixed n equaling 1; $\boldsymbol{AT}$ has the continuum power otherwise. V. For every $\langle A, R\rangle\in K$, A infinite, there is a $\langle B, S\rangle\equiv\langle A, R\rangle$ with B of any prescribed infinite power. All these notions and results extend to systems with many relations and operations. (Received October 12, 1948.)

75t. Alfred Tarski: *Metamathematical aspects of arithmetical classes and types.* Preliminary report.

For notions below see Tarski, Fund. Math. vols. 25, 26. Using symbolism of Bull. Amer. Math Soc. Abstract 54-11-74, consider any $L\subseteq K$. Let $\mathrm{Th}(L)$ be the *arithmetical* (*elementary*) *theory of L*, that is, the part of the theory of systems $\langle A, R\rangle\in L$ formalizable within the lower predicate calculus. $\mathrm{Th}(L)$ and $\mathrm{Th}(\overline{L})$ clearly coincide. ($\mathrm{Th}(L)$ is consistent)$\leftrightarrow(L\neq 0)$ [Godel's result related to II of preceding abstract]; ($\mathrm{Th}(L)$ has a finite system of non-logical axioms)$\leftrightarrow(\overline{L}\in\boldsymbol{AC})$; ($\mathrm{Th}(L)$ is consistent and complete)$\leftrightarrow(\overline{L}\in\boldsymbol{AT})$. V of abstract 54-11-74 in metamathematical translation presents an extension of Löwenheim-Skolem's theorem. Call M an *arithmetical L-class*,

$M \in \mathbf{AC}(L)$, or *L-type*, $M \in \mathbf{AT}(L)$, if $M = L \cdot N$ for some $N \in \mathbf{AC}$, or $M = L \cdot N \neq 0$ for some $N \in \mathbf{AT}$. If $L \in \mathbf{AC}$, then $(M \in \mathbf{AC}(L)) \leftrightarrow (L \supseteq M \in \mathbf{AC})$ and $(M \in \mathbf{AT}(L)) \leftrightarrow (L \supseteq M \in \mathbf{AT})$. $\mathbf{AC}(L)$ is a Boolean algebra under class addition and complementation, with unit class L; its isomorphism type determines uniquely the structural type of $\mathrm{Th}(L)$ and conversely. If one succeeds in establishing a decision procedure for $\mathrm{Th}(L)$ by means of the so-called method of eliminating quantifiers, one also succeeds in exhaustively describing $\mathbf{AC}(L)$ and $\mathbf{AT}(L)$; if $L \in \mathbf{AC}$, the description of $\mathbf{AT}(L)$ amounts to that of all consistent and complete extensions of $\mathrm{Th}(L)$, that is, consistent and complete theories obtained by enriching $\mathrm{Th}(L)$ with new non-logical axioms. (Received October 12, 1948.)

76t. Alfred Tarski: *Arithmetical classes and types of Boolean algebras.* Preliminary report.

For notations see Bull. Amer. Math. Soc. Abstracts 54-11-74 and 54-11-75. L denotes the class of all Boolean algebras. Clearly $L = \overline{L} \in \mathbf{AC}$. The decision procedure for the theory $\mathrm{Th}(L)$ (found by the author) leads to the following description of $\mathbf{AC}(L)$ and $\mathbf{AT}(L)$. Given $\mathfrak{B} \in L$, let $\mathfrak{B}_0 = \mathfrak{B}$; let $\mathfrak{B}_{k+1}$ be the quotient algebra $\mathfrak{B}_k | I$ where I is the ideal of the elements x in $\mathfrak{B}_k$ such that the join of all atoms included in x exists. M, $N(n)$, $P(n)$, $Q(n, p)$, where $n, p = 0, 1, \cdots$, are respectively classes of all $\mathfrak{B} \in L$ such that (i) none of the algebras $\mathfrak{B}_0, \mathfrak{B}_1, \cdots, \mathfrak{B}_k, \cdots$ is atomistic; (ii) $\mathfrak{B}_n$ is atomistic; (iii) $\mathfrak{B}_n$ has infinitely many atoms while $\mathfrak{B}_{n+1}$ has only one element; (iv) $\mathfrak{B}_n$ has exactly p atoms. Theorems: $\mathbf{AC}(L)$ is the smallest family containing L, all $N(n)$'s and $Q(n, p)$'s, and closed under class addition and subtraction. $\mathbf{AC}(L)$ as a Boolean algebra under class operations has a well ordered basis of the type ω^2 (cf. Mostowski-Tarski, Fund. Math. vol. 32). $\mathbf{AT}(L) \cdot \mathbf{AC}(L)$ consists of all the classes $Q(0, 0) \cdot N(0)$, $Q(n, p+1) \cdot N(n)$ and $Q(n, p) - N(n)$; $\mathbf{AT}(L) - \mathbf{AC}(L)$ consists of all the classes M, $P(n) \cdot N(n)$, and $P(n) - N(n)$. The families $\mathbf{AC}(L)$, $\mathbf{AT}(L) \cdot \mathbf{AC}(L)$, $\mathbf{AT}(L) - \mathbf{AC}(L)$ are countably infinite. The results above were obtained in 1940. (Received October 12, 1948.)

77t. Alfred Tarski: *Arithmetical classes and types of algebraically closed and real-closed fields.* Preliminary report.

For notations see Bull. Amer. Math. Soc. abstracts 54-11-74, 54-11-75. L being the class of all algebraically closed fields $\langle A, +, \cdot \rangle$, we have $L = \overline{L} \notin \mathbf{AC}$. A decision procedure for $\mathrm{Th}(L)$ has been found. The resulting description of $\mathbf{AC}(L)$ and $\mathbf{AT}(L)$ follows: $M(p)$ being the class of all $\mathfrak{A} \in L$ with characteristic p, $\mathbf{AC}(L)$ is the smallest family containing all $M(p)$'s, p a prime, and closed under class addition and complementation with respect to L; $\mathbf{AT}(L) \cdot \mathbf{AC}(L)$ consists of all $M(p)$'s, p a prime; $\mathbf{AT}(L) - \mathbf{AC}(L)$ consists only of $M(0)$. Hence $\mathbf{AC}(L)$, $\mathbf{AT}(L)$, and $\mathbf{AT}(L) \cdot \mathbf{AC}(L)$ are countably infinite. As a Boolean algebra under class operations, $\mathbf{AC}(L)$ has a well ordered basis of the type ω. $\mathfrak{A} \equiv \mathfrak{B}$ (for $\mathfrak{A}, \mathfrak{B} \in L$) if, and only if, $\mathfrak{A}$ and $\mathfrak{B}$ have the same characteristic. Let now L' be the class of all real-closed fields. Sturm's theorem for L' is extended from one equation in one unknown to arbitrary systems of algebraic equations and inequalities in many unknowns. By means of this, a decision procedure for $\mathrm{Th}(L')$ is established; $\mathrm{Th}(L')$ proves to be consistent and complete. Hence $L' = \overline{L'} \in \mathbf{AT} - \mathbf{AC}$; $\mathbf{AC}(L')$ consists only of L' and 0, and $\mathbf{AT}(L')$ of L'; finally $\mathfrak{A} \equiv \mathfrak{B}$ for arbitrary $\mathfrak{A}, \mathfrak{B} \in L'$ (for example, the field $\mathfrak{A}$ of real numbers and the field $\mathfrak{B}$ of real algebraic numbers). (Received October 12, 1948.)

78*t*. Andrzej Mostowski and Alfred Tarski: *Arithmetical classes and types of well ordered systems*. Preliminary report.

For notations see Bull. Amer. Math. Soc. Abstract 54-11-74, 54-11-75. The set A being ordered by the relation $\leqq$, $\tau(A)$ denotes the order type of the system $\mathfrak{A}=\langle A, \leqq\rangle$. Let L be the class of all $\langle A, \leqq\rangle$ with $A\neq 0$ well ordered by $\leqq$. A decision procedure for $\mathrm{Th}(L)$ is established. Given any ordinal γ, let $M(\gamma)$, $N(\gamma)$, $P(\gamma)$, $Q(\gamma)$ be respectively classes of all $\mathfrak{A}\in L$ with: $\tau(\mathfrak{A})=\gamma+\eta$ for some ordinal $\eta\neq 0$; $\tau(\mathfrak{A})=\xi+\gamma+\eta$ for some ξ and η, $\eta<\gamma\cdot\omega$; $\tau(\mathfrak{A})=\gamma$; $\tau(\mathfrak{A})=\omega^\omega\cdot\xi+\gamma$ for some $\xi\neq 0$. Theorems : $\boldsymbol{AC}(L)$ is the smallest family containing all the classes L, $M(\omega^n)$, $N(\omega^n\cdot p)$, where $n, p=0, 1, \cdots$, and closed under class addition and subtraction. $\boldsymbol{AC}(L)$ as a Boolean algebra under class operations has a well ordered basis of the type $\omega^\omega\cdot \boldsymbol{AT}(L)\cdot\boldsymbol{AC}(L)$ consists of all $P(\gamma)$'s, and $\boldsymbol{AT}(L)-\boldsymbol{AC}(L)$ of all $Q(\gamma)$'s, with $\gamma<\omega^\omega$. The families $\boldsymbol{AC}(L)$, $\boldsymbol{AT}(L)$, $\boldsymbol{AT}(L)\cdot\boldsymbol{AC}(L)$, $\boldsymbol{AT}(L)-\boldsymbol{AC}(L)$ are countably infinite. For $\mathfrak{A}, \mathfrak{B}\in L$, we have $\mathfrak{A}\equiv\mathfrak{B}$ if, and only if, either $\tau(\mathfrak{A})=\tau(\mathfrak{B})<\omega^\omega$ or $\tau(\mathfrak{A})=\omega^\omega\cdot\alpha+\gamma$, $\tau(\mathfrak{B})=\omega^\omega\cdot\beta+\gamma$ for some $\alpha\neq 0$, $\beta\neq 0$, $\gamma<\omega^\omega$. If $\mathfrak{A}\in L$, $\tau(\mathfrak{A})\geqq\omega$ then $\mathfrak{A}\equiv\mathfrak{B}$ for some $\mathfrak{B}\notin L$; take, for example, $\tau(\mathfrak{B})=\omega+\omega^*\cdot+\tau(\mathfrak{A})$. There is a $\mathfrak{B}\notin\overline{L}$ such that $\mathfrak{B}\in M$ whenever $L\subseteq M\in \boldsymbol{AC}$; take, for example, $\tau(\mathfrak{B})=\omega+\omega^*$. $L\neq\overline{L}$ and $L, \overline{L}\notin \boldsymbol{AC}$ (Tarski, Fund. Math. vol. 26, p. 301). The results above were obtained in 1941. (Received October 12, 1948.)

79*t*. Wanda Szmielew: *Arithmetical classes and types of Abelian groups*. Preliminary report.

Let L be the class of all Abelian groups. Obviously, $L=\overline{L}\in \boldsymbol{AC}$ (see Bull. Amer. Math. Soc. Abstracts 54-11-74, 54-11-75). A decision procedure for the theory $\mathrm{Th}(L)$ has been established. The resulting description of $\boldsymbol{AC}(L)$ follows: Given $\langle A, +\rangle\in L$ and $n, p=1, 2, \cdots$, the elements $a_1, \cdots, a_n\in A$ are called independent modulo p if the formula $p_1\cdot a_1+\cdots+p_n\cdot a_n=p\cdot a$ with any $a\in A$ and $0\leqq p_1<p, \cdots, 0\leqq p_n<p$ implies $p_1=\cdots=p_n=0$; they are called p-independent if they are of order p and the formula $p_1\cdot a_1+\cdots+p_n\cdot a_n=0$ with $0\leqq p_1<p, \cdots, 0\leqq p_n<p$ implies $p_1=\cdots=p_n=0$. Let $M(n, p)$, $N(n, p)$, and $P(n, p)$ be classes of all $\langle A, +\rangle\in L$ with at least n different elements which are respectively (i) p-independent, (ii) independent modulo p, and (iii) of order p and independent modulo p; let $Q(n)$ be the class of all $\langle A, +\rangle\in L$ with $n\cdot a=0$ for every $a\in A$. Theorems: $\boldsymbol{AC}(L)$ is the smallest family containing all the classes L, $M(n, p^k)$, $N(n, p^k)$, $P(n, p^k)$, $Q(n)$ $[k, n=1, 3, \cdots, p$ any prime$]$, and closed under class addition and subtraction. (Hence a description of $\boldsymbol{AT}(L)$ can also be derived.) $\boldsymbol{AC}(L)$ as a Boolean algebra under class operations has no well ordered basis. The families $\boldsymbol{AC}(L)$ and $\boldsymbol{AT}(L)\cdot\boldsymbol{AC}(L)$ are countably infinite; $\boldsymbol{AT}(L)$ and $\boldsymbol{AT}(L)-\boldsymbol{AC}(L)$ have the power of the continuum. Analogous (though simpler) results apply to simply ordered Abelian groups $\langle A, +, \leqq\rangle$. (Received October 12, 1948.)

80*t*. Andrzej Mostowski: *A general theorem concerning the decision problem*. Preliminary report.

Let $\mathfrak{A}$ be a mathematical system formed by a set A and arbitrary relations and operations $R_1, \cdots, R_n$. Let $\mathfrak{B}$ be the direct (cardinal) product of arbitrarily many factors each identical with $\mathfrak{A}$. The term "direct product" can be understood in the strong or the weak sense; in the latter case any element $s\in A$ definable in terms of $R_1, \cdots, R_n$ can be taken as 0. (See, for example, A. Tarski, *Cardinal algebras*, New

York, 1948.) Let $\mathrm{Th}(\mathfrak{A})$ and $\mathrm{Th}(\mathfrak{B})$ be the arithmetical theories of $\mathfrak{A}$ and $\mathfrak{B}$ respectively (cf. Bull. Amer. Math. Soc. Abstract 54-11-75). Theorem: if the theory $\mathrm{Th}(\mathfrak{A})$ is decidable (that is. if the set of all theorems in $\mathrm{Th}(\mathfrak{A})$ is generally recursive), the same applies to $\mathrm{Th}(\mathfrak{B})$. Example: Let $\mathfrak{A} = \langle A, + \rangle$ and $\mathfrak{B} = \langle B, \cdot \rangle$, where A is the set of all non-negative integers, B the set of all positive integers, and $+$ and $\cdot$ have their usual arithmetical meanings. A decision procedure for $\mathrm{Th}(\mathfrak{A})$ has been established (by Presburger); by the theorem stated above it implies the existence of a decision procedure for $\mathrm{Th}(\mathfrak{B})$ (a result obtained in a different way by Skolem). (Received October 12, 1948.)

81. F. B. Thompson: *A note on the unique factorization of abstract algebras.*

In their monograph, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures, No. 5, 1947, Jónsson and Tarski establish the unique factorization theorem and several related results for finite algebras consisting of a set A of elements closed under a binary operation $+$, such that A contains an element z which is a zero element for $+$. They raised the problem whether these results can be extended to algebras differing from those mentioned above in that the element z need only be idempotent under $+$ (that is, $z+z=z$). The solution of this problem proves to be negative. In fact, let $\bigcirc$ and $\square$ be operations on integers such that (i) $m \bigcirc n = 0$ for all m and n, (ii) $m \square n = 2m'+n'$ where $0 \leqq m', n' < 2$, $m \equiv m' \pmod 2$, $n \equiv n' \pmod 2$. Let $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_1$, $\mathfrak{C}_2$ be respectively the algebras constituted by: (1) the set $\{0, 1\}$ and the operation $\bigcirc$; (ii) the set $\{0, 1, 2, 3, 4, 5\}$ and the operation $\square$; (iii) the set $\{0, 1, 2\}$ and the operation $\bigcirc$; (iv) the set $\{0, 1, 2, 3\}$ and the operation $\square$. It is seen that each of these algebras has an idempotent element and is indecomposable, and that $\mathfrak{B}_1 \times \mathfrak{B}_2 \cong \mathfrak{C}_1 \times \mathfrak{C}_2$ This contradicts the unique factorization theorem. (Received September 22, 1948.)

ANALYSIS

82*t*. Paul Civin: *Approximation in* Lip (α, p).

If $f(x) \in \mathrm{Lip}\ (\alpha, p)$ and $\{P_n(x)\}$ is a sequence of trigonometric polynomials of order n such that $(\int_{-\pi}^{\pi} |f(x)-P_n(x)|^p dx)^{1/p} \leqq Kn^{-\alpha}$, then it is shown that $D = (\int_{-\pi}^{\pi} |P_n'(x)|^p dx)^{1/p} \leqq A(1-\alpha)^{-1} n^{1-\alpha}$ for $0<\alpha<1$, $D \leqq A \log n$ for $\alpha = 1$, and $D \leqq A(\alpha-1)^{-1}$ for $\alpha > 1$, where in each case A depends only on α and the sequence $P_n(x)$ but not on n. The corresponding result for functions in Lip α was established by M. Zamansky (C. R. Acad. Sci. Paris vol. 224 (1947) pp. 704–706). (Received August 10, 1948.)

83. G. E. Forsythe: *Certain inequalities concerning the Legendre polynomials.* Preliminary report.

For integers $n \geqq 0$, $k \geqq h \geqq 1$ and a real variable x, let $\Delta = \Delta(n, h, k; x) = P_n(x)P_{n+h+k}(x) - P_{n+h}(x)P_{n+k}(x)$, where $P_r(x)$ is the Legendre polynomial of rth degree. When $h+k$ is an even [odd] number, Δ is an even [odd] function of x. For given n, h, k, the form Δ is said to have property T when $0<x<1$ implies that $\Delta<0$. The general purpose of this investigation is to see which of the forms Δ have property T. Turán showed that $\Delta(n, 1, 1; x)$ has property T for each $n \geqq 0$ (see G. Szegö, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 401–405). The present author proves that $\Delta(n, 1, 2; x)$ and $\Delta(2n+1, 2, 2; x)$ both have property T for each $n \geqq 0$. The proofs involve applying Szegö's first two methods of proof (loc. cit.) to Δ, $d\Delta/dx$, or $d^2\Delta/dx^2$ in various subintervals of (0, 1). On the other hand, the present author shows that Δ has one or more zeros x_0 with $0<x_0<1$ for the following values of the parameters: (i) for each $n \geqq 0$, when $h+k=4M+1$ ($M\geqq 1$); (ii) for each even $n\geqq 0$, when $h+k=2M$ ($M\geqq 2$), except when $h-k=$ odd number $\geqq 3$; (iii) for each odd $n\geqq 1$, when $h+k=2M$ ($M\geqq 2$), except when $h=k\geqq 2$; (iv) for $n=0$, $h=1$, when $k\geqq 3$; (v) for $n=0$, $h=2$ or 3, when $k\geqq h$. By computation when necessary it is further shown that for $n+h+k\leqq 10$ the only Δ with property T are those of types $\Delta(n, 1, 1; x)$, $\Delta(n, 1, 2; x)$, and $\Delta(2n+1, 2, 2; x)$. (Received October 16, 1948.)

84. P. R. Garabedian (National Research Fellow): *The sharp form of the principle of hyperbolic measure.*

Let D be a Riemann surface of genus g bounded by n continua, let G be a Riemann surface with a universal covering surface of hyperbolic type, and let z_0 be a point of D, w_0 a point of G. Denote by E the class of functions $f(z)$, analytic in D, which asassume values in G and satisfy $f(z_0)=w_0$, and denote by $F(z)$ a function in E with $|F'(z_0)| = \max |f'(z_0)|$ for f in E. It is shown that $F(z)$ maps D upon a covering surface S of G which has no boundary components over G, that is, upon which any point can move indefinitely, provided its projection lies interior to G. While S may have infinitely many sheets, it has at most $4g+2n-2$ branch points, with multiple branch points counted according to their multiplicities. These branch points lie at the zeros of a quadratic differential $R(t)dt^2$ which is regular on S except for a possible double pole at $t=w_0$ and which is positive on the boundary of S. These results can be extended to problems of Pick-Nevanlinna type. The proofs are based on variational methods. The variations are defined by Green's formula and the construction of simply- and doubly-connected covering surfaces of G which correspond to the boundary components of D. (Received October 1, 1948.)

85*t*. M. R. Hestenes: *A note on the Lagrange multiplier rule in the calculus of variations.*

Consider a Hilbert space H. Let $Q(x, y)$ be a real symmetric bilinear form on HH and denote by H_0 the set of all y such that $Q(y, x)=0$ for all x in H. Suppose that there is a positive number m such that $Q(x, x)\geqq m|x|^2$ on the subset of H that is orthogonal to H_0. Then every linear form $L(x)$ that vanishes on H_0 is expressible in the form $L(x)=Q(y, x)$. In the present paper it is shown that the Lagrange multiplier rule for the problem of Bolza is a simple consequence of this result. (Received October 15, 1948.)

86*t*. R. M. Lakness: *Green's theorems in multiple leaved domains.*

A multiple leaved space $\mathfrak{M}$ in 3 dimensions is the analog of a Riemann surface in the plane. Let Ω be a bounded domain on $\mathfrak{M}$ with boundary consisting of a finite number of branch curves $\{s_i\}$ and a bounded exterior frontier Ω^* such that Ω is topologically equivalent to an m-leaved sphere where the $\{s_i\}$ correspond to branch circles within the sphere, no two of which loop or have a point in common. The $\{s_i\}$ are to be of zero capacity as closed sets in space. Since the boundary Ω^* is not involved by the branch curves we may take the parts of the boundary to be regular surfaces. Let u, v be functions which are linear combinations of bounded functions of class C_1 whose Laplacians exist almost everywhere and are non-negative. Then $\nabla^2 u$, $(\nabla u)^2$, $(\nabla u \cdot \nabla v)$ are summable over Ω and the usual Green's theorems are valid in which the boundary integral involves only Ω^* and not the branch curves. The main tool used in the proof is Kellogg's uniqueness theorem for such domains, which has been proved by Professor G. C. Evans (Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) p. 272). (Received October 11, 1948.)

87. S. F. Neustadter: *Multiple-valued harmonic functions with circle as branch-curve.*

The three-dimensional Euclidean space is referred to toroidal coordinates. A multiple-leaved Riemann space with the fundamental circle of the coordinate system as branch curve is obtained by joining a finite or infinite number of ordinary spaces along the disk spanned by the fundamental circle. One is concerned with the multiple-leaved domain T of which the branch-circle is the sole boundary. There is obtained in the form of an integral the solution of Laplace's equation which has an ordinary pole at some point of T, continuous derivatives everywhere else in T, and vanishes at infinity in all the leaves of T. It is shown that if a point on the branch-circle is approached, the function which results for the finitely-leaved case approaches one nth of the reciprocal of the distance from the pole to the point on the branch-circle. In the infinitely-leaved case this limit value is zero. Analogous results hold for the limits of the derivative of the function with respect to the toroidal coordinates, while the directional derivatives are shown to be unbounded near the branch-circle. (Received October 12, 1948.)

88. A. E. Taylor: *Spectral theory for closed distributive operators.*

Let T be a closed distributive operator with domain and range in a complex Banach space X. Let the resolvent set $\rho(T)$ be nonempty. Let $G(T)$ be the class of complex functions $f(\lambda)$, each analytic in some domain containing the spectrum $\sigma(T)$ and a neighborhood of $\lambda=\infty$; it is assumed that f is regular at $\lambda=\infty$. Define $f(T)=f(\infty)I+(1/2\pi i)\int f(\lambda)(\lambda I-T)^{-1}d\lambda$, the integration being over the boundary of a certain region containing $\sigma(T)$ and a neighborhood of $\lambda=\infty$. The correspondence $f(\lambda)\rightarrow f(T)$ defines a homomorphism of the ring $G(T)$ into the ring $[X]$ of bounded linear operators on X into itself. Polynomials in T are defined in the natural way. If $P(\lambda)$ is a polynomial of degree n and if $f\in G(T)$ has a zero of order at least n at $\lambda=\infty$, the operator corresponding to $P(\lambda)f(\lambda)$ is $P(T)f(T)$. The operational calculus and spectral theory founded in this way include the work of Dunford on bounded operators (Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217) and afford a uniform development of the theory in which the role of boundedness at particular places is made clear. Various spectral criteria for T to be in $[X]$ are obtained. If $T\notin[X]$, it is proved that there exists no polynomial $P(\lambda)\neq 0$ for which $P(T)$ is the 0 operator on its domain. (Received October 16, 1948.)

89. A. L. Whiteman: *Theorems analogous to Jacobstahl's theorem.*

For any prime p of the form $8n+1$ the diophantine equation $p=x^2+2y^2$ has a unique solution in x and y except for signs. It is proved in this paper that $x=4^{-1}\sum_{m=1}^{p-1}(m/p)((m^4+1)/p)$, where (m/p) denotes the Legendre symbol. From this result it follows that the number of solutions of the congruence $y^2\equiv x^5+x \pmod{p}$ ($p\equiv 1 \pmod{8}$) is $p+O(p^{1/2})$. The method is based upon the theory of the division of the circle and is used to obtain similar formulas for primes of other forms. In particular, the analogous theorems of E. Jacobstahl (J. Reine Angew. Math. vol. 132 (1907) pp. 238–245) and S. Chowla (Proceedings of the Lahore Philosophical Society vol. 7

(1945) 2 pp.) for primes of the forms $4n+1$ and $3n+1$, respectively, are derived. (Received September 28, 1948.)

APPLIED MATHEMATICS

90. H. W. Becker: *The symmetric reciprocity theorem.*

A source and receiver of any internal impedances s and z may be bodily interchanged in a network θ without change of current through z, if they are in the same summand of the branch symmetry partition, and both impedances are the same in each pair of branches permuted by the transformation $\theta_{-+} \rightarrow \theta_{+-}$. The latter are the here topologically identical networks to which θ reduces, when s is shorted and z opened, and vice versa. Moreover the same equalities (graphically deduced, with no time lost in any calculations whatever) yield the same interchangeability in the dual network ${}^{-}\theta$, since ${}^{-}(\theta_{\mp\pm}) = ({}^{-}\theta)_{\pm\mp}$. If s and z are not symmetrical in θ, $\theta_{-+} \neq \theta_{+-}$ topologically; and the conditions for their algebraic equality, by adjustment of any one branch between s and z, are neither simple, dual, nor necessarily realizable. (Received October 16, 1948.)

91. R. S. Phillips and Henry Malin. *A helical wave guide.* Preliminary report.

This paper is concerned with an idealized helical wave guide. This guide consists of a circular cylindrical surface which has perfect conductivity in a given helical direction and zero conductivity in the direction normal to this on the surface. For each n ($n=0, \pm 1, \pm 2, \cdots$) there is a general solution of the wave equation in cylindrical coordinates involving nth order Bessel functions. For each such solution there are precisely as many non-attenuated modes as there are real and imaginary solutions in v of the following characteristic equation: $\delta/\alpha = -(n/v^2)[1+(v/\alpha)^2]^{1/2} \pm \{-[I_n'(v)K_n'(v)]/[v^2 I_n(v)K_n(v)]\}^{1/2}$. It is found that there are no imaginary solutions. The paper is thus devoted to a study of the real solutions of this equation for all possible α and n and their relations to one another. The results are obtained from a study of certain ordinary differential equations of the Riccati type. In the course of the study numerous approximations for and inequalities between Bessel functions are found. (Received October 20, 1948.)

92*t*. Edmund Pinney: *Elastic waves from a point source within a sem-infinite elastic solid.*

A solution is given to the problem of harmonic elastic waves originating at a point source in the interior of a semi-infinite elastic solid. (Received October 18, 1948.)

93*t*. C. J. Thorne: *Symmetrically loaded rectangular plates fixed at points.*

The work begun in the previous paper *Square plates fixed at points* (Journal of Applied Mechanics (1948)) is extended to include additional design data for the square plate and similar results for plates with side ratios 2:1 and 3:1. Deflection functions for thin plates with symmetric loads are given as a sum of biharmonic polynomials with coefficients determined by the slopes and deflections at equally spaced points on each edge of the plate for what are called full fixed and not full fixed edge conditions. Results and design data are then given numerically and graphically for a center point load and a uniform load. (Received October 18, 1948.)

GEOMETRY

94. Alfred Horn: *Some generalizations of Helly's theorem on convex sets.*

Let S_n be the surface of the unit sphere $x_1^2+\cdots+x_n^2=1$ in n-dimensional space R_n. A subset A of S_n is convex if whenever a and b are linearly independent points of A, then either the minor or the major arc of the great circle through a and b lies in A. Let S_p denote the intersection of S_n with a p-dimensional plane through the origin. Theorem: Let F be a family of closed convex subsets of S_n which is such that every p members of F have a point in common. Then through every S_{n-p} there passes an S_{n-p+1} which intersects every member of F, $1 \leqq p \leqq n$. From this can be deduced the following theorem concerning convex sets in the ordinary sense: Let F be a family of compact convex subsets of R_n which is such that every p members of F have a point in common. Then through every $(n-p)$-dimensional plane, there passes an $(n-p+1)$-dimensional plane which intersects every member of F, $1 \leqq p \leqq n$. The case $p=n+1$ is Helly's theorem, if a -1-dimensional plane is defined as the empty set. (Received May 17, 1948.)

LOGIC AND FOUNDATIONS

95*t*. Julia Robinson: *General recursive functions.*

A function will be called general recursive if it can be obtained from the constant, identity and successor functions by repeated use of substitution, primitive recursion and the μ-rule. Functions obtained without using the μ-rule are called primitive recursive. The μ-rule has the form $F(x_1, \cdots, x_n)=\mu y\{A(x_1, \cdots, x_n, y)=0\}$, where the expression on the right denotes the smallest y such that the condition in braces holds, it being assumed that for every $x_1, \cdots, x_n$ there exists a y satisfying this condition. The purpose of this paper is to give a mathematical treatment of general recursive functions; part of the results obtained were previously established metamathematically by Kleene. In particular, the question whether all general recursive functions are obtained by restricted operations is discussed. One result is that the recursions may be omitted, if suitable functions are adjoined to the initial functions. On the other hand, using primitive recursions, a single application of the μ-rule is sufficient. It is also shown that all general recursive functions of one variable may be obtained without using functions of more than one variable, by starting with just two primitive recursive functions and using only substitution and the μ-rule in the form $F(x)=\mu y\{A(y)=x\}$. (Received October 11, 1948.)

96. A. C. Sugar: *A rational reconstruction of dimensional analysis.* Preliminary report.

A dimensional number is defined as a class (N, P) where N is a real number and the dimensional part P is the class $((D_1, \alpha_1), \cdots, (D_n, \alpha_n))$, $D_i \neq D_j$, when $i \neq j$; the D_i are dimensions, and the α_i rational numbers. In the usual interpretation D_i is a standard physical or geometric object such as the standard meter bar, the standard gram, or a radian. As a consequence of this definition, the so-called dimensionless quantity can be regarded as a special kind of dimensional number, that is, $(N, (D_1, 0), \cdots, (D_n, 0))$. The set $(D_1, \cdots, D_n)$, $D_i \neq D_j$, when $i \neq j$, is called a dimensional system of order n. The order of a system is fixed in a given analysis. Using these concepts and the definitions of multiplication and division of dimensional numbers it is possible to analyze the fundamental functions of the physical sciences. Three typical examples of fundamental physical functions are length, L, mass, M, and acceleration, A. A fundamental physical function will have a dimensional number for its value and may have several arguments one of which may be a physical object. Consider, for example, the mass function, M. Here, the value of the function, $M(Q)$, is a dimensional number and the argument, Q, is a physical object. Included is a logical critique of the methods of obtaining physical laws by dimensional analysis. (Received October 6, 1948.)

NUMERICAL AND GRAPHICAL METHODS

97t. C. J. Thorne: *A table of biharmonic polynomials.*

The polynomials $z^n/n!$ and $\bar{z}(z^n/n!)$ satisfy $\nabla^4 W=0$. The values of the first twenty-five such polynomials both real and imaginary are given at intervals of .05 for $y=0$, $0\leqq x\leqq 1$; $x=1$, $0\leqq y\leqq 1$; $x=y$, $0\leqq y\leqq 1$; $x=2y$, $0\leqq y\leqq 1/2$; $x=3y$, $0\leqq y\leqq 1/3$; $y=1/2$, $0\leqq x\leqq 1$; $y=1/3$, $0\leqq x\leqq 1$. The partial derivatives to the third order are given for the real parts of $\bar{z}(z^n/n!)$. The construction of the tables and their use is discussed. (Received October 18, 1948.)

STATISTICS AND PROBABILITY

98t. A. L. Whiteman. *Distribution theory of unrestricted runs.*

A run is a sequence of like elements preceded and succeeded by different elements. An unrestricted run is a sequence of like elements without regard to the elements which precede or follow the sequence. The number of elements in a run is its length. Consider a random arrangement of n_1 a's and n_2 b's with $n_1+n_2=n$. Let r_{1j} denote the number of runs of a's of length j and r_{2j} denote the number of runs of b's of length j. Let r_1 and r_2 denote the total number of runs of a's and b's, respectively. In a similar manner, define u_{1j}, u_{2j}, u_1 and u_2 for unrestricted runs. Typical theorems are the following. The expected value and variance of u_{12} are given by the following expressions: $E(u_{12})=n_1(n_1-1)(n_1-2)/n(n-1)$ and $\sigma^2(u_{12})=n_1(n_1-1)(n_1-2)n_2(n_2+1)\cdot(5n^2-4n_2n-19n+6n_2+12)/n^2(n-1)^2(n-2)(n-3)$. When r_1 is given, the expected value and variance of u_{12} are given by the following expressions: $E(u_{12})=(n_1-r_1)(n_1-r_1-1)/(n_1-1)$ and $\sigma^2(u_{12})=(n_1-r_1)(n_1-r_1-1)r_1(r_1-1)/(n_1-1)^2(n_1-2)$. New derivations of some theorems of A. M. Mood (*The distribution theory of runs*, Ann. Math. Statist. vol. 11 (1940) pp. 367–392) are obtained by expressing the r's as linear functions of the u's. (Received May 4, 1948.)

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GODFREY HAROLD HARDY

1877–1947

Last year English mathematics suffered the loss of that mathematician who has more than any other been identified with the English School from the early 1900s to the present.

Godfrey Harold Hardy died of heart failure in 1947. For a considerable period his work had been restricted by bad health, which came as a special misfortune to a man whose career for many years had been identified not only with great mental activity, but also with a large measure of interest in all those games in which a ball takes part, and for whom this interest involved a very considerable share of personal participation. He was an encyclopedia of information concerning cricket, and to the very end nothing delighted him more than to witness a match.

Hardy came from a family with artistic and intellectual traditions. He went to Winchester and then to Trinity College, Cambridge. The milieu in which he developed as a mathematician is one which it is particularly difficult for those outside of the English tradition to understand, and even rather difficult for those belonging to the newer English tradition which Hardy himself had so much hand in establishing.

It all goes back to the disputes between Newton and Leibniz concerning the invention of the calculus. At present we have not much doubt of the fact that Newton invented the differential and integral calculus, that Leibniz' work was somewhat later but independent, and that Leibniz' notation was far superior to Newton's. At the beginning the relations between the Leibnizian and the Newtonian schools were not hostile, but it was not long before patriotic and misguidedly loyal colleagues of both discoverers instigated a quarrel, the effects of which have scarcely yet died out. In particular, it became an act of faith and of patriotic loyalty for the British mathematicians to use the less flexible Newtonian notation and to affect to look down on the new work done by the Leibnizian school on the Continent. For a while there was no scarcity of able English mathematicians of the strictly Newtonian school. For example, we must mention Taylor and Maclaurin. However, when the great continental school of the Bernoullis and Euler arose (not to mention Lagrange and Laplace who came later) there were no men of comparable calibre north of the Channel to compete with them on anything like a plane of equality. Part of this must be attributed to the fallen status of the English Universities during the 18th century. In the 17th century the English Universities were seats of learning comparable with the greatest schools of the Continent, but in the 18th century the grasping new Whig aristocracy that had risen out of the prosperous middle class (the nabobs) took over the older English institutions, the common land, public schools, universities, lock, stock and barrel, as their private property. The public schools were transformed from institutions of a semi-charitable nature to the place where the children of the new aristocracy were formed after its own pattern. The universities became nests of sinecures for dependent clergymen. In this atmosphere creative scholarship did not and could not flourish, and it is not until the 19th century is well under way that we find the signs of a new awareness of what the continental scholars, particularly Laplace and Lagrange, had done in mathematics. Among the English names belonging to this tentative reformation we may mention Boole, Peacock and DeMorgan. DeMorgan in particular is associated with the new University College at London which by its pressure did so much to bring the older universities back to a sense of intellectual responsibility.

This reform of English education was far from complete. The level of mathematics at Oxford was for many years scarcely more than contemptible, and even at Cambridge the training was devoted to the passing of severe examinations, the Triposes, rather than to the development of original mathematical workers. What mathematical talent there was in the British Isles went rather to the formation of a great school of mathematical physicists. Even here Cambridge entered the game rather late. Clerk Maxwell owes more to Faraday, the self-taught practical experimentor, than to any Cambridge man, and neither George Green nor Hamilton was in the Cambridge tradition. Sylvester, as a Jew, was not permitted to enter the older universities till towards the end of his life, and is another of those seminal figures who center around the University of London. Cayley is the first real great Cambridge pure mathematician of the 19th century. He certainly was in touch with those continental scholars whose interest was primarily in algebra, but algebra was at that time an important secondary mathematical subject rather than one in the main stream of development.

It is not remarkable that in such an environment, secluded from the central activity of world mathematics, mathematical study should be devoted rather to the formation of public school ushers or a trial intellectual run for promising barristers than to research activities. As a matter of fact, the Tripos was made such an ordeal, at least in difficulty though in general not in originality, that it marked the culminating point in the intellectual life of many of those who participated in it, and their subsequent activity became retrospective rather than creative.

This was the state of English mathematics to about the turn of the century, when an awareness of the great work of the continental mathematicians smuggles itself into England by non-academic bypaths. The English generation of pure mathematicians of the 19th century and the first decade of the 20th century is curiously tentative. It has many important names, such as A. N. Whitehead, Andrew Forsyth, E. A. Hobson and W. H. Young. These all carry to some degree a mathematical style and ethos formed under the older English tradition into a period when the topics of interest were far more continental.

Whitehead developed the new stream of postulationalism into a tool for handling the problems of multiple algebra. Together with Bertrand Russell he formed the incipient mathematical logic of Boole and of Frege into something much more scholarly and complete. Nevertheless, he did not quite arrive at the present expanded technique for handling questions of modern algebra and of topology for which he paved the way, and his later career suggests a definite abandonment of mathematics for metaphysics.

Forsyth wrote several extremely inadequate books on the theory of functions and on differential equations, but their inadequacy must not obscure the fact that they represent a transfer of English mathematical interest to problems significant to the contemporary French and Germans. Hobson, after an initial career in the classical British tradition, shifted his interest to the new theories of the integration of Lebesgue, and published a book which is a *tour de force* of the correct use of the modern notions, although the whole feeling of the book belongs to earlier days. Before this W. H. Young had almost shared with the Frenchman Lebesgue the honors of inventing the new theory of integration, for which he lost much of the contemporary credit through a certain difficulty of style and through personal idiosyncrasies.

It was into this environment of transition that Hardy and his immediate associates were born as scientists. They represent the first generation to have had contact from the beginning of their training with modern continental analysis and point-set theory. They also represent the first generation, unless one makes a partial exception here of Professor Young, to have familiar personal contact with all the leaders of their work on the continent and to be regarded by the latter as friendly equals.

Their students, and this is particularly the case with Professor Hardy, are scattered over the whole world. I need only mention Vigayaraghavan, and especially the late mathematician Ramanujan of India, Hua Loo Keng of China, Bohr of Denmark, Landau and Heilbronn of Germany, Levinson, Widder and the reviewer himself from the United States, and there are many others whom I could add.

While it was a great achievement to accomplish the integration of British mathematics into world mathematics it would not alone have placed Hardy where he stands if it were not for certain more intellectual personal characteristics.

In the first place, Hardy's approach to mathematics was that of a sportsman. To be a sportsman means to take a joy in difficulty, and to overcome difficulties according to a meticulously exact code of what is permissible and what is not permissible.

Hardy chose as his field the analytic theory of numbers, in which the theory of the Riemann zeta function had been developed sufficiently to make the field promising, and in which important triumphs had already been achieved by de La Vallée Poussin and Hadamard, but where the greater part of the more precise results remained wide-open to speculation. In the closely related field of Waring's problem, the problem of the representation of all numbers by the sum of a fixed number of nth powers, Hilbert had already obtained important results with the aid of a method distinctly algebraic rather than analytic. In those fields Hardy and his colleague, Littlewood, showed a virtuosity of rigorous mathematical technique which led to the solution of wide classes of problems. That these methods have been supplemented and in many cases superseded by the work of Vinogradov and his school in no way detracts from the subtlety and the power of the methods of the English school.

Your true sportsman is always a little aloof from worldly matters. He prefers the skilful use of a tennis racquet to the skilful use of an axe. In this respect Hardy is true to form. How much of the emotional response which led him to become a conscientious objector in the first World War is due to the abhorrence of the social destructiveness of war itself and how much to his feeling of the perversion of a beautiful thing like mathematics to unworthy ends one cannot say. Perhaps the two motives are not fully distinct. It is, however, certain that Hardy carried his hostility to applied mathematics to the extent of a real doubt that much work, which attached itself to his own but claimed an engineering motivation, really was genuinely associated with engineering.

Besides the foreign scholars whom we have listed, Hardy was a great teacher and a fellow worker of the younger leaders of English mathematics. It may not be fair to call Littlewood his pupil, but it is certainly fair to attribute to the collaboration between the two much of the stimulus which led Littlewood to develop his own vein. This vein is so closely related to Hardy's that it may be difficult for those not thoroughly familiar with the work of both to separate them. I think it is fair to say that throughout their long collaboration the extremes of technical facility belong to Littlewood, but that much of the nexus of leading ideas and the philosophical unity is that of Hardy.

If I were to single out any of the English disciples of those two men as the continuers of their tradition I should name Titchmarsh, Ingham and Paley. In his early twenties Paley had already distinguished himself as sharing with Littlewood some of the latter's power and virtuosity. His early death in a skiing accident in the Canadian Rockies was a tragedy which has been felt in the entire later development, not only of English, but of world mathematics.

The Hardy school came into being in a time of mathematical transition, not only in England but elsewhere. In the period between the wars there were signs of another change of interest which was felt all over the world, but perhaps most intensely in the United States. It presented on the one hand something of which Hardy would thoroughly approve, namely, the separation between pure and applied mathematics; on the other hand, largely under the influence of Veblen, the development of topological and algebraic fields led to somewhat a cavalier assumption that the days of analysis were over. The dominance of the Princeton school over the many English scholars who have studied there has carried these new movements back to England. I need not mention the younger Whitehead, Newman, Hodge and Mordell to indicate the trend. Indeed, Mordell, who like Hardy is a number-theory man but unlike Hardy thinks much more in algebraico-geometrical terms, is Hardy's successor.

How long these new interests will hold the upper hand is no man's prediction. I myself do not share the easy assumption that analysis is played out, and I believe that if Paley had lived these algebraic and topological schools would still find a serious competitor along the lines of Hardy's analysis. I think, however, that Hardy's opposition to applied mathematics might well give way to a recognition of the important pure mathematical problems presented by applications.

Here a parenthetic remark is permissible. The extreme abstractness of some of the modern schools tends to become rather empty abstractness unless it is supplemented by a study of special cases where the difficulty is found to be a real difficulty even when the abstract problem is studied in a concrete case. There is nothing better than concrete instances for the morale of the mathematician. Some of these concrete cases are to be found in mathematical physics and the closely related mathematical engineering, but there is a branch within mathematics which has similar merits as a source of actual problems. This field is that of number theory. It is here that such concrete cases arise with the greatest frequency and where very precise problems which are easy to formulate may demand the mathematician's greatest power and skill to resolve. Here Hardy found the central core of his work. It is precisely because Hardy's analytical tools are applied to number theory that his work has a freshness and exactness which much, although by no means all, of the fashionable work of the present day fails to exhibit. In short, Hardy had his feet on the ground.

In addition to his accomplishments in research and teaching, Hardy contributed greatly to the reform of mathematical instruction. He was bitterly opposed to the rigid and unmathematical Tripos system and is unquestionably in a large part responsible for the fact that the order of rank of the Wranglers, those who obtain first class in the Tripos, has not been published since 1912. The present mathematical Tripos and indeed the whole system of training at Cambridge has been modified in the sense of conforming very closely to the actual work and career of the mathematicians of this day. Even this change, which has spread from Cambridge to all the British Universities, is a compromise between the old system and a system where research should even more completely take the place of examinations.

Hardy accepted the shift of the focus of modern interest away from his own fields with resignation, and yet with a certain sadness. He regarded it as inevitable, and still he felt that with this inevitable change his own day was done. There is much of this sadness in the book which he published under the title *A mathematician's apology*.

Hardy was a staunch friend of all mathematicians, and especially of all young mathematicians. There is no man to whom so many of us owe so much personally. His loss, although we had been prepared for it for some years, gave us a real shock, and brought us the sense of the passing of a great age.

NORBERT WIENER

BOOK REVIEWS

Guide to the literature of mathematics and physics including related works on engineering science. By N. G. Parke. New York, McGraw-Hill, 1947. 15+205 pp. $5.00.

This book has been written in order to enable mathematicians, physicists, and engineers to have entry into the important literature of fields which are related to their own specialty. Two-thirds of the volume are devoted to a listing under subject headings of the leading treatises and texts. Judging by the books in mathematics, one would conclude that the author is exceedingly well informed as to important titles. Probably, some of the works are somewhat too advanced for the outsider. For example—examples are always unfair—E. H. Moore's *General analysis* is more of a curiosity than a reference text for the electronics specialist. Some typical headings are: aeronautics, 27 titles; algebraic geometry, 10 titles; electric discharge through gases, 16 titles; relativity, 23 titles.

In part I of this book, the author discusses the principles of reading and study, self-directed education, and the search of the literature. The young student will find here sound advice, warmly written. The scientist who wishes to collect a library, a desire which this reviewer regards with strict neutrality, will obtain valuable guidance from this work.

E. R. Lorch

Eleven and fifteen-place tables of Bessel functions of the first kind, to all significant orders. By E. Cambi. New York, Dover Publications, 1948. 5+154 pp. $3.95.

Bessel functions are in great favor, as usual, with the makers of mathematical tables. This is a handy set for $J_n(x)$, $0 \leqq x \leqq 10.5$, $\Delta x = .01$, $n = 0, 1, \cdots, 29$, 11 decimals. A supplementary table gives $J_n(x)$, $0 \leqq x \leqq .5$, $\Delta X = .001$, $n = 0, 1, \cdots, 11$, 15 decimals. A question may be raised as to whether the simultaneous publication of these tables and similar ones by the Annals of the Computation Laboratory of Harvard University represents a duplication of effort.

E. R. Lorch

Tables of spherical Bessel functions. Vol. II. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 20+232 pp. $7.50.

The functions tabulated here are $(\pi/2x)^{1/2}J_\nu(x)$ where ν is one-half an odd integer. Volume I treated the range $V = \pm 1/2, \pm 3/2, \cdots, \pm 27/2$. In the present volume we find tables corresponding to $\nu = \pm 29/2, \pm 31/2, \cdots, \pm 61/2$. The entries are given to 8–10 significant figures for $x \leqq 10$ and to 7 figures for $x > 10$. For $|\nu| \leqq 43/2$, x varies from 0 to 10 at intervals of .01 and from 10 to 25 at intervals of .1. For $|\nu| > 43/2$ the entries correspond to the range $10 \leqq x \leqq 25$. Second and sometimes fourth differences are given. The introduction contains instructions for interpolation.

E. R. LORCH

Tables of Bessel functions of fractional order. Vol. I. Prepared by the Computation Laboratory of the National Bureau of Standards. New York, Columbia University Press, 1948. 42+413 pp. $7.50.

The above tables list $J_n(x)$ for $n = \pm 3/4, \pm 2/3, \pm 1/3, \pm 1/4$, $0 < x < 25$, $\Delta x = .01$, 10 decimals. For values of x close to zero, the tables are refined by taking $\Delta x = .001$. Second differences are tabulated. The differential equations of wave theory provide an example in which these functions arise.

E. R. LORCH

Tables of the Bessel functions of the first kind of orders four, five, six, seven, eight, and nine. (Annals of the Computation Laboratory of Harvard University, vols. 5 and 6.) Cambridge, Harvard Uni-University Press, 1947. $10.00.

The staff of the computation laboratory at Harvard University has undertaken to produce tables of the function $J_n(x)$ for $n = 0, 1, 2, \cdots, 100$. In volumes III and IV the values $n = 0, 1$ and $n = 2, 3$ were treated. The present volumes take us up to $n = 10$. And even as you and I sleep, the indefatigable Automatic Sequence Controlled Calculator is pouring forth endless decimals which will shortly complete this series. Future publications in the set will not receive notice on these pages. We invite the interested reader to peruse the last page of the Bulletin which lists new publications. The functions $J_n(x)$ are listed for $0 \leqq x < 25$ with $\Delta x = .001$ and for $25 \leqq x < 100$ with $\Delta x = .01$. The tables are printed to 10 decimal places.

E. R. LORCH

NOTES

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Committee, who have been consulted regarding papers submitted for publication: R. P. Agnew, L. V. Ahlfors, Stefan Bergman, D. H. Blackwell, L. S. Bosanquet, A. T. Brauer, Herbert Busemann, J. W. Calkin, K. Chandrasekharan, I. S. Cohen, M. M. Day, J. L. Doob, C. H. Dowker, H. W. Emmons, Arthur Erdélyi, Herbert Federer, N. J. Fine, R. H. Fox, Orrin Frink, J. H. Giese, V. G. Grove, P. R. Halmos, M. H. Heins, Einar Hille, F. B. Jones, E. R. Kolchin, E. P. Lane, Norman Levinson, Charles Loewner, L. H. Loomis, C. C. MacDuffee, G. W. Mackey, G. R. MacLane, C. V. Newsom, J. C. Oxtoby, Hans Rademacher, C. E. Rickart, A. E. Ross, Herman Rubin, Raphael Salem, A. C. Schaeffer, R. D. Schafer, M. M. Schiffer, I. J. Schoenberg, I. E. Segal, Atle Selberg, Max Shiffman, Frank Smithies, Ernst Snapper, E. H. Spanier, D. C. Spencer, Gabor Szegö, A. H. Taub, T. Y. Thomas, R. M. Thrall, W. J. Trjitzinsky, A. D. Wallace, Henry Wallman.

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have referred papers: R. P. Agnew, L. V. Ahlfors, C. B. Allendoerfer, Richard Arens, Reinhold Baer, M. A. Basoco, P. T. Bateman, E. F. Beckenbach, P. O. Bell, R. H. Bing, Garrett Birkhoff, Salomon Bochner, A. T. Brauer, Richard Brauer, H. W. Brinkmann, R. C. Buck, E. W. Chittenden, R. V. Churchill, I. S. Cohen, H. V. Craig, John DeCicco, J. L. Doob, C. H. Dowker, Samuel Eilenberg, H. W. Emmons, Paul Erdös, H. P. Evans, Herbert Federer, N. J. Fine, R. H. Fox, Philip Franklin, M. J. Gottlieb, W. H. Gottschalk, Marshall Hall, P. R. Halmos, G. A. Hedlund, A. E. Heins, E. D. Hellinger, M. R. Hestenes, T. H. Hildebrandt, J. D. Hill, Einar Hille, G. P. Hochschild, L. K. Hua, R. L. Jeffery, R. E. Johnson, Bjarni Jónsson, Mark Kac, I. N. Kagno, Irving Kaplansky, S. C. Kleene, J. R. Kline, E. R. Kolchin, Norman Levinson, A. N. Lowan, N. H. McCoy, C. C. MacDuffee, G. W. Mackey, Saunders MacLane, H. W. March, Morris Marden, A. M. Mark, W. T. Martin, W. E. Milne, David Moskovitz, M. E. Munroe, Zeev Nehari, C. V. Newsom, Oystein Ore, Gordon Pall, Harry Pollard, W. T. Puckett, Hans Rademacher, E. D. Rainville, J. F. Randolph, M. O. Reade, R. M. Robinson, A. E. Ross, Hans Samelson, A. C. Schaeffer, I. J. Schoenberg, W. T. Scott, I. E. Segal, Abraham Seidenberg, I. M. Sheffer, M. F. Smiley, R. D. Specht, J. J. Stoker, M. H. Stone, W. C. Strodt, Gabor Szegö, A. H. Taub, Olga Taussky-Todd, T. Y. Thomas, W. J. Thron, C. A. Truesdell, Paul Turán, J. L. Vanderslice, D. F. Votaw, R. J. Walker, J. L. Walsh, Hassler Whitney, D. V. Widder, Albert Wilansky, R. L. Wilder, J. E. Wilkins, John Williamson, G. S. Young, J. W. T. Youngs, Oscar Zariski, Leo Zippin, Max Zorn.

Applications are invited for the Harry Bateman Research Fellowships at the California Institute of Technology. These are post-doctoral fellowships carrying a stipend of $3600 for the academic year. Recipients will teach one course and devote the remainder of their time to research.

Professor Tomlinson Fort of the University of Georgia has been appointed Distinguished Service Professor.

Professor L. M. Graves of the University of Chicago has been honored with the presidential certificate of merit for his work on the panel of applied mathematics of the National Defense Research Committee.

Professor Tibor Rado of Ohio State University has been elected a corresponding member of the Academy of Science of Bologna.

Professor I. S. Sokolnikoff of the University of California at Los Angeles has been awarded the presidential certificate of merit for "outstanding contributions in aid of the war effort against the common enemies of the United States and its allies in World War II."

Dr. J. A. Schouten has been appointed to a professorship at the University of Amsterdam.

Professor R. B. Seth of Hindu College, Delhi, India, has been appointed to a visiting professorship at the Iowa State College of Agriculture and Mechanic Arts.

Dr. Mary-Elizabeth L. Solari has been appointed lecturer at Chelsea Polytechnic, London, England.

Dr. B. L. van der Waerden has been appointed to a professorship at the University of Amsterdam.

Associate Professor J. A. Clarkson of the University of Pennsylvania has been appointed to a professorship at Tufts College.

Dr. G. M. Conwell of St. Paul's School, Concord, New Hampshire, has been appointed to an associate professorship at the University of Georgia.

Dr. J. T. Culbertson of the University of Chicago has been appointed to an assistant professorship at the University of Southern California.

Mrs. Stewart Dismuke has accepted a position as physicist at the Oak Ridge National Laboratory.

Mr. L. A. Fine has accepted a position in business management with the United Trading Company, Boston, Massachusetts.

Mr. H. L. Herrick of Yale University has accepted a position as mathematician with the International Business Machines Corporation, New York, New York.

Mr. Joseph Hilsenrath of the Naval Ordnance Laboratory has accepted a position as scientific educational advisor with the National Bureau of Standards.

Mr. Jaime Lifschitz has been appointed to a professorship at the Technical Institute of Monterrey, Monterrey, Mexico.

Professor J. C. C. McKinsey of the Oklahoma Agricultural and Mechanical College has been granted a leave of absence to participate in the Rand Project at Douglas Aircraft, Santa Monica, California.

Assistant Professor J. K. L. MacDonald of Cooper Union has been appointed head of Applied Mathematics Research, Naval Ordnance Test Station, China Lake, California.

Mr. Paul Meier of Princeton University has been appointed to an assistant professorship at Lehigh University.

Mr. C. A. de Moraes of Rensselaer Polytechnic Institute has accepted a position with the National Advisory Committee for Aeronautics, Langley Field, Hampton, Virginia.

Dr. T. S. Motzkin, on leave from the Hebrew University, Jerusalem, has been appointed to an associate professorship at Boston College.

Mr. A. T. Myers of the University of Airzona has accepted a position at the Tucumcori High School, Tucumcori, New Mexico.

Assistant Professor D. H. Rock of the Iowa State College of Agriculture and Mechanic Arts has accepted a position as mathematician with the United States Air Forces.

Mr. M. L. Slater of Harvard University has been appointed senior mathematician for the Bureau of Ordnance at the Museum of Science and Industry, Chicago, Illinois.

Sister Helen Sullivan of Mount Saint Scholastica College has been appointed to a visiting professorship at Loyola University.

Assistant Professor D. B. Sumner of the Louisiana State University and Agricultural and Mechanical College has been appointed special lecturer at the University of Toronto.

Associate Professor C. J. Thorne, on leave from the University of Utah, has been appointed to an associate professorship at the University of California at Los Angeles.

Mr. Herschel Weil of Brown University has accepted a position as mathematician with the General Electric Company, Schenectady, New York.

The following promotions have been announced:

Leonard Greenstone, University of California at Los Angeles, to an assistant professorship.

Howard Levi, Columbia University, to an assistant professorship.

K. F. McLaughlin, United States Naval Academy, to an assistant professorship.

P. V. Reichelderfer, Ohio State University, to a professorship.

R. C. Simpson, United States Naval Academy, to an assistant professorship.

E. G. Swafford, United States Naval Academy, to an assistant professorship.

Annita Tuller, Hunter College, to an assistant professorship.

The following appointments to instructorships are announced: University of Georgia: Mr. James Bercos; Harvard University: Dr. John Gurland; University of Illinois: Dr. R. G. Langebartel; North Texas Agricultural College: Miss Merle Mitchell; University of Oklahoma: Mr. E. V. Greer; Princeton University: Mr. E. E. Floyd; Rockford College: Miss Margaret Owchar; Tufts College: Dr. Shepard Bartnoff (physics); Wilson Junior College: Mr. J. H. Siedband.

Dr. J. T. Rorer of Temple University died August 13, 1948. He had been a member of the Society for thirty-two years.

Professor Emeritus C. T. Sullivan of McGill University died September 17, 1948. He had been a member of the Society for thirty-six years.

Dean R. C. Tolman of the Graduate School, California Institute of Technology, died September 5, 1948. He had been a member of the Society for twenty-five years.

Professor Emeritus J. H. M. Wedderburn of Princeton University died October 10, 1948. He had been a member of the Society for thirty-five years.

Correction: The note appearing on page 870 of the Bulletin of the American Mathematical Society, vol. 54, no. 9, stating that Dr. P. H. Anderson has been appointed to an associate professorship of marketing at Loyola University, should read "professorship."

NEW PUBLICATIONS

Bell, A. E. *Christian Huygens and the development of science in the seventeenth century.* London, Arnold, 1947. 220 pp. 18s.

Blanc, C. *Les équations différentielles de la technique.* Neuchatel, Griffon, 1947. 315 pp.

Bochner, S. and Martin, W. T. *Several complex variables.* (Princeton Mathematical Series, no. 10.) Princeton University Press, 1948. 10+216 pp. $4.00.

Casanova, G. *Mathématiques et materialisme dialectique.* Paris, Editions Sociales, 1947. 196 pp. 225 fr.

Cohen, M. R., and Drabkin, I. E. *A source book in Greek Science.* New York, McGraw-Hill, 1948. 22+597 pp. $9.00.

Coxeter, H. S. M. *Regular polytopes.* London, Methuen, 1948. 20+231 pp. 8 plates. 50s.

Drabkin, I. E. See Cohen, M. R.

Gonseth, F. *La géometrie et le problème de l'espace.* Parts 1–3. Neuchatel, Griffon, n.d. 70+90+110 pp.

Hille, E. *Functional analysis and semi-groups.* (American Mathematical Society Colloquium Publications, vol. 31.) New York, American Mathematical Society, 1948. 7+528 pp. $7.50.

Kachanow, L. N., and others. *Plastic deformation.* New York, Mapleton, 1948. 192 pp. $8.00.

Locher-Ernst, L. *Differential- und integralrechnung im hinblick auf ihre anwendungen.* Basel, Verlag, 1948. 594 pp. 48 fr.

von Kármán, T. See Mises, R.v.

Martin, W. T. See Bochner, S.

Miller, K. S. See Salvadori, M. G.

Mises, R. v. and von Kármán, T. *Advances in applied mechanics.* New York, Academic Press, 1948. 8+293 pp. $6.80.

Rashevsky, N. *Mathematical biophysics.* Rev. ed. University of Chicago Press, 1948. 23+669 pp. $7.50.

Rosser, J. B. *Theory and application of* $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy\int_0^y e^{x^2}dx$. Part 1: *Methods of computation.* New York, Mapleton, 1948. 4+192 pp. $8.00.

Salvadori, M. G., and Miller, K. S. *The mathematical solution of engineering problems.* New York, McGraw-Hill, 1948. 10+245 pp. $3.50.

Segre, B. *Lezioni di geometria moderna.* Bologna, Zanichelli, 1948. 4+195 pp.

Severi, F. *Funzioni quasi-abeliane.* (Pontificiae Academiae Scientiarum Scripta Varia 4.) Vatican City, Pontifical Academy of Sciences, 1947. 327 pp.

Struik, D. J. *A concise history of mathematics.* 2 vols. New York, Dover, 1948. 18+299 pp. $3.00.

Wiener, N. *Cybernetics.* New York, Wiley, 1948. 194 pp. $3.00.

Vranceanu, G. *Leçons de géometrie différentielle.* Bucarest, Rotativa, 1947. 422 pp.

Wilks, S. S. *Elementary statistical analysis.* Princeton University Press, 1948. 11+284 pp. $2.50.

A CHARACTERIZATION OF CONFORMALLY FLAT SPACES

A. H. TAUB[1]

1. Introduction. The purpose of this paper is to show some analogies between the theory of groups of infinitesimal conformal transformations and that of groups of motions in Riemannian spaces. The main part of the paper will be devoted to Theorem I: *The necessary and sufficient condition for an n-dimensional* ($n \geqq 3$) *Riemannian space,* V_n, *to admit an* $(n+1)(n+2)/2$ *parameter group of infinitesimal conformal transformations is that the* V_n *be conformally flat.*

This theorem is the analogue of the well known theorem concerning groups of motions of n-dimensional Riemannian spaces V_n, namely,[2] *A group of motions of a* V_n *has at most* $n(n+1)/2$ *parameters and this number only in case* V_n *has constant curvature.* Sasaki[3] has shown part of Theorem I by using the formalism of the conformal connection. He has shown that a space with a conformal connection will admit a maximal number of independent infinitesimal conformal transformations only if it is conformally flat but has not given this number. The proof given here of Theorem I will be carried out in a fashion similar to Eisenhart's proof of the latter one. We shall use the notation of R. G. throughout this paper.

In a V_n $(n>3)$ for which the Weyl conformal curvature tensor vanishes, there exists a coordinate system[4] where

$$g_{ij} = e^{2\sigma(x)} e_i \delta_{ij}, \qquad e_i = \pm 1. \tag{1.1}$$

That is, the V_n is conformal to a flat space S_n with the metric

$$a_{ij} = e_i \delta_{ij}.$$

The S_n admits[5] the conformal group of $(n+1)(n+2)/2$ parameters generated by dilatation, inversions, translations and orthogonal transformations. The part of this group connected to the identity is

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[1] Guggenheim post-Service Fellow at the Institute for Advanced Study, on leave from the University of Washington.

[2] L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, 1926, p. 239. This work will be referred to as R. G. hereafter.

[3] Shigeo Sasaki, *Geometry of the conformal connexion*, Science Reports of the Tôhoku Imperial University (1) vol. 29 (1940) pp. 219–267. I am indebted to J. L. Vanderslice for this reference and a valuable suggestion regarding the integrability condition of equation (3.5).

[4] R. G. p. 92.

[5] R. G. p. 94, problem 16.

generated by an infinitesimal group of $(n+1)(n+2)/2$ parameters. This infinitesimal group is simply isomorphic to a group in V_n which is a group of infinitesimal conformal transformations of V_n into itself. Hence if a V_n $(n \geqq 3)$ is conformally flat, it admits an $(n+1)(n+2)/2$ parameter group of infinitesimal conformal transformations.

2. **Infinitesimal conformal transformations.** The proof of Theorem I will be completed by examining the differential equations satisfied by the vectors determining an infinitesimal conformal transformation of a V_n. These are[6]

$$h_{ij} = \xi_{i,j} + \xi_{j,i} = \psi g_{ij} \tag{2.1}$$

where the comma denotes covariant differentiation.

It is obvious from these equations that if $\xi_{\alpha|i}$ and $\xi_{\beta|i}$ are solutions with $\psi=\psi_\alpha$ and $\psi=\psi_\beta$ respectively then so is $\xi_i = a^\alpha\xi_{\alpha|i}+a^\beta\xi_{\beta|i}$ (α and β not summed) where a^α and a^β are arbitrary constants and $\psi = a^\alpha\psi_\alpha + a^\beta\psi_\beta$. Hence we have[7]

If $\xi_{\sigma|i}$ *for* $\sigma=1, \cdots, r$ *are components of infinitesimal conformal transformations of a* V_n *so are* $a^\sigma\xi_{\sigma|i}$, *where the a's are arbitrary constants.*

A straightforward calculation using (2.2) given below shows that under the same hypothesis as above

$$\xi_{\beta\alpha|i} = \overset{j}{\xi}_{\beta|} \frac{\partial\xi_{\alpha|i}}{\partial x^j} - \overset{j}{\xi}_{\alpha|} \frac{\partial\xi_{\beta|i}}{\partial x^j} = \overset{j}{\xi}_{\beta|}\xi_{\alpha|i,j} - \overset{j}{\xi}_{\alpha|}\xi_{\beta|i,j}$$

is also a solution of (2.1) with $\psi=\psi_{\beta\alpha}=\overset{k}{\xi}_{\beta|}\psi_{\alpha,k}-\overset{k}{\xi}_{\beta|}\psi_{\beta,k}$. Thus we have the result:[7] *If* $X_\sigma f \equiv \overset{i}{\xi}_{\sigma|}\partial f/\partial x^i$ *for* $\sigma=1, 2, \cdots, r$ *are the generators of infinitesimal conformal transformations of a* V_n, *so are* $(X_\sigma, X_\tau)f$ *for* $\sigma, \tau = 1, \cdots, r$ $(\sigma \neq \tau)$.

These two results show that the set of all generators of infinitesimal conformal transformations of V_n are closed under the operations of taking linear combinations and forming commutators, that is, these generators form a Lie algebra just as in the case of generators of motions of V_n.

From equation (2.1) it follows that

$$h_{ij,k} + h_{ik,j} - h_{jk,i} = g_{ij}\psi_{,k} + g_{ik}\psi_{,j} - g_{jk}\psi_{,i}.$$

By using the procedure given in R. G. it may be shown that

$$\xi_{i,jk} = -\xi_m R^m_{kij} + 2^{-1}(\psi_{,k}g_{ij} + \psi_{,j}g_{ik} - \psi_{,i}g_{jk}) \tag{2.2}$$

---

[6] R. G. p. 231, equation (69.1).

[7] Cf. R. G. §71 p. 237.

and the integrability conditions of these equations are

$$
\begin{aligned}
(2.3)\qquad & \xi_m(R^{m}_{kij,l} - R^{m}_{lij,k}) + \xi_{m,l}R^{m}_{kij} - \xi_{m,k}R^{m}_{lij} + \xi_{i,m}R^{m}_{jkl} + \xi_{m,j}R^{m}_{ikl} \\
& \qquad + 2^{-1}(g_{il}\psi_{,jk} - g_{ik}\psi_{,jl} + g_{jk}\psi_{,il} - g_{jl}\psi_{,ik}) = 0.
\end{aligned}
$$

3. **Transformation of the integrability conditions.** We now transform equations (2.3) into another form. From the Bianchi identity

$$(3.1)\qquad R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

it follows that equation (2.3) may be written as

$$
\begin{aligned}
(3.2)\qquad & -\xi^{m}R_{ijkl,m} + \xi_{m,l}R^{m}_{kij} - \xi_{m,k}R^{m}_{lij} + \xi_{i,m}R^{m}_{jkl} + \xi_{m,j}R^{m}_{ikl} \\
& \qquad + 2^{-1}(g_{il}\psi_{,jk} - g_{ik}\psi_{,jl} + g_{jk}\psi_{,il} - g_{jl}\psi_{,ik}) = 0.
\end{aligned}
$$

To eliminate ψ from equation (3.2) we multiply by g^{il} and sum. The resulting expression is multiplied by g^{jk} and summed. We thus obtain

$$(3.3)\qquad -\xi^{m}R_{jk,m} - \xi_{m,k}R^{m}_{j} - \xi_{m,j}R^{m}_{k} + 2^{-1}(n-2)\psi_{,jk} + 2^{-1}g_{jk}\Delta_2\psi = 0,$$

$$(3.4)\qquad \Delta_2\psi = \frac{1}{n-1}(\xi^{m}R_{,m} + 2\xi_{m,k}R^{mk})$$

where $\Delta_2\psi = g^{ij}\psi_{,ij}$. Substituting for $\Delta_2\psi$ from (3.4) into (3.3) we obtain

$$
\begin{aligned}
(3.5)\qquad \frac{1}{2}(n-2)\psi_{,jk} = {} & \xi^{m}\left(R_{jk,m} - \frac{1}{2(n-1)}g_{jk}R_{,m}\right) \\
& + \xi_{m,k}R^{m}_{j} + \xi_{m,j}R^{m}_{k} - \frac{1}{(n-1)}g_{ik}\xi_{m,l}R^{ml}.
\end{aligned}
$$

When this value of $\psi_{,jk}$ is substituted into (3.2) we obtain

$$(3.6)\qquad -\xi^{m}C_{ijkl,m} + \xi_{m,p}(\delta^{p}_{l}C^{m}_{kij} - \delta^{p}_{k}C^{m}_{lij} + \delta^{m}_{i}C^{p}_{jkl} + \delta^{p}_{j}C^{m}_{ikl}) = 0$$

as the integrability conditions of equations (2.2) where C^{h}_{ijk} is the Weyl conformal curvature tensor and is defined in terms of the Riemann tensor, Ricci tensor and the scalar curvature by

$$
\begin{aligned}
(3.7)\qquad C^{h}_{ijk} = {} & R^{h}_{ijk} + \frac{1}{n-2}(\delta^{h}_{j}R_{ik} - \delta^{h}_{k}R_{ij} + g_{ik}R^{h}_{j} - g_{ij}R^{h}_{k}) \\
& + \frac{R}{(n-1)(n-2)}(\delta^{h}_{k}g_{ij} - \delta^{h}_{j}g_{ik}).
\end{aligned}
$$

The integrability conditions of equations (3.5) are

$$\xi^m R_{jkl,m} + \xi^m_{,l} R_{jkm} - \xi^m_{,k} R_{jlm} - 2^{-1}(n-2)\psi_{,p} C^p_{jkl} = 0 \tag{3.8}$$

where

$$R_{jkl} = \left(R_{jk} - \frac{R}{2(n-1)} g_{jk}\right)_{,l} - \left(R_{jl} - \frac{R}{2(n-1)} g_{jl}\right)_{,k}. \tag{3.9}$$

Equations (2.2), (3.5) and the equations

$$\frac{\partial \xi_i}{\partial x^j} = \xi_{i,j} + \xi_m \left\{ {m \atop ij} \right\}, \qquad \frac{\partial \psi}{\partial x^i} = \psi_{,i}$$

constitute a system of equations of the form

$$\frac{\partial \theta^\alpha}{\partial x^i} = \overset{\alpha}{\psi_i}(x, \theta), \qquad \alpha = 1, 2, \cdots, (n+1)^2;\ i = 1, 2, \cdots, n, \tag{3.10}$$

where θ^α represents the set of $(n+1)^2$ functions ψ; ξ_i; $\psi_{,i}$; and $\xi_{i,j}$. The functions θ^α are subject to the $n(n+1)/2$ algebraic relations (2.1). The system (3.10) is of a standard form[8] and it is well known that the maximum number of linearly independent ξ_i is $(n+1)(n+2)/2$ and that this number obtains when the integrability conditions of (3.10) are identically satisfied in virtue of (2.1). That is, if a space admits this maximum number of linearly independent infinitesimal conformal transformations equations (3.6) and (3.8) must be satisfied for arbitrary choices of the ξ_i and $\xi_{i,j}$ satisfying (2.1). Thus

$$C_{ijkl,m} = 0 \tag{3.11}$$

and at least the antisymmetric part with respect to p and m of the coefficient of $\xi_{m,p}$ in equation (3.6) must vanish. That is, we must have

$$\delta^p_l C^m_{kij} - \delta^p_k C^m_{lij} + \delta^m_i C^p_{jkl} + \delta^p_j C^m_{ikl} = \delta^m_l C^p_{kij} - \delta^m_k C^p_{lij} + \delta^p_i C^m_{jkl} + \delta^m_j C^p_{ikl}.$$

Setting $p=l$, summing, and using the fact that $C^m_{i\ jm}=0$ we obtain

$$(n-1)C^m_{kij} = C^m_{ijk} + C^m_{jki} + C^m_{kij} = 0$$

the last equality being a consequence of the symmetry properties of C^m_{ijk}. Hence if a $(n+1)(n+2)/2$ parameter group of infinitesimal conformal transformations is to exist we must have

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[8] Cf. L. P. Eisenhart, *Non-Riemannian geometry*, Amer. Math. Soc. Colloquium Publications, vol. 8, New York, 1927, p. 14.

$$C^{m}_{kij} = 0. \tag{3.12}$$

This is the necessary and sufficient condition for a V_n ($n>3$) to be conformally flat.

If $n>3$ then it follows from

$$C^{h}_{ijk,h} = \frac{n-3}{n-2} R_{ijk}$$

that equations (3.8) are satisfied when (3.11) holds. If $n=3$, equations (3.6) are automatically satisfied since the Weyl conformal curvature tensor vanishes in this case. If a V_3 admits a 10 parameter group of infinitesimal conformal transformations then the argument used above applied to (3.8) leads to

$$R_{jkl,m} = 0$$

and

$$g_{pl}R_{jkm} - g_{ml}R_{jkp} - g_{pk}R_{jlm} + g_{mk}R_{jlp} = 0.$$

Multiplying the last equation by g^{pl} and summing gives

$$R_{jkm} = 0.$$

This is the necessary and sufficient condition for a V_3 to be conformally flat.

INSTITUTE FOR ADVANCED STUDY

THE DIFFERENTIAL INVARIANTS OF A TWO-INDEX TENSOR

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Riemannian geometry, based upon a metric form $ds^2 = g_{ij}dx^i dx^j$, gives us the curvature tensor R^i_{jkl} as the sole basic differential invariant of the space, and of the symmetric tensor g_{ij}. The general tensor g_{ij} can be broken up into the sum of two irreducible components, namely the symmetric and antisymmetric portions defined respectively by $2g_{(ij)} = g_{ij} + g_{ji}$ and $2g_{[ij]} = g_{ij} - g_{ji}$. The latter disappears in constructing ds^2; but the general differential invariants of g_{ij} must necessarily be composed of those derivable from $g_{(ij)}$ (the curvature tensor above), from $g_{[ij]}$, and a group of mixed invariants dependent upon both. It is proposed to investigate the general problem by use of a well known and easily proved fundamental lemma of the calculus of variations: *The Euler equations derived from a variational principle are tensor-invariant under the group of transformations which leaves the original integral invariant.* Actually the equations as directly obtained state that a certain covariant vector vanishes.

Given the tensor $g_{ij}(x^1 \cdots x^n)$ we first introduce two (implicit) absolute parameters u, v, and construct the variational problem

$$(1) \qquad \delta \int g_{ij}\overset{i}{x}_u \overset{j}{x}_v du dv = 0; \qquad \overset{i}{x}_u = \frac{\partial x^i}{\partial u}, \quad \text{and so on.}$$

Only x-transformations will be allowed for the present. The Euler equations become

$$(2) \qquad \begin{gathered} 2\{g_{(ij)}\overset{j}{x}_{uv} + L_{jki}\overset{j}{x}_u \overset{k}{x}_v\} = 0; \\ L_{jki} = (g_{ik,j} + g_{ji,k} - g_{jk,i})/2; \qquad g_{ij,k} = \partial g_{ij}/\partial x^k. \end{gathered}$$

These L_{ijk} must, therefore, have the law of transformation of Christoffel symbols of the first kind. In fact

$$(3) \qquad \begin{aligned} L_{jki} &= \{g_{(ik),j} + g_{(ij),k} - g_{(jk),i}\}/2 + \{g_{[ik],j} + g_{[ji],k} + g_{[kj],i}\}/2 \\ &= \Gamma_{jki} + \Omega_{jki}. \end{aligned}$$

Here Γ_{jki} are precisely Christoffel symbols of the first kind associated with $g_{(ij)}$, and Ω_{jki} *is the fully covariant form of the Cartan torsion tensor.* If now the discriminant $|g_{(ij)}| \neq 0$, we may construct $g^{(ij)}$ as usual to raise indices, and then obtain the coefficients of a general

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affine connection:

$$L^i_{jk} = g^{(ir)} L_{jkr} = \Gamma^i_{jk} + \Omega^i_{jk}, \tag{4}$$

with

$$g_{(ij)|k} \equiv g_{(ij),k} - g_{(ir)}L^r_{jk} - g_{(rj)}L^r_{ik} = 0.$$

That is, *the covariant derivative of* $g_{(ij)}$ *with respect to the affine connection defined by* (2) *vanishes*. This follows from the vanishing of the covariant derivative of $g_{(ij)}$ with respect to Γ^i_{jk}, and the antisymmetry of Ω_{ijk} in any pair of indices.

One may note in passing that Ω_{ijk} *is the only basic differential invariant dependent upon* $g_{[ij]}$ *alone*. Any differential invariant constructed from $g_{[ij]}$ must be based upon some form of tensor differentiation. If we take the covariant derivative with respect to any symmetric affine connection, we may write

$$\begin{aligned} \Omega_{ijk} &\equiv g_{[ij]|k} + g_{[jk]|i} + g_{[ki]|j} \\ &\equiv g_{[ij],k} + g_{[jk],i} + g_{[ki],j}. \end{aligned} \tag{5}$$

The antisymmetry of $g_{[ij]}$ and the symmetry of the connection coefficients make the latter cancel out of (5). The tensor $g_{[ij]|k}$ has only one other irreducible component besides Ω_{ijk}, namely $g_{[ij]|k}+g_{[kj]|i}$ and this is clearly dependent upon the connection coefficients; hence, in this case, the remaining tensor-invariants are mixed. The condition $\Omega_{ijk}=0$ is necessary and sufficient for $g_{[ij]}$ to be a curl, that is, of form $\lambda_{i,j}-\lambda_{j,i}$, as is well known.

In the foregoing, however, we obtain no indication as to the intrinsic geometry of a purely antisymmetric g_{ij}, that is, when $g_{(ij)}=0$, though this is the only case where the integral in (1) is invariant under parameter transformation. This defect may again be remedied by utilizing another variational principle. We take a single absolute parameter t, and look for the geodesics of a metric of higher order (2) associated with $g_{ij}\dot{x}^i\ddot{x}^j$. But this is not an invariant under x-transformations, as $\ddot{x}^j$ is not a vector. We must therefore add to $\ddot{x}^i$ the terms $A^i_{jk}\dot{x}^j\dot{x}^k$, where A^i_{jk} are as yet unspecified coefficients of affine connection symmetric, without loss of generality. Still further, we have to utilize the fact that a variational problem is unchanged by the addition of a perfect differential to the integral. If we integrate $g_{ij}\dot{x}^i\ddot{x}^j/2$ by parts, we have left under the sign of integrationn $-g_{ji}\dot{x}^i\ddot{x}^j/2-g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k/2$. The actual statement, therefore, is:

$$\delta\int\left\{g_{[ij]}\dot{x}^i\ddot{x}^j + (-g_{ij,k}/2 + g_{ir}A^r_{jk})\dot{x}^i\dot{x}^j\dot{x}^k\right\}dt = 0. \tag{6}$$

The terms in parentheses are completely symmetric in all three lower indices ijk, and we may therefore take $A^r_{jk}=\Gamma^r_{jk}$, the Christoffel symbols associated with $g_{(ij)}$; then the terms in $g_{(ij),k}/2$ cancel out with those in $g_{(ir)}A^r_{jk}$, leaving only $g_{[ir]}A^r_{jk}\dot{x}^i\dot{x}^j\dot{x}^k$. We may then drop the brackets in subscript, taking $g_{ij}=g_{[ij]}$ for we have finished with $g_{(ij)}$, which enters hereafter only through Γ^i_{jk}, its Christoffel symbols.

We make the further assumption that $|g_{ij}|\neq 0$ which implies that n, the number of dimensions of the space, is even. This is to allow raising and lowering of indices, along with a solution of the Euler equations explicitly for dx^i/dt. The equations themselves are

$$(7)\qquad 2g_{ij}\{dx^j/dt + 3x^r\dot{x}^k\gamma^j_{rk} + \beta^j_{krs}\dot{x}^k\dot{x}^r\dot{x}^s\} = 0,$$

where

$$(8)\quad \begin{aligned} &\text{(a)}\quad g_{ij} + g_{ji} = 0;\\ &\text{(b)}\quad 3g_{ir}\gamma^r_{jk} = 3g_{ij,k}/2 + (g_{ik,j} + g_{jk,i})/2 - H_{ijk};\\ &\text{(c)}\quad 2g_{il}\beta^l_{jkr} = g_{i(j,k,r)} - H_{i(jk,r)} + H_{rjk,i};\\ &\text{(d)}\quad \text{with } H_{ijk} = g_{ir}\Gamma^r_{jk} + g_{jr}\Gamma^r_{ki} + g_{kr}\Gamma^r_{ij}, \end{aligned}$$

where the bracketed subscripts in c denote only the completely symmetric portion. The tensor analysis of equations $d\ddot{x}^i/dt+\alpha^i(x, \dot{x}, \dot{x}, t)=0$ has been discussed elsewhere. *The basic set of differential invariants in our present case would be, besides the vector $\dot{x}^i$ itself, the two tensors*

$$\dot{P}^i_r = \alpha^i_{;r} - 3\left(\frac{d}{dt}\gamma^i_r + \gamma^i_k\gamma^k_r\right);$$

$$(9)\qquad Q^i_r = \alpha^i_{,r} - \alpha^i_{;k}\gamma^k_r - \frac{d^2}{dt^2}\gamma^i_r + 2\gamma^k_r\frac{d}{dt}\gamma^i_k - 2\gamma^i_k\frac{d}{dt}\gamma^k_r + 2\gamma^i_k\gamma^k_l\gamma^l_r;$$

where

$$\alpha^i = 3x^r\dot{x}^k\gamma^i_{rk} + \beta^i_{krs}\dot{x}^k\dot{x}^r\dot{x}^s,$$

$$\alpha^i_{;r} = \frac{\partial\alpha^i}{\partial\dot{x}^r},\quad\text{and}\quad \gamma^i_j = \frac{1}{3}\,\frac{\partial\alpha^i}{\partial x^j} = \gamma^i_{jk}\dot{x}^k.$$

The remaining (basic) differential invariants all vanish identically. Moreover, γ^i_{jk} have the transformation law of coefficients of an affine connection, while there are three tensorial differential operators:

$$(10)\qquad \frac{\partial}{\partial x^i},\qquad \frac{\partial}{\partial\dot{x}^i} - 2\gamma^r_i\frac{\partial}{\partial x^r},$$

and the third corresponding full covariant differentiation

$$\begin{aligned}\nabla_r u^i \equiv u^i_{,r} &- \gamma^k_r\left(\frac{\partial u^i}{\partial \dot{x}^k} - 2\gamma^s_k\frac{\partial u^i}{\partial x^s}\right)\\ &+ \left\{2\frac{d}{dt}\gamma^k_r - 2\gamma^k_l\gamma^l_r - \alpha^k_{;r}\right\}\frac{\partial u^i}{\partial x^k} + \gamma^i_{jr}u^r.\end{aligned} \tag{11}$$

We are primarily interested in those differential invariants obtained from (9)–(11) which are independent of $\dot{x}$ and x.

These invariants may be calculated very simply as follows. Wherever $\ddot{x}^i$ occurs, we add thereto the terms $\Gamma^i_{jk}\dot{x}^j\dot{x}^k$, and subtract the corresponding terms elsewhere in the same tensor. The tensor is thereby resolved into its invariant components, namely the coefficients of $(\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k)$ and the homogeneous polynomials of various degree in $\dot{x}^r$.

The actual calculation is avoided almost entirely by the following considerations. It is known and easily proved that the connection coefficients γ^i_{jk} in (8b) have the same law of transformation as Christoffel symbols of the second kind. Moreover, we have

$$g_{ij,k} - g_{ir}\gamma^r_{jk} - g_{rj}\gamma^r_{ik} = 0, \text{ identically,} \tag{12}$$

whence *the covariant derivative of the basic tensor* g_{ij} *vanishes with respect to its own "Christoffel symbols."* Now the torsion, that is, the anti-symmetric portion, is seen to be

$$g_{ir}\gamma^r_{[jk]} = (g_{ij,k} + g_{jk,i} - g_{ik,j})/2 = \Omega_{ijk}/6, \tag{13}$$

as before. Since any tensor of the proper rank may be added to or subtracted from connection coefficients without affecting their transformation law, we have only to investigate invariants of the symmetric portion

$$\gamma^r_{(jk)} = g^{ir}(g_{ij,k} + g_{ik,j} - H_{ijk})/3. \tag{14}$$

From these, again, one may eliminate $g_{ij,k}+g_{ik,j}$ by using their covariant derivatives with respect to Γ^i_{jk}, that is, $g_{ij|k}+g_{ik|j}$, which is seen to be the irreducible component of $g_{ij|k}$ mentioned before. We then have to investigate the curvature tensor formed by means of the symbols

$$L^i_{jk} = \left\{\Gamma^r_{jk} + 2g^{ir}(\Gamma^s_{ik}g_{sj} + \Gamma^s_{ij}g_{sk})\right\}/3. \tag{15}$$

Choosing special coordinates in which Γ^i_{jk} (but not their derivatives) vanish at a point,.the tensor in question is seen to be

$$R^{i}_{jlk}/3 + 2(g^{ir}g_{sj}R^{s}_{ilk} + g^{ir}g_{sk}R^{s}_{ilj})/3, \tag{16}$$

where R^{i}_{jkl} is the Riemann-Christoffel curvature tensor formed as usual from the Γ^{i}_{jk}. The derivation proves that the sole invariant depending only on the g_{ij} is Ω_{ijk}, as before.

To round out the discussion we have to consider other tensors of rank two, such as g^{ij} and $g_{j}{}^{i}$, as well as tensors of weight p other than zero. The weight can be reduced to zero by division with a suitable power of the determinant except when the determinant vanishes, or $np+2=0$, in which case nothing further can be done. Furthermore, we have the usual algebraic method of taking the normalized cofactors of the transposed matrix for associating a contravariant tensor g^{ij} with a covariant g_{ij}. In this case, however, the geometry is associated with the variational principle $\delta\int g^{ij}\phi_{,i}\phi_{,j}dx^{1}\cdots dx^{n}=0$ and therewith a generalized Laplace equation. No new invariants are obtained. Finally, for the mixed tensor g^{i}_{j}, one may look for the extremum $\delta\int g^{i}_{j}\phi_{,i}\dot{x}^{j}dt$ but it can be seen that this leads neither to a geometrical interpretation nor to differential invariants; though $\dot{x}^{i}$ may be eliminated from the resulting equations, the equations themselves cannot be solved explicitly for $\partial^{2}\phi/\partial x^{i}\partial x^{j}$. Thus, we obtain only the trivial cases (1) $g^{i}_{j}=\delta^{i}_{j}$ which is then an invariant of every affine connection, the integral above reducing to that of a perfect differential, hence independent of the path, and (2) the obvious invariant $g^{r}_{r,i}$.

BOMBAY, INDIA

COVERINGS AND BETTI NUMBERS

BENO ECKMANN

1. **Introduction.** Let the finite polyhedron P be a regular covering of the polyhedron $\bar{P}$, and let G be the corresponding (finite) group of covering transformations of P. Then G acts as operator group on the homology groups H_n of P. If we consider homology with real coefficients, H_n is a real vector group of finite rank $p_n = n$th Betti number of P, and G operates in H_n as a group of linear transformations; we denote by $s_n(x)$, $x \in G$, the character of this linear representation of G of degree p_n. Let g be the order of G.

In this note we shall prove:

THEOREM 1. *The nth Betti number $\bar{p}_n$ of $\bar{P}$ is given by*

$$\bar{p}_n = \frac{1}{g} \sum_{x \in G} s_n(x).$$

In the Princeton Bicentennial Conference W. Hurewicz raised the question whether the homology groups of a polyhedron $\bar{P}$ are determined by those of a regular covering P given as groups with operators. According to Theorem 1 the answer is affirmative in the case of *finite* polyhedra P, $\bar{P}$ and real (or rational) coefficients. We shall show elsewhere—in a general theory of complexes with automorphisms—that the same is true for arbitrary (finite or infinite) polyhedra, provided that the group G is finite. Since the proof in our present case, based upon "harmonic chains," is very simple and yields the explicit formula of Theorem 1, which has interesting applications, we give it here independently of other more general considerations.

2. **Simplicial covering and harmonic chains.** To compute the homology groups of P and $\bar{P}$, let K and $\bar{K}$ be finite simplicial complexes which are subdivisions of P and $\bar{P}$ respectively, such that each oriented simplex of K covers one oriented simplex of $\bar{K}$. Then G acts as an *automorphism group*[1] on K; the automorphisms $x \in G$ are permutations of the oriented simplices σ_n of K in each dimension n and preserve all incidence relations. The set of all simplices σ_n covering a simplex $\bar{\sigma}_n$ of $\bar{K}$ is a transitivity domain of G; that is, it contains with any simplex σ_n' all $x\sigma_n'$, $x \in G$, and only those. Furthermore, since in an automorphism $x \neq e$ no simplex is fixed,

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[1] For details see for example [1, §6]. Numbers in brackets refer to the references at the end of the paper.

$x\sigma_n' = x_1\sigma_n'$ implies $x = x_1$. The simplicial map which projects each σ_n of K onto the $\bar{\sigma}_n$ it covers is denoted by U; obviously, $Ux\sigma_n = U\sigma_n$ for all $x \in G$.

In the simplicial complexes K and $\bar{K}$ we shall only use chains with *real coefficients*, and no distinction will be made between chains and cochains. An n-chain a_n in K is a linear form $a_n = \sum_i \alpha_i \sigma_{ni}$ with real coefficients, α_i, where the n-simplices σ_{ni} of K play the role of indeterminates. All n-chains a_n in K form a vector space C_n (a real linear space) of finite rank, in which the scalar product of $a_n = \sum_i \alpha_i \sigma_{ni}$ and $b_n = \sum_i \beta_i \sigma_{ni}$ is defined by $a_n \cdot b_n = \sum_i \alpha_i \beta_i$. The boundary operator ∂ is a linear mapping of C_n into C_{n-1}, $n = 0, 1, \cdots$ $(C_{-1} = 0)$; the coboundary operator δ is the linear mapping of C_{n-1} into C_n which is *dual* to ∂, that is, δ is defined by $\delta a_{n-1} \cdot b_n = a_{n-1} \cdot \partial b_n$ for all $(n-1)$ chains a_{n-1} and n-chains b_n. If $\partial a_n = \delta a_n = 0$, a_n is called a *harmonic chain*; as in each homology or cohomology class there is exactly one harmonic chain,[3] the vector group H_n of all harmonic n-chains (a linear subgroup of C_n) is isomorphic both to the nth homology and the nth cohomology group of K (the isomorphism is given by representing a homology or cohomology class by the harmonic chain it contains). The rank of H_n is the nth Betti number p_n of K. All chains, groups, and so on, of $\bar{K}$ will be denoted by the same symbols as in K, with a bar (for example, $\bar{a}_n$, $\bar{C}_n$, $\bar{H}_n$), the boundary and coboundary operator as in K by ∂ and δ.

The linear mappings of the chain groups and the homology groups induced by the simplicial maps U and $x \in G$ will also be denoted by U and x respectively. x is an isomorphism of C_n onto itself, U a homomorphism of C_n onto $\bar{C}_n$, for all n.

The simplicial map U of K onto $\bar{K}$ is locally one-one; that is, if the simplices σ_n and σ_n' of K are both incident with a simplex σ_k, then $U\sigma_n = U\sigma_n'$ implies $\sigma_n = \sigma_n'$. For each simplex σ_n of K one has therefore not only $U\partial\sigma_n = \partial U\sigma_n$, but also

$$U\delta\sigma_n = \delta U\sigma_n; \tag{1}$$

in other words, the linear mapping U of C_n onto $\bar{C}_n$ commutes with ∂ and δ. Let U^* be the dual mapping of $\bar{C}_n$ into C_n, defined by $U^*\bar{a}_n \cdot b_n = \bar{a}_n \cdot Ub_n$ for all n-chains $\bar{a}_n \in \bar{C}_n$ and $b_n \in C_n$; it also commutes with ∂ and δ. Hence

(2) *U and U^* both map harmonic chains onto harmonic chains.*

Since U is a mapping of C_n onto $\bar{C}_n$, the dual mapping U^* is an isomorphism of $\bar{C}_n$ into C_n. For, $U^*\bar{a}_n = 0$ implies $U^*\bar{a}_n \cdot b_n = \bar{a}_n \cdot Ub_n = 0$

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[3] See for example [2, pp. 245–246]. Other references are given in [2].

for all $b_n \in C_n$, hence $\bar{a}_n \cdot \bar{b}_n = 0$ for all $\bar{b}_n \in \bar{C}_n$, hence $\bar{a}_n = 0$. This together with (2) yields

THEOREM 2. *U^* induces an isomorphism of $\bar{H}_n$ into H_n, $n = 0, 1, \cdots$.*

COROLLARY 1. *The nth Betti number $\bar{p}_n$ of $\bar{K}$ is not greater than the nth Betti number p_n of K.*

We remark further that U maps $U^*\bar{C}_n \subset C_n$ isomorphically onto $\bar{C}_n$. For, $UU^*\bar{a}_n = 0$ implies $UU^*\bar{a}_n \cdot \bar{a}_n = U^*\bar{a}_n \cdot U^*\bar{a}_n = 0$, hence $U^*\bar{a}_n = 0$. In particular U maps $U^*\bar{H}_n \subset H_n$ isomorphically into $\bar{H}_n$, and since $U^*\bar{H}_n$ is isomorphic to $\bar{H}_n$, this must be a map onto $\bar{H}_n$. Hence

(3) *U maps H_n onto $\bar{H}_n$.*

REMARK. UU^* is not the identity mapping of $\bar{C}_n$ or $\bar{H}_n$ but, as is easily seen, multiplies each chain $\bar{a}_n$ by the order g of G.

3. **Invariant chains.** In the linear mapping of C_n onto itself induced by an automorphism $x \in G$ the scalar product of two n-chains a_n, b_n remains unchanged:

$$xa_n \cdot xb_n = a_n \cdot b_n. \tag{4}$$

This follows simply from the fact that x is a permutation of the oriented simplices (that is, of the basis vectors of C_n). In other words, x is an orthogonal transformation of C_n.

A chain $a_n \in C_n$ will be called *invariant* if $xa_n = a_n$ for all $x \in G$. This is the case if and only if

$$a_n \cdot b_n = a_n \cdot xb_n \tag{5}$$

for all $x \in G$ and $b_n \in C_n$. For, if a_n is invariant, $a_n \cdot b_n = xa_n \cdot xb_n = a_n \cdot xb_n$; and conversely, if (5) holds for all $x \in G$, then $a_n \cdot b_n = a_n \cdot x^{-1}b_n = xa_n \cdot b_n$ for all $b_n \in C_n$, hence $a_n = xa_n$. The invariant chains constitute a linear subgroup C_n^i of C_n.

(6) *The isomorphism U^* maps $\bar{C}_n$ onto C_n^i.*

PROOF. For any $\bar{a}_n \in \bar{C}_n$, $U^*\bar{a}_n$ is invariant; for

$$U^*\bar{a}_n \cdot xb_n = \bar{a}_n \cdot Uxb_n = \bar{a}_n \cdot Ub_n = U^*\bar{a}_n \cdot b_n$$

for all $b_n \in C_n$. Conversely, if a_n is invariant, the relation

$$\bar{a}_n \cdot U\sigma_n = a_n \cdot \sigma_n \qquad \text{for all } \sigma_n \text{ of } K$$

defines an n-chain $\bar{a}_n$ of $\bar{K}$ without ambiguity (for replacing σ_n by $\sigma_n' = x\sigma_n$ does not change either side of this equation). Then $U^*\bar{a}_n \cdot \sigma_n = \bar{a}_n \cdot U\sigma_n = a_n \cdot \sigma_n$ for all σ_n, hence $U^*\bar{a}_n = a_n$.

The invariant harmonic n-chains of K form a linear subgroup H_n^i of H_n. For any $a_n \in H_n^i$ there is, by (6), an $\bar{a}_n$ such that $a_n = U^*\bar{a}_n$;

since U^* commutes with ∂ and δ and is an isomorphism, it follows that $\bar{a}_n$ is harmonic. Combining this with Theorem 2 and (6) we obtain the following result.

THEOREM 3. *U^* induces an isomorphism of $\bar{H}_n$ onto H_n^I.*

4. **Proof of Theorem 1.** Since the automorphisms $x\in G$ of K permute the n-simplices, for each n, and preserve the incidence relations, the linear mappings x of C_n onto itself, $n=0, 1, \cdots$, commute with both ∂ and δ. Hence they induce linear mappings of H_n onto itself; these mappings describe *the operation of G on the n-dimensional homology and cohomology groups of K.* By (4) these linear mappings of H_n onto itself are orthogonal. Hence for each n one has an orthogonal representation $\mathfrak{R}_n$ of the group G in the vector group H_n. The subspace H_n^I of H_n consists of those elements of H_n which are fixed under this representation; by Theorem 3 it may be considered as the homology or cohomology group $\bar{H}_n$ of $\bar{K}$. This proves that the latter *is determined by H_n and by the operation of G in H_n* (cf. §1).

The rank of H_n^I, which is the nth Betti number $\bar{p}_n$ of $\bar{K}$, is equal to the number of trivial irreducible representations of G contained in the representation $\mathfrak{R}_n$ of G in H_n, hence by a well known character relation[3] is equal to the average over the group G of the character $s_n(x)$ of the representation $\mathfrak{R}_n$. This proves Theorem 1. Replacing $s_n(e)$ by the degree p_n of the representation, where e is the unit element of G, we may write the formula in the form

$$g\cdot\bar{p}_n = p_n + \sum_{x\in G,\, x\neq e} s_n(x). \tag{7}$$

REMARK. $p_n=\bar{p}_n$ holds if and only if $H_n=H_n^I$, hence if all harmonic n-chains of K are invariant under the operations of G; in other words, if each n-dimensional homology class is mapped into itself by all $x\in G$. This result may be stated in the following way:

THEOREM 4. *The nth Betti numbers p_n and $\bar{p}_n$ of K and $\bar{K}$ are equal if and only if in the dimension n all automorphisms $x\in G$ are of the homology type of the identity (for real coefficients).*[4]

5. **Products,[5] manifolds.** The Alexander $\cup$-product associates with two cohomology classes of K of dimensions n and k a third one of di-

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[3] Cf. [3, p. 201, formula (16)].

[4] The sufficiency of the condition has previously been proved by G. Hirsch, with an application to closed Lie groups; see [4, p. 226].

[5] For definitions and properties see [5]; for products in a covering complex the remarks of [1, §6] have to be used.

mension $n+k$, hence with two harmonic chains a_n and b_k a unique harmonic $(n+k)$-chain $a_n \cup b_k$. The Čech-Whitney $\cap$-product associates with a cohomology class of dimension k and a homology class of dimension $k+p$ $(p \geqq 0)$ a homology class of dimension p, hence with two harmonic chains b_k and c_{k+p} a unique harmonic p-chain $b_k \cap c_{k+p}$. From the fact that all incidence relations are preserved by the automorphisms $x \in G$ it may easily be deduced that

$$xa_n \cup xb_k = x(a_n \cup b_k), \tag{8}$$

$$xb_k \cap xc_{k+p} = x(b_k \cap c_{k+p}) \tag{9}$$

for all harmonic chains a_n, b_k, c_{k+p} and all $x \in G$.

It follows that the $\cup$- or $\cap$-product of invariant harmonic chains is again an invariant harmonic chain. The direct sum of all H_n^i, $n=0, 1, \cdots$, together with the $\cup$-product, constitutes a subring R^i of the Alexander ring (the cohomology ring) R of K. Since U^* preserves the $\cup$-product, it follows from Theorem 3:

(10) *U^* induces a ring-isomorphism of the cohomology ring $\bar{R}$ of $\bar{K}$ onto R^i.*

We now assume K to be a (closed) orientable N-dimensional manifold. Let m_N be the fundamental harmonic N-chain corresponding to an orientation of K; that is, m_N is a chain with coefficients ± 1 generating H_N. The *duality operator* D in K may be defined by

$$Da_n = a_n \cap m_N \tag{11}$$

for harmonic chains a_n; if a_n is considered as a representative of an n-dimensional cohomology class, Da_n is the harmonic representative of the dual $(N-n)$-dimensional homology class. As is well known, D is an isomorphism of H_n onto H_{N-n}.

(12) *For any $x \in G$ one has $xDa_n = \gamma \cdot Dxa_n$, where $\gamma = \pm 1$ is the (topological) degree of x.*

Proof. $xDa_n = x(a_n \cap m_N) = xa_n \cap xm_N = xa_n \cap \gamma m_N = \gamma \cdot (xa_n \cap m_N)$ $= \gamma \cdot Dxa_n$.

As a consequence we have:

Theorem 5. *If K is an orientable N-dimensional manifold, then the characters $s_n(x)$ and $s_{N-n}(x)$ of the representations of G in H_n and H_{N-n} are related by*

$$s_n(x) = \gamma \cdot s_{N-n}(x),$$

where γ is the degree of $x \in G$.

6. **An application.** Let $\bar{K}$ be a non-orientable N-dimensional manifold, and K an orientable two-sheeted covering of K. The cover-

ing transformation group consists of the unit element e and an element y of degree -1. Various relations between the Betti numbers of $\bar{K}$ and K may be deduced from (7) and (12). (7) becomes in this case (we write s_n for $s_n(y)$)

$$2\bar{p}_n = p_n + s_n, \qquad n = 0, 1, \cdots, N. \tag{13}$$

Since y has degree -1, one has, by (12), $s_n = -s_{N-n}$, hence $2\bar{p}_{N-n} = p_{N-n} + s_{N-n} = p_n - s_n = 2\bar{p}_n - 2s_n$, hence

$$\bar{p}_n - \bar{p}_{N-n} = s_n, \tag{14}$$

$$\bar{p}_n + \bar{p}_{N-n} = p_n. \tag{15}$$

This is a duality theorem for non-orientable N-dimensional manifolds. We shall use (13)–(15) to prove the following theorem.

THEOREM 6. *The following relations hold between the Betti numbers* $\bar{p}_n$ *of a closed non-orientable N-dimensional manifold* $\bar{K}$ *and the Betti numbers* p_n *of a two-sheeted orientable covering* K *of* $\bar{K}$:

(16) *for* $N = 3$, $\quad p_1 = 2\bar{p}_1 - 1$;[6]

(17) *for* $N = 2L$, $\quad p_L = 2\bar{p}_L$;

(18) *for* $N = 2L + 1$, $\quad \sum_{n=0}^{L} (-1)^n p_n = 2 \sum_{n=0}^{L} (-1)^n \bar{p}_n$ *and*

$$\sum_{n \text{ even}} p_n = 2 \sum_{n \text{ even}} \bar{p}_n.$$

PROOF. (a) If $N=3$, the Euler characteristic $\bar{p}_0 - \bar{p}_1 + \bar{p}_2 - \bar{p}_3 = 0$, where $\bar{p}_0 = 1$, $\bar{p}_3 = 0$, hence $\bar{p}_2 = \bar{p}_1 - 1$; by (15), $\bar{p}_1 + \bar{p}_2 = 2\bar{p}_1 - 1 = p_1$. (b) (17) follows immediately from (15). (c) If $N = 2L+1$, the characteristics of K and $\bar{K}$ are 0 and we deduce from (13)

$$\sum_{n=0}^{N} (-1)^n s_n = 0, \tag{19}$$

hence $\sum_{n=0}^{L}(-1)^n s_n = -\sum_{n=L+1}^{N} (-1)^n s_n$; since $s_n = -s_{N-n}$, this is equal to $-\sum_{n=L+1}^{N}(-1)^{N-n} s_{N-n} = -\sum_{n=0}^{L}(-1)^n s_n$, hence $\sum_{n=0}^{L} (-1)^n s_n = 0$. Using $s_n = -s_{N-n}$ again, we may also write this as $\sum_{n \text{ even}} s_n = 0$. These two results together with (13) yield (18). We remark that since the characteristics of K and $\bar{K}$ are 0 ($N = 2L+1$) the first of the formulas (18) is equivalent to $\sum_{n=L+1}^{N}(-1)^n p_n = 2\sum_{n=L+1}^{N}(-1)^n \bar{p}_n$, the second one to $\sum_{n \text{ odd}} p_n = 2\sum_{n \text{ odd}} \bar{p}_n$.

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[6] For 3-dimensional manifolds $\bar{K}$ satisfying certain restrictive conditions (16) is proved in a note by T. H. Kiang [6].

REMARK. The formula (19), which here was proved directly, may also be deduced from a well known result concerning the number of fixed simplices in a simplicial self-mapping.[7] In the general case of an arbitrary automorphism group G and arbitrary finite complexes K and $\bar{K}$ it follows from this result that the "Lefschetz number" $\lambda(x) = \sum_{n=0}^{N} (-1)^n s_n(x)$ is 0 for any automorphism $x \neq e$.

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[7] Cf. [7, p. 530].

THE MAYER HOMOLOGY THEORY

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1. Introduction. In 1942 W. Mayer [4][2] defined new homology groups based on a boundary operator whose pth power (p a prime) is zero, instead of the usual one whose square is zero. As a coefficient group an abelian group G with only elements of order p is used. The Mayer homology groups $H_{n,q}$ depend on two integers: $n \geqq 0$ and $0 < q < p$. Mayer has established the topological invariance of these groups but left unsettled the question of their relation with the classical homology groups. This question is settled in this paper. The answer is embodied in the following theorem which is the main result of the paper.

THEOREM 1.1. *Let p be a prime and G an abelian group with all elements of order p. The Mayer homology groups $H_{n,q}$ (over G) are then related to the classical homology groups H_r (over G) as follows:*

(1) *If $n \equiv q-1 \pmod{p}$, then $H_{n,q} \approx H_r$ for $r = 2(n-q+1)/p$.*

(2) *If $n \equiv -1 \pmod{p}$, then $H_{n,q} \approx H_r$ for $r = 2(n+1)/p - 1$.*

(3) *In all other cases, $H_{n,q} = 0$.*

Conversely, the classical groups H_r can be expressed in terms of the Mayer groups as follows:

(4) *If r is even, then $H_r \approx H_{n,q}$ provided $n - q = pr/2 - 1$.*

(5) *If r is odd, then $H_r \approx H_{n,q}$ provided $n = p(r+1)/2 - 1$.*

The theorem implies that the Mayer groups do not yield new topological invariants but lead instead to rather interesting alternative definitions of the classical homology groups.

The theorem is valid for the absolute and relative homology groups in simplicial complexes. It also remains valid for arbitrary spaces provided the Čech limiting process is used to define both $H_{n,q}$ and H_r.

The proof of the theorem is *not* obtained by a direct construction of the requisite isomorphisms but is an application of the axiomatic characterization of homology theory of Eilenberg and Steenrod (sketched in [1] and fully developed in a forthcoming book). Roughly speaking, the procedure is the following. It is shown that certain collections of the Mayer groups, suitably relabeled and together

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[2] Numbers in brackets refer to bibliography at the end of the paper.

with suitably defined homomorphisms, satisfy the axioms for homology. The uniqueness theorem in the axiomatic theory then implies that these groups are isomorphic with the classical homology groups based on the same coefficient groups. The theorem follows from a discussion of the coefficient groups and the relabeling process.

The main interest of this paper lies in the fact that it offers an application of the axiomatic approach to a problem which did not seem easy to handle by direct methods. Of course, an analysis of the proof of the uniqueness theorem will yield a direct, though most likely complicated, definition of the isomorphisms.

The next section summarizes all the needed definitions and results from the book being prepared by Eilenberg and Steenrod, to whom the author is indebted for allowing him access to the manuscript.

2. **The axioms.** Elienberg and Steenrod [1] have characterized homology theory on a certain class of spaces by means of a set of axioms which are topological in nature. In addition, another set (to appear in their forthcoming book) is considered which is combinatorial in nature and characterizes homology theory for simplicial complexes. This latter set is the one we employ in the following.

DEFINITION 2.1. Two simplicial maps f and g of a simplicial pair (K, L) (consisting of a simplicial complex K and a closed subcomplex L) into a simplicial pair (K', L') are said to be *contiguous* if for every simplex s of K (or L) the simplexes $f(s)$ and $g(s)$ are contained in a simplex of K' (or L').

DEFINITION 2.2. A sequence of groups and homomorphisms

$$G_0 \xleftarrow{g_0} \cdots \longleftarrow G_{n-1} \xleftarrow{g_{n-1}} G_n \xleftarrow{g_n} \cdots$$

is said to be *exact* if for $n \geqq 1$ the kernel of g_{n-1} is the image of g_n, and g_0 is a homomorphism onto.

DEFINITION 2.3. Consider a system $H = \{H_r(K, L), \partial, f_*\}$ consisting of the following concepts:

(a) For every simplicial pair (K, L) and integer $r \geqq 0$, $H_r(K, L)$ is an abelian group called the nth homology group of K mod L.

(b) For $r \geqq 1$, $\partial : H_r(K, L) \rightarrow H_{r-1}(L)$ is a homomorphism called the boundary operator.

(c) For every simplicial map $f : (K, L) \rightarrow (K', L')$ there is a homomorphism $f_* : H_r(K, L) \rightarrow H_r(K', L')$ called the homomorphism induced by f.

The system H is called a *simplicial homology theory* if it satisfies the following axioms:

Axiom 1. *If f is the identity simplicial map of (K, L) onto itself, then f_* is the identity isomorphism of $H_r(K, L)$ onto itself.*

Axiom 2. *If $f:(K, L)\rightarrow(K', L')$ and $g:(K', L')\rightarrow(K'', L'')$ are simplicial, then $(gf)_*=g_*f_*$.*

Axiom 3. *If $f:(K, L)\rightarrow(K', L')$ is simplicial and $n\geqq 1$, then commutativity holds in the diagram*

$$\begin{tikzcd}
H_r(K, L) \arrow[r, "f_*"] \arrow[d, "\partial"'] & H_r(K', L') \arrow[d, "\partial"] \\
H_{r-1}(L) \arrow[r, "(f|L)_*"] & H_{r-1}(L').
\end{tikzcd}$$

Axiom 4 (*Exactness axiom*). *If $i:L\rightarrow K$ and $j:K\rightarrow(K, L)$ denote identity simplicial maps, then the following sequence is exact*

$$H_0(K, L) \xleftarrow{j_*} \cdots \xleftarrow{j_*} H_r(K) \xleftarrow{i_*} H_r(L) \xleftarrow{\partial} H_{r+1}(K, L) \xleftarrow{j_*} \cdots .$$

Axiom 5 (*Contiguity axiom*). *If $f, g:(K, L)\rightarrow(K', L')$ are contiguous simplicial maps, then $f_*=g_*$.*

Axiom 6 (*Excision axiom*). *If K_1, K_2 are closed subcomplexes of a complex K, the homomorphisms*

$$i_*:\ H_r(K_1, K_1\cap K_2)\rightarrow H_r(K_1\cup K_2, K_2)$$

induced by the identity map $i:(K_1, K_1\cap K_2)\rightarrow(K_1\cup K_2, K_2)$ are isomorphisms onto.

Axiom 7 (*Dimension axiom*). *If P is a complex consisting of a single vertex, then $H_r(P)=0$ for $r>0$.*

If a simplicial homology theory is given, the group $H_0(P)$ is called the coefficient group of the theory. If G is a fixed abelian group, the groups $H_r(K, L)=H_r(K, L; G)$, defined in terms of the cellular structure of K as in [3, chap. 3] together with suitable definitions of ∂ and f_* form a simplicial homology theory with G as coefficient group. This homology theory will be referred to as the *classical homology theory*.

The above shows that simplicial homology theories with arbitrary coefficient group exist. The following uniqueness theorem, which will appear in the book on the axiomatic theory mentioned earlier, shows that any simplicial homology theory is isomorphic to the classical homology theory with a suitable coefficient group.

Theorem 2.4. *Given two simplicial homology theories H and $\bar{H}$ and given an isomorphism*

$$h_0\colon\ G \approx \bar{G}$$

of their coefficient groups, there exists an isomorphism

$$h\colon\ H \approx \bar{H}$$

which is an extension of h_0.

By an isomorphism $h: H \approx \bar{H}$ between two simplicial homology theories is meant an isomorphism between corresponding homology groups $H_r(K, L)$ and $\bar{H}_r(K, L)$ which commutes with the boundary operator and induced homomorphism.

3. **Definition of the Mayer groups.** Let p be a fixed prime and let G be a fixed abelian group all of whose elements are of order p.

DEFINITION 3.1. Let K be a finite simplicial complex with vertices $\{v_i\}$. An *n-cell* σ^n of K is defined to be an $(n+1)$-tuple $v_0, \cdots, v_n$ of vertices v_i (not necessarily distinct) such that the vertices $v_0, \cdots, v_n$ all belong to some simplex of K. Two such $(n+1)$-tuples differing only in the arrangement of the v's are to be considered the same n-cell. There are no cells of dimension k for $k<0$.

DEFINITION 3.2. For $n \geqq 0$ the *group* $C_n(K)$ *of n-chains of* K with coefficients in G is the group of linear forms $\sum_i g_i\sigma_i^n$ in the n-cells σ_i^n of K with coefficients $g_i \in G$. An *elementary n-chain* is a chain of the form $g\sigma_i^n$. Clearly the elementary n-chains span $C_n(K)$.

For $n \geqq 1$, the *boundary homomorphism*

$$F\colon\ C_n(K) \to C_{n-1}(K)$$

is defined as follows. It suffices to define F for elementary n-chains and extend to $C_n(K)$ by linearity. Define $F(g(v_0 \cdots v_n))$ by

$$F(g(v_0 \cdots v_n)) = \sum_i g(v_0 \cdots \hat{v}_i \cdots v_n)$$

where $(v_0 \cdots \hat{v}_i \cdots v_n)$ denotes the $(n-1)$-cell obtained from the cell $v_0 \cdots v_n$ by deleting v_i. The fact that every element of G has order p implies, as shown by Mayer [4, p. 371], that for $n \geqq p$, F^p maps $C_n(K)$ into the zero of $C_{n-p}(K)$.

DEFINITION 3.3. Let q be an integer such that $0<q<p$. If $n \geqq q$, consider the diagram

$$C_{n+p-q}(K) \xrightarrow{F^{p-q}} C_n(K) \xrightarrow{F^q} C_{n-q}(K).$$

The *group* $Z_{n,q}(K)$ of qth *n-dimensional cycles of* K is defined to be the kernel of F^q, and the group $B_{n,q}(K)$ *of* qth *n-dimensional boundaries of* K is defined to be the image of F^{p-q}. If $0 \leqq n < q$, we define

$Z_{n,q}(K)$, to be $C_n(K)$ and $B_{n,q}(K)$ to be the image of $C_{n+p-q}(K)$ under F^{p-q}. Since $F^p=0$, it follows that $B_{n,q}(K)\subset Z_{n,q}(K)$. The factor group

$$H_{n,q}(K) = Z_{n,q}(K)/B_{n,q}(K)$$

is defined to be the qth *n-dimensional homology group of K*. (This notation differs slightly from that used by Mayer in that he uses H_q^n to denote the group $H_{n,q}$.)

THEOREM 3.4. *If P is a complex consisting of a single vertex v, then*

$$H_{q-1,q}(P) \approx G, \text{ and } H_{n,q}(P) = 0 \text{ for } n \neq q-1.$$

PROOF. If $n\geqq 0$, there is only one n-cell σ^n of P. Hence, $C_n(P)$ consists solely of the elementary chains $g\sigma^n$ and is isomorphic to G under the mapping $g\sigma^n\rightarrow g$. If $n>0$,

$$F(g\sigma^n) = ((n+1)g)\sigma^{n-1}.$$

Since p is a prime, elements of G can be divided by integers m if $m\not\equiv 0 \pmod{p}$. It follows that if $n+1\not\equiv 0 \pmod{p}$, $F((1/(n+1))g\sigma^n) = g\sigma^{n-1}$. Therefore, for $n>0$,

$$\begin{aligned} F(C_n(P)) &= 0 && \text{if } n \equiv -1 \pmod{p},\\ F(C_n(P)) &= C_{n-1}(P) && \text{if } n \not\equiv -1 \pmod{p}. \end{aligned}$$

It follows that for $n\geqq q$

$$\begin{aligned} F^q(C_n(P)) &= 0 \quad \text{if } n \equiv k \text{ with } -1 \leqq k < q-1,\\ F^q(C_n(P)) &= C_{n-q}(P) \quad \text{otherwise.} \end{aligned}$$

Therefore,

$$Z_{n,q}(P) = \begin{cases} C_n(P) & \text{if } n = q-1 \text{ or } n \equiv k \text{ with } -1 \leqq k < q-1,\\ 0 & \text{otherwise,} \end{cases}$$

$$B_{n,q}(P) = \begin{cases} 0 & \text{if } n+p-q \equiv m \text{ with } -1 \leqq m < p-q-1,\\ C_n(P) & \text{if } n+p-q \equiv m \text{ with } p-q-1 \leqq m < p-1. \end{cases}$$

Then $H_{n,q}(P)=0$ unless $n=q-1$ or $n\equiv k$ with $-1\leqq k<q-1$. If $n\equiv k$ with $-1\leqq k<q-1$, then $n+p-q\equiv k+p-q$ and $p-q-1\leqq k+p-q<p-1$, so $B_{n,q}(P)=C_n(P)$, and $H_{n,q}(P)=0$. Therefore, $H_{n,q}(P)=0$ if $n\neq q-1$. If $n=q-1$, then $n+p-q=p-1\equiv -1$, so $B_{q-1,q}(P)=0$. Since $Z_{q-1,q}(P)=C_{q-1}(P)$, it follows that $H_{q-1,q}(P) = C_{q-1}(P)\approx G$.

4. **Relative theory.** The axioms stated in §2 require that a group $H_n(K, L)$ be defined for every simplicial pair (K, L). In this section

we extend the Mayer definition to the relative case so that later we will have the groups needed in verifying the axioms.

Let L be a (closed) subcomplex of K. Then any cell of L is also a cell of K so that $C_n(L) \subset C_n(K)$. It is clear that F defined for chains of L is the same homomorphism as F defined for chains of K when restricted to $C_n(L)$.

DEFINITION 4.1. For $n \geqq q$, the *group* $Z_{n,q}(K, L)$ *of qth n-dimensional cycles of* K mod L is defined to be the subgroup of $C_n(K)$ which is the inverse image of $C_{n-q}(L)$ under F^q, while for $n<q$, $Z_{n,q}(K, L)$ is defined to be $C_n(K)$. The *group* $B_{n,q}(K, L)$ of qth *n-dimensional boundaries of* K mod L is defined to be the subgroup of $C_n(K)$ spanned by the two subgroups $C_n(L)$ and $F^{p-q}(C_{n+p-q}(K))$. Then $B_{n,q}(K, L) \subset Z_{n,q}(K, L)$ and the factor group

$$H_{n,q}(K, L) = Z_{n,q}(K, L)/B_{n,q}(K, L)$$

is defined to be the qth *n-dimensional homology group of* K mod L.

Note. If $L=0$ is the null set, then $H_{n,q}(K, 0)=H_{n,q}(K)$.

In the following we use $\phi: Z_{n,q}(K, L) \to H_{n,q}(K, L)$ to denote the natural homomorphism of $Z_{n,q}(K, L)$ onto its factor group $H_{n,q}(K, L)$.

DEFINITION 4.2. For $n \geqq q$, define a homomorphism

$$F^q \colon H_{n,q}(K, L) \to H_{n-q,p-q}(L)$$

as follows. Let $z \in Z_{n,q}(K, L)$. Then from the definition of $Z_{n,q}(K, L)$ it follows that $F^q z \in C_{n-q}(L)$. If $n-q \geqq p-q$, then $F^{p-q}(F^q z) = F^p z = 0$. If $n-q<p-q$, $C_{n-q}(L) = Z_{n-q,p-q}(L)$. In any event we see that $F^q z \in Z_{n-q,p-q}(L)$ so that F^q is a homomorphism

$$F^q \colon Z_{n,q}(K, L) \to Z_{n-q,p-q}(L).$$

Let $b \in B_{n,q}(K, L)$. Then there is $u \in C_{n+p-q}(K)$ and $v \in C_n(L)$ such that

$$F^{p-q}u + v = b.$$

Since $F^q b = F^p u + F^q v = F^q v \in B_{n-q,p-q}(L)$, it follows that F^q maps $B_{n,q}(K, L)$ into $B_{n-q,p-q}(L)$ and so induces a homomorphism

$$F^q \colon H_{n,q}(K, L) \to H_{n-q,p-q}(L)$$

such that $F^q \phi z = \phi F^q z$ for $z \in Z_{n,q}(K, L)$.

DEFINITION 4.3. Let $f\colon (K, L) \to (K', L')$ be a simplicial map. If $v_0, \cdots, v_n$ is an n-cell of K (or L), $f(v_0, \cdots, v_n) = f(v_0), \cdots, f(v_n)$ is clearly an n-cell of K' (or L'). Therefore, f induces a homomorphism $f_n\colon C_n(K) \to C_n(K')$ such that $f_n(\sum_i g_i \sigma_i^n) = \sum_i g_i f(\sigma_i^n)$.

LEMMA 4.4. *For $n \geqq 1$, commutativity holds in the diagram*

$$\begin{tikzcd}
C_n(K) \arrow[r, "F"] \arrow[d, "f_n"'] & C_{n-1}(K) \arrow[d, "f_{n-1}"] \\
C_n(K') \arrow[r, "F"] & C_{n-1}(K').
\end{tikzcd}$$

PROOF. It suffices to prove the commutativity for elementary n-chains. Let $v_0 \cdots v_n$ be an n-cell of K. Then

$$\begin{aligned}
Ff_n(gv_0 \cdots v_n) &= Fgf(v_0) \cdots f(v_n) \\
&= \sum_i gf(v_0) \cdots \widehat{f}(v_i) \cdots f(v_n)
\end{aligned}$$

and

$$\begin{aligned}
f_{n-1}F(gv_0 \cdots v_n) &= f_{n-1}\left(\sum_i gv_0 \cdots \hat{v}_i \cdots v_n \right) \\
&= \sum_i gf(v_0) \cdots \widehat{f}(v_i) \cdots f(v_n).
\end{aligned}$$

It follows from Lemma 4.4 that f_n induces a homomorphism

$$f_*: H_{n,q}(K, L) \to H_{n,q}(K', L')$$

such that $f_*\phi z = \phi f_n z$ for $z \in Z_{n,q}(K, L)$.

THEOREM 4.5. *If $f:(K, L) \to (K, L)$ is the identity simplicial map, then f_* is the identity isomorphism.*

PROOF. The theorem follows immediately from the fact that f_n is the identity isomorphism of $C_n(K)$ onto itself.

THEOREM 4.6. *If $f:(K, L) \to (K', L')$ and $g:(K', L') \to (K'', L'')$ are simplicial, then $(gf)_* = g_*f_*$.*

PROOF. gf is simplicial because both f and g are, and $(gf)_n = g_nf_n$ which implies the theorem.

THEOREM 4.7. *If $f:(K, L) \to (K', L')$ is simplicial and if $n \geqq q$, commutativity holds in the diagram*

$$\begin{tikzcd}
H_{n,q}(K, L) \arrow[r, "f_*"] \arrow[d, "F^q"'] & H_{n,q}(K', L') \arrow[d, "F^q"] \\
H_{n-q,p-q}(L) \arrow[r, "(f|L)_*"] & H_{n-q,p-q}(L').
\end{tikzcd}$$

PROOF. It follows from Lemma 4.4 that

$$F^q f_n = (f \mid L)_{n-q} F^q$$

and this, together with the commutativity of F^q and ϕ, implies the theorem.

THEOREM 4.8. *If f, g: $(K, L) \to (K', L')$ are contiguous, then $f_* = g_*$.*

PROOF. This is proved for the absolute case by Mayer [4, p. 379]. The same proof carries over to the relative case without any essential change and so is omitted here.

THEOREM 4.9. *If K_1, K_2 are closed subcomplexes of K, and if i: $(K_1, K_1 \cap K_2) \to (K_1 \cup K_2, K_2)$ is the identity simplicial map, then*

$$i_*: H_{n,q}(K_1, K_1 \cap K_2) \approx H_{n,q}(K_1 \cup K_2, K_2).$$

PROOF. Since $C_n(K_1 \cap K_2) = C_n(K_1) \cap C_n(K_2)$ and $C_n(K_1 \cup K_2) = C_n(K_1) + C_n(K_2) =$ the subgroup of $C_n(K)$ spanned by $C_n(K_1)$ and $C_n(K_2)$, it follows from the Noëther isomorphism theorem that

$$C_n(K_1)/C_n(K_1 \cap K_2) \approx C_n(K_1 \cup K_2)/C_n(K_2).$$

More precisely, if τ: $C_n(K_1 \cup K_2) \to C_n(K_1 \cup K_2)/C_n(K_2)$ is the natural homomorphism, then τi_n maps $C_n(K_1)$ onto $C_n(K_1 \cup K_2)/C_n(K_2)$ and has kernel $C_n(K_1 \cap K_2)$.

If $n \geqq q$ and $z \in Z_{n,q}(K_1 \cup K_2, K_2)$, then $z \in C_n(K_1 \cup K_2)$ such that $F^q z \in C_{n-q}(K_2)$. Choose $z' \in C_n(K_1)$ such that $\tau i_n z' = \tau z$. Then $i_n z' = z + c$ for some $c \in C_n(K_2)$ and so

$$\tau F^q i_n z' = \tau F^q(z + c) = \tau(F^q z + F^q c) = 0$$

because $F^q z$ and $F^q c$ both belong to $C_{n-q}(K_2)$. Therefore, $\tau i_{n-q} F^q z' = \tau F^q i_n z' = 0$, so $F^q z'$ is an $(n-q)$-chain of $K_1 \cap K_2$ from which it follows that z' is a qth n-cycle of K_1 mod $K_1 \cap K_2$. Then we have $\phi z' \in H_{n,q}(K_1, K_1 \cap K_2)$ and

$$i_* \phi z' = \phi i_n z' = \phi(z + c) = \phi z,$$

the last equality because $c \in B_{n,q}(K_1 \cup K_2, K_2)$. Therefore, i_* maps $H_{n,q}(K_1, K_1 \cap K_2)$ onto $H_{n,q}(K_1 \cup K_2, K_2)$ if $n \geqq q$.

If $n < q$, the same proof as above can be used, the only difference being that it is no longer necessary to show that z' is a qth n-cycle of K_1 mod $K_1 \cap K_2$ because any n-chain of K_1 is such a cycle by definition. Hence, i_* is onto in all cases.

To prove the kernel of i_* is zero let $i_* \phi z = 0$. Then $i_n z = F^{p-q} c + d$ where $c \in C_{n+p-q}(K_1 \cup K_2)$ and $d \in C_n(K_2)$. Choose $c' \in C_{n+p-q}(K_1)$ such that $\tau i_{n+p-q} c' = \tau c$. Then $c - i_{n+p-q} c' \in C_{n+p-q}(K_2)$. Now $z - F^{p-q} c' \in Z_{n,q}(K_1, K_1 \cap K_2)$ and

$$\tau i_n(z - F^{p-q}c') = \tau F^{p-q}c + \tau d - \tau F^{p-q} i_{n+p-q}c'$$

which equals zero because $F^{p-q}(c - i_{n+p-q}c')$ and d both belong to $C_n(K_2)$. Since the kernel of τi_n is $C_n(K_1 \cap K_2)$, we see that $z - F^{p-q}c' \in C_n(K_1 \cap K_2)$. Hence $z = F^{p-q}c' + d'$ for some $d' \in C_n(K_1 \cap K_2)$. It follows that $\phi z = 0$, so i_* is an isomorphism.

5. **Verification of the axioms.** All of the concepts necessary to verify the axioms have been defined, and the main theorem can now be proved.

A pair of integers (q, k) is called *admissible* if $0 < q < p$ and $0 \leqq k < q$. For a fixed admissible pair (q, k) we define a simplicial homology theory $H^{q,k} = \{H_r^{q,k}, \partial, f_*\}$ as follows.

Define

$$H_r^{q,k}(K, L) = \begin{cases} H_{n,q}(K, L) & \text{where } n = rp/2 + k \qquad\qquad \text{for } r \text{ even,} \\ H_{m,p-q}(K, L) & \text{where } m = p(r+1)/2 + k - q \text{ for } r \text{ odd.} \end{cases}$$

The boundary operator $\partial : H_r^{q,k}(K, L) \to H_{r-1}^{q,k}(L)$ is defined to be the homomorphism

$$F^q : H_{n,q}(K, L) \to H_{n-q,p-q}(L) \qquad \text{if } r \text{ is even,}$$
$$F^{p-q} : H_{m,p-q}(K, L) \to H_{m-p+q,q}(L) \qquad \text{if } r \text{ is odd.}$$

The homomorphism

$$f_* : H_r^{q,k}(K, L) \to H_r^{q,k}(K', L')$$

induced by a simplicial map $f : (K, L) \to (K', L')$ is defined to be the homomorphism

$$f_* : H_{n,q}(K, L) \to H_{n,q}(K', L') \qquad \text{if } r \text{ is even,}$$
$$f_* : H_{m,p-q}(K, L) \to H_{m,p-q}(K', L') \qquad \text{if } r \text{ is odd.}$$

It will be shown that $\{H_r^{q,k}, \partial, f_*\}$ is a simplicial homology theory. Theorems 4.5–4.9 and 3.4 imply all the axioms except for Axiom 4, the exactness axiom. To prove exactness, consider the following sequence

$$\cdots \xrightarrow{F_{p-q}} C_{rp/2+k}(K) \xrightarrow{F_q} C_{rp/2+k-q}(K) \quad \xrightarrow{F_{p-q}} \cdots \xrightarrow{F_{p-q}} C_k(K)$$

and subsequence

$$\cdots \xrightarrow{F_{p-q}} C_{rp/2+k}(L) \xrightarrow{F_q} C_{rp/2+k-q}(L) \quad \xrightarrow{F_{p-q}} \cdots \xrightarrow{F_{p-q}} C_k(L).$$

The groups $H_r^{q,k}(K, L)$ have been defined to be the homology groups of the upper sequence modulo the lower sequence (in the notation of Kelley-Pitcher [2]). It follows that the homology sequence of (K, L) is exact [2, Theorem 3.3].

We have now shown that for any admissable pair (q, k) there is a corresponding homology theory. The uniqueness theorem stated in §2 implies that this homology theory is the same as the classical homology theory based on $H_0^{q,k}(P)$ as coefficient group. By Theorem 3.4,

$$H_0^{q,k}(P) = H_{k,q}(P) = \begin{cases} G & \text{if } k = q - 1, \\ 0 & \text{if } k \neq q - 1. \end{cases}$$

Hence, for $k \neq q-1$, the homology theory $H^{q,k}$ is trivial because it is isomorphic to the classical one based on a trivial coefficient group. For $k=q-1$, the homology theory $H^{q,q-1}$ is isomorphic to the classical one based on G as coefficient group. Hence, if $H_r(K, L)$ denotes the classical homology group of K mod L based on G as coefficient group, we see that

$$H_r(K, L) = \begin{cases} H_r^{q,q-1}(K, L) = H_{n,q}(K, L) \\ \qquad\qquad \text{where } n = pr/2 + q - 1 \text{ for } r \text{ even}, \\ \\ H_r^{p-q,p-q-1}(K, L) = H_{m,q}(K, L) \\ \qquad\qquad \text{where } m = p(r + 1)/2 - 1 \text{ for } r \text{ odd}, \end{cases}$$

thus parts (4) and (5) of Theorem 1.1 have been proved.

Let $H_{n,q}(K, L)$ be any Mayer group. We shall determine this group in terms of the classical groups by showing that it belongs to some homology theory $H^{m,k}$. Find integers m, t such that

$$n = mp + t \qquad \text{where } 0 \leqq t < p.$$

If $t<q$, (q, t) is an admissible pair, and

$$H_{2m}^{q,t}(K, L) = H_{mp+t,q}(K, L) = H_{n,q}(K, L).$$

Hence, if $n \equiv t \pmod{p}$ with $0 \leqq t < q$,

$$\begin{aligned} H_{n,q}(K, L) &= H_{2(n-t)/p}^{q,t}(K, L) \\ &= \begin{cases} 0 & \text{if } t \neq q - 1, \\ H_r(K, L) & \text{for } r = 2(n - q + 1)/p \text{ if } t = q - 1. \end{cases} \end{aligned}$$

This proves (1) of Theorem 1.1.

If $t \geqq q$, then $0 \leqq t-q < p-q$ so $(p-q, t-q)$ is an admissible pair, and

$$H_{2m+1}^{p-q,t-q}(K, L) = H_{mp+p+t-p,q}(K, L) = H_{n,q}(K, L).$$

Hence, if $n \equiv t \pmod{p}$ with $q \leqq t < p$

$$\begin{aligned} H_{n,q}(K, L) &= H_{2(n-t)/p+1}^{p-q,t-q}(K, L) \\ &= \begin{cases} 0 & \text{if } t \neq p-1, \\ H_r(K, L) \quad \text{where } r = 2(n+1)/p - 1 & \text{if } t = p-1, \end{cases} \end{aligned}$$

thus proving (2) and (3) of Theorem 1.1.

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INSTITUTE FOR ADVANCED STUDY

A NOTE ON HOMOGENEOUS PLANE CONTINUA

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In his Doctoral Dissertation (Texas, 1947), E. E. Moise proved that there exists a compact plane continuum M (not an arc) which is homeomorphic to each of its subcontinua [1].[1] Subsequently, R. H. Bing showed that M is homogeneous [2]. Bing's result flatly contradicts the previously announced result of G. Choquet to the effect that a homogeneous, compact, plane continuum must be a simple closed curve [3]. It is the purpose of this note to show that had Choquet assumed in addition to homogeneity that the continuum was aposyndetic[2] at some point, or that some point of the continuum failed to be a weak cut point[3] of it, then his conclusion would have been valid.

THEOREM 1. *If a compact, plane continuum M is both homogeneous and aposyndetic, then M is a simple closed curve.*

PROOF. If a point of M is of order 2 in M, then M is a simple closed curve [4]. So suppose that *no* point of M is of order 2 in M.

Let G denote the collection of all the complementary domains of M. Because M is homogeneous and contains a non-separating point, no point of M separates M. Since M is aposyndetic, M is semi-locally-connected [5]. Hence each element of G is a simple domain [6, 7]. Let the simple closed curve J denote the boundary of an element D of G.

Case 1. If $M-J$ is connected, then each point of M belongs to some such simple closed curve lying in M. Consequently each point of M belongs to the boundary of an element of G. But G is countable. Hence $M=\sum J_i$ $(i=1, 2, 3, \cdots)$, such that for each i, J_i is the boundary of an element of G. Since no point of M is of order 2 in M, each point of J_i is a limit point of $M-J_i$. This contradicts a well known theorem (Baire).

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

[2] A continuum M is said to be *aposyndetic* at a point x of M if for each point y of $M-x$, there exists a subcontinuum of M and an open subset U of M such that $M-y\supset H\supset U\supset x$. If a continuum is aposyndetic at each of its points, then it is said to be aposyndetic.

[3] A point p of a continuum M is a *weak cut point* of M if $M-p$ is not strongly (continuum-wise) connected. For other definitions the reader is referred to Moore's book or Whyburn's book, volumes 13 and 28, respectively, of the American Mathematical Society Colloquium Publications.

Case 2. If $M-J$ is not connected, then $M-J=H+K$ such that $H\cdot\overline{K}=\overline{H}\cdot K=0$. There exists an arc T lying in the complement of $\overline{D}$ which is irreducible from H to K. Except for its end points, T lies in an element U of G. Let C denote the boundary of U and let the point x of C be one of the end points of T. The component AxB of $C-C\cdot J$ which contains x is an arc-segment whose end points lie on J. It follows that $M-(A+B)$ is *not* connected, but $M-B$ *is* connected. Hence A is a local separating point of M. Since M is homogeneous, every point of M is a local separating point of M. But all except countably many of the local separating points of M must be of order 2 in M [8].

THEOREM 2. *If a homogeneous, compact, plane continuum M contains no weak cut point, it is a simple closed curve.*

PROOF. If M contains no weak cut point, it is aposyndetic at some point [9]. Being aposyndetic at one point, M must be aposyndetic at each of its points. Hence Theorem 2 follows from Theorem 1.

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THE UNIVERSITY OF TEXAS

ON THE CHARACTERISTIC EQUATIONS OF CERTAIN MATRICES

W. V. PARKER

In a recent paper Brauer[1] proved the following theorem credited to R. v. Mises.

THEOREM. *Let* $A=(a_{ij})$, $B=(b_{ij})$, *and* $C=(c_{ij})$ *be square matrices of order* n. *If the elements of* A *and* C *satisfy the conditions*

$$r_i = \sum_{j=1}^{n} a_{ij} = 0 \qquad (i = 1, 2, \cdots, n), \tag{1}$$

$$s_j = \sum_{i=1}^{n} a_{ij} = 0 \qquad (j = 1, 2, \cdots, n), \tag{2}$$

$$c_{ij} = c_i + c_j \qquad (i, j = 1, 2, \cdots, n), \tag{3}$$

where $c_1, c_2, \cdots, c_n$ *are arbitrary numbers, then the matrices* AB *and* $A(B+C)$ *have the same characteristic equation.*

Write $C_1=c'e$ where $c=(c_1, c_2, \cdots, c_n)$ and $e=(1, 1, \cdots, 1)$ then conditions (1), (2), and (3) are $AC_1'=0$, $C_1A=0$ and $C=C_1+C_1'$. This is a special case of the following theorem.

THEOREM. *Let* A, C_1, *and* C_2 *be n-rowed square matrices such that* $C_1A=AC_2=0$. *If* $C=C_1+C_2$ *and* B *is an arbitrary n-rowed square matrix, then* AB *and* $A(B+C)$ *have the same characteristic equation.*

The theorem is trivial if A is nonsingular, for then $C=0$. The proof will be based on the well known lemma:

LEMMA. *If* A *and* B *are square matrices,* AB *and* BA *have the same characteristic equation.*

Since $AC_2=0$, $A(B+C)=A(B+C_1)$ and from the lemma it follows that $A(B+C_1)$ has the same characteristic equation as $(B+C_1)A=BA$, and BA has the same characteristic equation as AB.

It may be readily shown that if A and C are matrices (not necessarily square) such that $ACA=0$, then $C=C_1+C_2$ where $AC_2=C_1A=0$. Also if A is an $m\times n$ matrix and B and C are $n\times m$ matrices and $ACA=0$, there exists a nonsingular matrix P, such that

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[1] Alfred Brauer, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 605–607.

$$PABP^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad PACP^{-1} = \begin{pmatrix} 0 & C_2 \\ 0 & 0 \end{pmatrix}$$

and hence in this case AB and $A(B+C)$ have the same characteristic equation.

University of Georgia

ON MATRICES WITH ELEMENTS IN A PRINCIPAL IDEAL RING

WILLIAM LEAVITT AND GEORGE WHAPLES

We prove the following theorem.

THEOREM 1. *Let* $\mathfrak{D}$ *be any commutative principal ideal ring without divisors of zero, and* A *any matrix with elements in* $\mathfrak{D}$ *whose characteristic equation factors into linear factors in* $\mathfrak{D}$. *Then there exists a unimodular matrix* T, *with elements in* $\mathfrak{D}$, *such that* $T^{-1}AT$ *has zeros below the main diagonal.*

This theorem was proved by Leavitt [1][1] for the special case of the ring $\mathfrak{H}$ of all functions of a complex variable holomorphic in, and on the boundary of, a closed bounded region R. His paper contains a proof that this set of functions forms such a ring; and gives an essentially algebraic construction for the transforming matrix T of the ring. Since this construction uses only properties of $\mathfrak{H}$ which are shared by all principal ideal rings [2; pp. 168–170] for [3; vol. 1, pp. 60–67], it can be carried out in all such rings.

The only changes necessary are those of terminology: "Holomorphic functions" must be replaced by "elements of $\mathfrak{D}$." Two elements are called *associated* if they differ by a unit factor. Since the prime ideals of $\mathfrak{H}$ are the ideals generated by the functions $(z-z_0)$, $z_0 \in R$, one must replace "$\alpha(z)$ has a zero of hth order at z_0" by "$\alpha \equiv 0 \bmod (\mathfrak{P}^h)$," and so on. Substituting a constant z_0 for z corresponds to mapping the ring $\mathfrak{H}$ into its homomorphic image $\mathfrak{H}/(z-z_0)$. Thus since $\mathfrak{D}/\mathfrak{P}$ is always a field or our rings $\mathfrak{D}$, the original arguments hold.

The one portion of the proof which might seem difficult to generalize is the use, in the final construction of the transforming matrix, of the theorem on the existence of a holomorphic function whose expansion at a finite number of points is specified to a finite number of terms. But this is simply the theorem that in any ring with unique factorization into prime ideals, we can find an element satisfying a finite number of simultaneous congruences, provided that the moduli are powers of different prime ideals. This can be proved for all principal ideal rings by the argument used to prove the Chinese remainder theorem in [4; p. 12].

It is also possible to give a much simpler proof of our theorem.

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[1] The numbers in brackets refer to the bibliography at the close of the paper.

Let $\mathfrak{D}$ be any principal ideal ring, A a matrix satisfying our assumption. Since A can be reduced to its Jordan form J by transformation in the quotient field of $\mathfrak{D}$, we conclude as in §2 of [1] that there exists a matrix S, with elements in $\mathfrak{D}$ and nonzero determinant, such that

$$AS = SJ. \tag{1}$$

Since J is in Jordan canonical form, it has elements in $\mathfrak{D}$, and zeros below the main diagonal. Now there always exists a unimodular matrix T such that TS has zeros below the main diagonal ([2; p. 228]; it is easy to make the slight necessary modification). Then T^{-1} also has elements in $\mathfrak{D}$, and

$$(TAT^{-1})(TS) = (TS)J.$$

It is easy to verify that the product XY of two matrices in normal form [that is, with zeros below the main diagonal] is again in normal form. Further if Y and XY are in normal form, and none of the diagonal elements of Y are zero, then X is in normal form. Since TS and J are already in normal form, and $|TS| \neq 0$, it follows that TAT^{-1} is in normal form. Clearly the diagonal elements of TAT^{-1} are the roots of $|TAT^{-1} - \lambda I| = 0$, which is the characteristic equation of A.

It should be pointed out that we have not yet defined a canonical form, since we have not described what can be done with the nondiagonal elements, and since two matrices with the same diagonal elements can still be dissimilar. For example

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & 3 & 0 \end{pmatrix},$$

are not similar even though their Jordan forms are identical.

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NONLINEAR NETWORKS. III

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The object of this paper is to show that a certain system of nonlinear differential equations has one and only one periodic solution. These equations are of interest in that they describe the vibrations of a common type of electric network; therefore, the physical origin of the equations will be discussed first.

A linear network is a collection of linear inductors, linear resistors, and linear capacitors arbitrarily interconnected. If a periodic electromotive force is applied to this network, a periodic system of currents can exist, provided that the network has no free vibration of the same period. This, of course, is well known. The main theorem of this paper states that if in such a network the linear resistors are replaced by *quasi-linear* resistors, a periodic system of currents can again exist.

A quasi-linear resistor is a conductor whose differential resistance lies between positive limits. Quasi-linear resistors have extensive practical applications. No other type of nonlinearity except this type of nonlinear damping is considered here.

For example, consider a linear network with one degree of freedom. An inductor of inductance L, a resistor of resistance R, and a capacitor of capacitance S^{-1} are connected in series. The current $i(t)$ flowing in this circuit must satisfy the following differential equation:

$$L\frac{di}{dt} + Ri + S\int idt = g.$$

Here $g(t)$ is the electromotive force impressed in the circuit and is a periodic function of time.

The corresponding nonlinear equation to be studied is obtained by replacing the linear relation Ri by a function $V(i)$ which for all values of i is such that $A^{-1} \leqq V'(i) \leqq A$, where A is a positive constant.

In a general network with m degrees of freedom, if a set of m independent circuits (meshes) is chosen, any distribution of current in the network may be uniquely specified by assigning suitable values to the cyclic currents $i_1, i_2, \cdots, i_m$ flowing in these circuits. Let $g_1, g_2, \cdots, g_m$ be the electromotive forces acting in these circuits. It is convenient to introduce the electric charge variables $y_1, y_2, \cdots, y_m$ such that $i_j = y_j'$. The linear network equations may

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then be written as $Ly''+Ry'+Sy=g$. Here L, R, and S are m-way matrices and y and g are vectors with components y_j and g_j. Well known arguments concerning electric and magnetic energy show that the matrices L, R, and S are symmetric semidefinite. The nonlinear network equations are $Ly''+V(y')+Sy=g$ where V is a vector function.

The central idea of this paper is to consider the network equations as a transformation of Hilbert space; that is, the equations give a transformation from the Hilbert space of electric charge to the Hilbert space of electromotive force. The existence theorem to be proved, then, is that this transformation has an inverse. The novelty of the proof arises when the linear transformation is replaced by a nonlinear transformation. Theorem 4, below, gives conditions which insure that a nonlinear transformation have an inverse. This theorem is analogous to the theorem that closure and completeness are equivalent for linear transformations. The casual reader may turn directly to this theorem and skip the algebraic complications which must be disposed of first.

It should be noted that Levinson, Chevalley, Lefschetz, and others have treated even more general types of nonlinearity in a single second-order differential equation. Therefore, in the case of one degree of freedom, the contribution of this paper is to furnish a method of proof which differs radically from theirs. It would be interesting to know if their techniques could be extended to treat the system of equations discussed here.

In a previous paper the transient solutions of the nonlinear system of equations were discussed. Combining the results of these two papers, it follows that for networks of the type considered, all solutions must approach the periodic solution. No appeal, however, will be made in this paper to results obtained in previous papers. Moreover, no appeal will be made to known theorems on differential equations. Theorem 3 of the present paper is believed to be a new theorem, even for linear networks.

Let E be a Euclidean space of m dimensions. A vector x will have components $x_1, x_2, \cdots, x_m$ which are real numbers. If x and y are vectors, then the inner product is given by $(x, y)=x_1y_1+x_2y_2+\cdots+x_my_m$ and the norm of x is $\|x\|=(x, x)^{1/2}$. All constants and functions shall be assumed real. The symbol A will stand for a positive constant whose value, however, may differ in different sections.

If B is a matrix, the manifold of all vectors of the form Bx will be designated as M_B. A semi-definite matrix B satisfies the relation $(Bx, x)\geq 0$ for all vectors x.

LEMMA 1. *If B is a symmetric matrix which satisfies the inequality $0 \leqq (Bx, x) \leqq A_1 \|x\|^2$ for all vectors x, then:*

(a) $\|Bx\|^2 \leqq A_1(Bx, x)$.

(b) $(Bx, x) \leqq A_2 \|Bx\|^2$ *where A_2 is a constant.*

(c) $\|z\|^2 \leqq A_2(Bz, z)$ *where $z = Bx$.*

PROOF. The inequalities of this lemma are obvious if B is a diagonal matrix; however, the expressions are invariant under an orthogonal transformation, and it is well known that there is an orthogonal transformation which reduces B to diagonal form.

LEMMA 2. *Let M_B and M' together span E. Then any vector v has a unique representation $v = Bb + v'$ where b is orthogonal to M' and v' is contained in M'. There is a constant A independent of v such that $\|b\|$, $\|v'\| \leqq A\|v\|$.*

PROOF. If the manifold spanned by the vectors of the form $Bb + v'$ were not E, there would be a nonzero vector c such that $(Bb + v', c) = 0$ for all b and v'. Let $b = 0$, then $(v', c) = 0$, so c is orthogonal to M'; therefore, it must be in M_B. Let $b = c$; therefore $(Bc, c) = 0$, and by Lemma 1 (c) it follows that $c = 0$, which is a contradiction.

From $v = Bb + v'$, we have $(v, b) = (Bb, b)$, so $(Bb, b) \leqq \|v\| \|b\|$. Then from Lemma 1 (c) we have $\|b\|^2 \leqq A\|v\| \|b\|$ or $\|b\| \leqq A\|v\|$. Because $v' = v - Bb$, it is obvious that v' satisfies an inequality of the same form. The uniqueness of the representation follows directly from these inequalities.

In what follows, L, R, and S will denote symmetric semi-definite matrices such that M_L, M_R, and M_S together span E. A simple application of Lemma 1 (a) reveals that the matrix $L + R + S$ is nonsingular.

THEOREM 1. *Any vector v has a unique representation in the form $v = Ll + Rr + Ss$ where $l \in M_L$, $r \in M_R$, and $s \in M_S$ and $Rl = Sl = Sr = 0$. Moreover, $\|l\|$, $\|r\|$, $\|s\| \leqq A\|v\|$ where the constant A is independent of v.*

PROOF. In Lemma 2 let $B = L$ and let M' be the manifold spanned by M_R and M_S. Then $v = Ll + v'$. The manifold M' is also a Euclidean space, so again by Lemma 2, $v' = Rr + v''$ where $v'' \in M_S$; hence $v'' = Ss$. The remainder of the statements of the theorem follow easily from Lemma 2.

We are to consider vector equations of the form

$$Ly'' + Ry' + Sy = g. \tag{1}$$

We first wish to show that it is possible to make a linear transformation of variables so that these equations break up into three independent sets, two of which are of very simple form; the remaining set we shall call the canonical form. First by an orthogonal transformation of y'', y', y, and g, we may employ a rotated coordinate system in which S is a diagonal matrix. Since the matrix elements of S are then non-negative, a second transformation, $D^{-1}y''$, $D^{-1}y'$, $D^{-1}y$, and Dg (where D is a nonsingular diagonal matrix), may be chosen in such wise that the new matrix elements of S are either 0 or 1. The new matrices L, R, and S are expressed in terms of the old by DLD, DRD, and DSD. Moreover, the new matrices are symmetric semi-definite. By Lemma 2, any vector y of E may be uniquely expressed in the form y_1+Ss where y_1 is in the manifold M_L+M_R and s is orthogonal to this manifold and is in the manifold M_S. But now $S^2=S$, so $Ss=s$. Let $y_2=s$, then we have a unique decomposition of the vector y in the form y_1+y_2. Then $Sy=Sy_1+Sy_2=Sy_1+y_2$ where $Sy_1 \in M_L+M_R$. To show the latter statement, let x be any vector orthogonal to M_L+M_R, so it is in M_S. Then $(Sy_1, x)=(y_1, Sx)=(y_1, x)=0$. The vectors y' and y'' are split in the same fashion. Noting that $Ly_2''=Ry_2'=0$, we may write the equations in the form $Ly_1''+Ry_1'+Sy_1+Sy_2=g$. Both sides of this relation can be split into components in the manifold M_L+M_R and its orthogonal complement. Thus $Ly_1''+Ry_1'+Sy_1=g_1$, and $y_2=g_2$. We could transform the first set of equations in an analogous fashion but splitting L instead of S; this gives the desired decomposition into the three sets $y_2=g_2$, $y_4''=g_4$, and $Ly_3''+Ry_3'+Sy_3=g_3$. *The latter set of equations is the canonical form and is such that if* E_3 *is the manifold of these equations, then* M_L *and* M_R *span* E_3 *and* M_R *and* M_S *also span* E_3. In other words, $L+R$ and $R+S$ are both nonsingular.

Let $v=V(x)$ be a vector function with corresponding differential transformation $dv=V'dx$ where V' is the matrix of differential coefficients. If R is a symmetric semi-definite matrix, we shall say that $V(x)$ is a *quasi-linear* replacement of the linear function Rx if:

(I) $V(0)=0$.

(II) $V'(x)$ *exists for all* x *as a symmetric matrix.*

(III) *There is a positive constant* A *such that for all vectors* x *and* y

$$A^{-1}(Ry, y) \leqq (V'y, y) \leqq A(Ry, y).$$

By use of Lemma 1 we may derive other inequalities such as $\|Ry\|^2 \leqq A_3(V'y', y')$, $\|V'y\|^2 \leqq A_4(V'y, y)$, and $\|V'y\| \leqq A_5\|Ry\|$. It is apparent that V' is semi-definite and that it is made up of uniformly bounded matrix elements. By the mean value theorem $V(x)-V(y)=V'(x-y)$ where V' is evaluated at the point $y+\theta(x-y)$; $0 \leqq \theta \leqq 1$. Setting $y=0$, we have $V(x)=V'x$.

In performing the transformation of equations (1) to a canonical form, we note that the new matrix R_n is given in terms of the old by the relation $R_n = B_t R B$ where B is a nonsingular matrix and B_t is its transpose. The quasi-linear replacement would become $V_n(x) = B_t V(Bx)$, and hence $V_n' = B_t V' B$. Clearly, then, $V_n(x)$ is a quasi-linear replacement of $R_n x$.

We now wish to show that all vectors $v = V(x)$ are contained in the manifold M_R. Because $V(x) = V'x$, the manifold of vectors $V'y$ contains v. If the manifold of vectors $V'y$ were not contained in the manifold M_R, there would be a vector z perpendicular to M_R; that is, $Rz = 0$ but $V'z \neq 0$. By virtue of the inequalities given for a quasi-linear replacement, this implies that $z = 0$. It follows, therefore, that the same transformation which reduces equations (1) to canonical form will also reduce the following equations:

$$Ly'' + V(y') + Sy = g. \tag{2}$$

We now regard the vectors of a space E to have components which are functions of the time t. The primes in equations (1) and (2) refer to differentiation with respect to time; thus $Ry' = (d/dt)Ry$. We adopt the convention that writing Ry' does not imply that all components of y are differentiable but only those in the manifold M_R. Moreover, in what follows, "differentiable" means absolutely continuous; that is, we differentiate only those functions which are the integrals of Lebesgue integrable functions. It follows, of course, that the derivatives may not exist everywhere, but only almost everywhere. Except where the contrary is indicated we shall require only that the equations (1) and (2) hold almost everywhere.

We define the real Hilbert space H (of electromotive force) to be the set of all vectors of E whose components are periodic functions of the time t of period 2π and which are of integrable square; that is, they belong to $L_2(0, 2\pi)$. The bilinear form of two vectors x and y is given by $((x, y)) = (1/2\pi)\int_0^{2\pi} (x, y)dt$. If x and y are constant vectors, it follows that $((x, y)) = (x, y)$. The norm of x is given by $\|x\|^2 = ((x, x))$. The context will prevent confusion between norms in the space E and H. The subspace of constant vectors will be denoted by C. The subspace H_0 will be defined as the space of all vectors orthogonal to C.

The normed linear space Q will be defined as the space of vectors of the form $y = y_0 + lt^2/2 + rt + s$ where l, r, and s satisfy Theorem 1 and where $y_0 \in H_0$. The norm of y is given by the relation $\|y\|_Q^2 = \|Ly''\|^2 + \|Ry'\|^2 + \|Sy\|^2$. We note that in equation (1) if $g \in C$, we can obtain a solution $y \in Q$ with $y_0 = 0$. This is by virtue of Theorem 1. If $y \in Q$, it follows that Ly'', Ry', and $Sy \in H$, and by the convention adopted above, Ly' and Ry are absolutely continuous.

In what follows it will be tacitly assumed that we are considering a canonical form, because a considerable simplification of the proofs is thereby achieved. It will be apparent, a posteriori, that the main theorems are independent of this assumption. The first simplification is that $l=0$, because by Theorem 1, $(R+S)l=0$, but $R+S$ is nonsingular in the canonical form. We now have that Ly' is periodic as well as being absolutely continuous, so $Ly' \in H$. Thus $(L+R)y' \in H$, and since $L+R$ is nonsingular, y' exists and belongs to H.

LEMMA 3. *If u, v, u', and v' all belong to H, then $((u, v')) = -((u', v))$.*

PROOF. Since $(d/dt)(u, v) = (u', v) + (u, v')$, and since (u, v) is absolutely continuous and of period 2π, it follows that integration of this equation on the left yields zero.

LEMMA 4. *If $y \in Q$, then $((Sy, y')) = 0$ and $((Ly'', y')) = 0$.*

PROOF. We have seen above that y' exists, so y is absolutely continuous, and so, also, is Sy. We now have (Sy, y) an absolutely continuous periodic function, and $(d/dt)(Sy, y) = 2(Sy, y')$, so the first part of the lemma is true. The second part follows from the identity $(d/dt)(Ly', y') = 2(Ly'', y')$.

A sequence f_n, $n = 1, 2, \cdots$, of H converges strongly to f if $\|f - f_n\|$ approaches zero as n approaches infinity, and a sequence converges weakly if $((x, f-f_n))$ approaches zero for every x of H. Strong convergence, of course, implies weak convergence. Pointwise convergence is ordinary convergence, and if such convergence is bounded (uniformly) it implies strong convergence by Lebesgue's theorem.

If a sequence f_n is uniformly bounded, it is easy to show, by using the "diagonal process" on an expansion of f_n in terms of a complete orthonormal set, that there is a sub-sequence which is weakly convergent.

LEMMA 5. *If f_n is a sequence of H_0 and f_n' is uniformly bounded and weakly convergent, then there is an $f \in H_0$ such that f_n converges pointwise and boundedly to f and f_n' converges weakly to f'. Conversely, if f_n converges pointwise to f, then f_n' converges weakly to f'.*

PROOF. Suppose f_n' converges weakly to f'. We may take f' to be the derivative of a function f belonging to H_0. Considering the sequence $f_n' - f'$, we see that there is no loss of generality in supposing in the beginning that f_n' converges weakly to zero. Thus $((x, f_n')) \rightarrow 0$, and by taking x to be a vector of one component, which component is the characteristic function of the interval a, b, we have, if u_n is the corresponding component of f_n, $\int_a^b u_n' \, dt \to 0$. But $|u_n(b)-u_n(a)|^2 = |\int_a^b u_n' \, dt|^2 \leqq |b-a| \int_a^b |u_n'|^2 dt \leqq (2\pi)^2 \|f_n'\|^2$. Thus $u_n(b)-u_n(a)$ is boundedly convergent, so $\int_0^{2\pi} (u_n(b)-u_n(a)) da \to 9$. But since $f_n \in H_0$, the integral of the second term vanishes, so $2\pi u_n(b) \to 0$. Thus all the components of f_n approach zero pointwise and boundedly. The converse is obvious.

Lemma 6. *If* $y \in Q$, *then* $\|y'\| \leqq A \|y\|_Q$ *where* A *is independent of* y.

Proof. Otherwise there is a sequence y_n such that $\|y_n'\| = 1$ and $\|y_n\|_Q \to 0$. Clearly r_n and s_n approach zero by Theorem 1, so without loss of generality we assume $y \in H_0$. Since Ly_n'' converges strongly to zero, it follows from Lemma 5 that Ly_n' converges to zero. Thus $(L+R)y_n'$ converges to zero but $L+R$ is nonsingular, so y_n' converges to zero, which is incompatible with $\|y_n'\| = 1$.

Theorem 2. *The space* Q *of electric charge is a Hilbert space.*

Proof. Let the bilinear form be $((x, y))_Q = ((Lx'', Ly'')) + ((Rx', Ry')) + ((Sx, Sy))$. If $\|y\|_Q = 0$, it follows from Lemma 6 that $y' = 0$. But by Theorem 1, $r = s = 0$, so $y = 0$. It is apparent, therefore, that the bilinear form has the necessary qualifications.

It remains to show that Q is closed, so consider a sequence y_n such that $\|y_n - y_m\|_Q^2 \to 0$. Without loss of generality we may suppose that $y_n \in H_0$. By Lemma 6, y_n' is convergent, and by Lemma 5, y_n converges pointwise to a function y. By the converse statement of Lemma 5, Ly_n'', Ry_n', and Sy_n converge to Ly'', Ry', and Sy respectively. This completes the proof.

We need the following further restriction on a quasi-linear replacement:

(IV) *If* V *is a function of the time* t, *then* $V(x, t) = V(x, t+2\pi)$ *and* $(V_1, V_1) \leqq A(V, V)$ *where* $V_1 = \partial V(x, t)/\partial t$.

It is evident that the inequalities given for a quasi-linear replacement remain unchanged when the bilinear form (,) is replaced by the bilinear form ((,)).

Lemma 7. *If* Rx_n *converges strongly to* Rx, *then* $V(x_n)$ *converges strongly to* $V(x)$. *The same is true for pointwise convergence.*

Proof. $V(x_n) - V(x) = V'(x_n - x)$, so

$$\|V(x_n) - V(x)\|^2 = \|V'(x_n - x)\|^2 \leqq A_5^2 \|Rx_n - Rx\|^2 \to 0.$$

The proofs below would be a lot simpler if we could say the same thing for weak convergence. However, it is easy to see that the lemma is almost always false under such conditions. Consider sin nt, which converges weakly to zero, but $f(C+\sin nt)$ would not converge weakly to $f(C)$ for all C unless $f(C)$ is a linear function.

The fundamental hypothesis which will be imposed on the matrices L, R, and S in what follows is the relation:

$$Ly'' + Ry' + Sy = 0 \text{ implies } y = 0 \text{ if } y \in Q. \tag{3}$$

An apparently less restrictive condition would be that these equations have no periodic solution in Q, but by virtue of Theorem 1 it is easy to see that these conditions are equivalent. It may be shown that a necessary and sufficient condition for this hypothesis is that the determinant $(-n^2L+(-1)^{1/2}nR+S)\neq 0$ for $n=\pm 1, \pm 2, \cdots$. No use shall be made of this fact, however.

THEOREM 3a. *Let* $Ly''+V'(x)y'+Sy=f$ *where* $y\in Q$ *and* $x\in H$. *Then* $\|y\|_Q\leqq A\|f\|$ *where* A *is independent of* x *and* y.

PROOF. It is to be noted that, in particular, $V'(x)y'$ could be $V(y')$. Because $\|y\|_Q$ is finite, it follows from the inequalities derived for a quasi-linear replacement that $\|f\|$ is finite. If the theorem is not true, it is clear that a sequence y_n exists with $\|y_n\|_Q=1$ and the corresponding sequence f_n converging strongly to zero. By Lemma 6, $\|y'\|\leqq A$, so it may be assumed that the sequence y_n' converges weakly. It may also be assumed that s_n converges; hence, by Lemma 5, y_n converges pointwise to some y.

Employing Lemma 4, we have $((V'y_n', y_n'))=((Ly_n''+V'y_n'+Sy_n, y_n'))=((f_n, y_n'))\to 0$. By the inequalities for a quasi-linear replacement, Ry_n' and $V'y_n'$ converge strongly to zero. Write $Ly_n''=f_n-V'y_n'-Sy_n$. The terms on the right converge strongly; hence, Ly_n'' converges strongly. Thus y_n converges in Q to, say, y, so $Ly''+Sy=0$ and $Ry'=0$. But $Ly''+Ry'+Sy=0$ implies $y=0$ by hypothesis (3). This is incompatible with $1=\|y_n\|_Q=\|y\|_Q$, so the contradiction proves the theorem.

Let $L_n=L+R/n$ where n is a positive integer. The equation $L_ny''+Ry'+Sy=0$ for $y\in Q$ implies $y=0$. This is seen by forming the inner product with y', yielding $((Ry', y'))=0$. Thus $Ry'=0$, and consequently $Ry''=0$. Thus $Ly''+Ry'+Sy=0$ and $y=0$.

THEOREM 3b. *Theorem* 3a *is valid if* L *is replaced by* L_n *for an* A *independent of* n *if* $f\in H$.

PROOF. For n fixed we may introduce a space Q_n. Then by the above remarks we see that Theorem 3 is applicable. Thus $\|L_ny''\|^2$

$+\|Ry'\|^2+\|Sy\|^2 \leqq A^2\|f\|^2$. Since L_n is nonsingular, $\|L_n y''\| \geqq A_1\|y''\| \geqq A_2\|Ly''\|$. We may assume $0 < A_2 \leqq 1$, so if $A_n = A/A_2$, then $\|y\|_Q \leqq A_n\|f\|$.

If it is not possible to obtain a uniform bound, there is a sequence of integers, say $\{m\}$, with a corresponding sequence y_m and f_m with the properties assumed in the proof of Theorem 3a. Then as before we have that $L_m y_m''$ is strongly convergent. But $L_m y_m' = Ly_m' + Ry_m'/m$, and this converges to Ly'. Thus by Lemma 5, $L_m y_m''$ converges strongly to Ly''. As before, we obtain the contradiction $Ly''+Ry'+Sy=0$.

THEOREM 4. *Let T be a transformation of a linear vector space Q on a Hilbert space H such that*: (a) *if y_n is a sequence of Q such that $\|Ty_n\| \leqq A$, then there is a sub-sequence y_m such that $T(y_m)$ converges weakly to $T(y)$ for some y of Q*; (b) *for each x and y of Q, $\lim_{h\to 0}(T(y+hx)-T(y))/h = T'(y, x)$ exists where h is real*; (c) *for each y the relation $((T'(y, x), f)) = 0$ for all x implies $f=0$.*

Then for any g of H the equation $g = T(y)$ is solvable.

PROOF. Let b be the greatest lower bound of $\|g-T(y)\|$ for all y of Q. Let y_n be a sequence such that $\|g-T(y_n)\| \to b$. Clearly $\|T(y_n)\|$ is uniformly bounded, so there is a weakly convergent subsequence, y_m; and by condition (a) there is a y such that $T(y_m) \to T(y)$ weakly. Thus $\|g-T(y)+T(y)-T(y_m)\|^2 = \|g-T(y)\|^2 + 2((g-T(y),\ T(y)-T(y_m))) + \|T(y)-T(y_m)\|^2$. The limit of the left side is b^2, and the limit of the right is not less than $\|g-T(y)\|^2$, so $b^2 \geqq \|g-T(y)\|^2$; hence $b = \|g-T(y)\|$. Let $f = T(y)-g$, then $\|g-T(y\pm hx)\|^2 = \|f+T(y\pm hx)-T(y)\|^2 = \|f\|^2 + 2((f,\ T(y\pm hx)-T(y))) + \|T(y\pm hx)-T(y)\|^2$. Thus $0 \leqq 2((f, T(y\pm hx)-T(y))) + \|T(y\pm hx)-T(y)\|^2$. Divide through by h taken to be positive, and let h approach zero. Then $0 \leqq 2((f, T'(y, x)))$, and $0 \leqq -2((f, T'(y, x)))$, so by (c) it follows that $f=0$. Q. E. D.

THEOREM 5. *For each g of H there is a unique y of Q such that $Ly''+V(y')+Sy=g$.*

PROOF. First assume that L is nonsingular. Let $T(y) = Ly'' + V(y') + Sy$, and consider the conditions of Theorem 4. Let $T(y_n)$ be a uniformly bounded and weakly convergent sequence. By Theorem 3a, Ly_n'' is uniformly bounded. There is a sub-sequence, therefore, over which Ly_n'' is weakly convergent, and since L is nonsingular, y_n'' is also weakly convergent. Also by Theorem 3a, Ry_n' and Sy_n are uniformly bounded, so we may assume that a sub-sequence has been selected in such wise that r_n and s_n converge. Thus by Lemma 5 there is a y of Q such that y_n converges pointwise boundedly to y, y_n' converges pointwise boundedly to y', and y_n'' converges weakly to

y''. By Lemma 7, $V(y_n')$ converges pointwise to $V(y')$. It follows, therefore, that $T(y_n)$ converges weakly to $T(y)$. This shows that (a) is satisfied. To show that (b) is satisfied, we note that $(V(y'+hx')-V(y'))/h$ has the limit $V'(y')x'$. Thus $T'(y, x)=Lx''+V'x'+Sx$, and T' is a linear transformation of x. To show that (c) is satisfied, consider the relation $((f, Lx''+V'x'+Sx))=0$. First let x be a constant vector; then $0=((f, Sx))$, so the component of f in C is orthogonal to the range of S. The same will be true of f_i where f_i is the first i terms of the Fourier series for f. An x may be chosen, therefore, so that $x'=f_i$; then $((f, Lx''))=((x', Lx''))=0$, and $((f, Sx))=((x', Sx))=0$. Thus we obtain $((f, V'x'))=0$, or $((f, V'f_i))=0$. Allowing i to approach infinity in this expression, we obtain $((f, V'f))=0$. By the properties of a quasi-linear replacement, this implies that $Rf=0$; thus, f belongs to H_0. Let $f=u''$, and let u and x belong to H_0; then $0=((u'', Lx''-Rx'+Sx))$. By Lemma 3, $((x'', Lu''+Ru'+Su))=0$, but x'' is an arbitrary element of H_0, so $Lu''+Ru'+Su=0$; thus $u=0$ and hence, $f=0$. This proves the existence of a solution under the assumption that L is nonsingular.

The matrix L_n is nonsingular, so let g be a function of H with a bounded first derivative. By what we have just proved, there is a y of Q such that $L_ny''+Vy'+Sy=g$. Thus $y''=L_n^{-1}(g-V(y')-Sy)$ almost everywhere. Since y_n' is absolutely continuous, $y'=\int_0^t L_n^{-1}(g-V(y')-Sy)dt+C$. But the integrand is continuous, so actually the expression for y'' holds in the pointwise sense. It is clear, therefore, that y'' has a first derivative which is uniformly bounded in the pointwise sense; hence, by a well known theorem of Lebesgue integration, y'' is absolutely continuous, and therefore $y'\in Q$. We may write $L_ny'''+V'y''+Sy'=g_1$ where $g_1=g'-V_1(y')$. By hypothesis IV, $\|V_1(y')\|\leqq A\|V(y')\|$ and from Theorem 3b applied to the equation for g, we have $\|V(y')\|\leqq A\|g\|$. Thus $\|g_1\|$ is uniformly bounded, so applying Theorem 3b to the equation for g_1, we have that $\|Ly'''\|$, $\|Ry''\|$, $\|Sy'\|$ are uniformly bounded. Allow n to approach infinity and consider the sequence y_n. We may suppose that the sequence is so chosen that Ly_n''', Ry_n'', and Sy_n' are weakly convergent. Thus Ly_n'', Ry_n', and Sy_n converge pointwise. Clearly, $V(y_n')$ converges pointwise. The equation for g_1 may be solved for Ry_n'''/n in terms of uniformly bounded expressions. It may be assumed, therefore, that it is uniformly bounded in H and that Ry_n''/n is pointwise convergent. But $\|Ry_n''\|$ is uniformly bounded, so Ry_n''/n must approach zero. If y is the limit of y_n, then $Ly''+V(y')+Sy=g$, *and this equation holds in the pointwise sense, and all terms appearing are continuous.*

If g does not have a bounded derivative, let g_i equal the first i terms in its Fourier series, and we can satisfy the equation for g_i. If y_i and y_j correspond to g_i and g_j, let $z=y_i-y_j$; then $Lz''+V'z'+Sz=g_i-g_j$. By Theorem 3a, $\|z\|_Q$ approaches zero as i and j approach infinity, so y_i converges in Q to some y. This y satisfies the equation for g, and the proof that a solution exists is complete.

To prove the uniqueness of the solution, we note that if z is the difference of two solutions, then $Lz''+V'z'+Sz=0$. Theorem 3a shows directly that $z=0$.

THEOREM 6. *Suppose that all solutions of equations* (3) *are such that* $y'\to 0$ *as* $t\to\infty$. *Let* y_1 *be the periodic solution of equations* (2) *when* g *is a function with a bounded first derivative. If* y_2 *is any other solution of these equations with a continuous first derivative, then* $\int_0^\infty \|y_1'-y_2'\|^2 dt < \infty$.

PROOF. This is a special case of a theorem which the writer proved in a previous note [1].[1] To apply this theorem here it is necessary to know that y_1' is continuous, but this is apparent from the proof of Theorem 5. In the previous note it was not explicitly stated that V could be a function of the time. The proof, however, remains unchanged for this generalization.

A later note will consider similar theorems for networks of linear resistors and iron-core inductors.

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ON THE SUMMABILITY OF CERTAIN ORTHOGONAL DEVELOPMENTS OF NONLINEAR FUNCTIONALS

R. H. CAMERON AND C. HATFIELD, JR.

1. Introduction. Using Wiener's[1] method of integration in function space, Cameron and Martin[2] have constructed a set of closed orthonormal functionals in the space of nonlinear functionals which are Wiener measurable and whose squares are Wiener summable. It is the purpose of this paper to investigate the infinite-dimensional Abel summability of the orthogonal development of a nonlinear functional in terms of these orthogonal functionals.

The space of functions over which Wiener's integral operates is the space C of functions $x(t)$ defined and continuous on $0 \leqq t \leqq 1$ and vanishing at $t=0$. The whole space has measure unity, and measure in this space is built up in exactly the same way as ordinary Lebesgue measure, except that the definition of an interval and its measure are different.

Wiener defines an interval Q, or as he calls it, a "quasi-interval" in his space by the inequalities

$$Q: \qquad \alpha_j < x(t_j) < \beta_j, \qquad j = 1, \cdots, n,$$

where $t_1, \cdots, t_n$ is any finite set of numbers such that $0 < t_1 < t_2 < \cdots < t_n \leqq 1$. He defines the measure of Q as

$$m(Q) = [\pi^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^{-1/2} \int_{\alpha_n}^{\beta_n} \cdots \int_{\alpha_1}^{\beta_1} \cdot \exp\left[-\frac{\xi_1^2}{t_1} - \frac{(\xi_2 - \xi_1)^2}{t_2 - t_1} - \cdots - \frac{(\xi_n - \xi_{n-1})^2}{t_n - t_{n-1}} \right] d\xi_1 \cdots d\xi_n,$$

and shows that this definition is self-consistent and leads to a satisfactory measure for general sets, in accordance with the usual definition of Lebesgue measure. After measure is defined, the usual Lebesgue procedure gives a satisfactory definition of integral with all the usual properties except invariance under translations and simple behavior under magnifications. Measurability and summabil-

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[1] *Generalized harmonic analysis*, Acta Math. vol. 55 (1930) pp. 117–258, esp. pp. 214–224.

[2] *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals*, Ann. of Math. vol. 48 (1947) pp. 385–392.

ity of functionals are defined as usual. It is convenient to denote the Wiener integral of a summable functional $F(x)$ over a measurable set S by

$$\int_S^w F(x)d_w x.$$

In particular, it has been shown that if

$$F(x) = f\left\{\int_0^1 \alpha_1(t)dx(t), \cdots, \int_0^1 \alpha_n(t)dx(t)\right\},$$

where $\{\alpha_j(t)\}$, $j=1, 2, \cdots, n$, are orthonormal and of bounded variation, then provided the second member exists,

$$\begin{aligned} &\int_C^w F(x)d_w x \\ (1.1)\qquad &= \pi^{-n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(u_1, \cdots, u_n) \exp\left(\sum_{k=1}^{n} u_k^2\right) du_1 \cdots du_n. \end{aligned}$$

Convenient Fubini theorems enable us to interchange order of two Wiener integrals or a Wiener integral and an n-dimensional Lebesgue integral in the ordinary way. Thus the integral has the usual satisfactory manipulative properties. We denote by $L_2(C)$ the space of measurable complex functionals $F(x)$ such that

$$\int_C^w |F(x)|^2 d_w x < \infty.$$

An orthonormal set of functionals $\{F_n\}$ is of course defined to be a set such that

$$\int_C^w F_j(x)\overline{F_k(x)}d_w x = \delta_{jk}, \qquad j, k = 1, 2, \cdots,$$

where δ_{jk} is the Kronecker delta, and such a set is closed if for each functional $G \in L_2(C)$ and a given $\epsilon > 0$, there exists a set of constants $C_1, C_2, \cdots, C_n$ such that

$$\int_C^w \left| G(x) - \sum_{j=1}^{n} C_j F_j(x)\right|^2 d_w x < \epsilon.$$

A C.O.N. (closed orthonormal) set of functionals can be constructed as follows. Let $\alpha_1(t), \alpha_2(t), \cdots$ be a C.O.N. set of real functions on $0 \leqq t \leqq 1$ and each $\alpha_j(t)$ of bounded variation, and let $H_n(u)$ be the partially normalized Hermite polynomials

$$H_n(u) = (-1)^n 2^{-n/2}(n!)^{-1/2} e^{u^2} \frac{d^n}{du^n}(e^{-u^2}), \qquad n = 0, 1, 2, \cdots .$$

Let

$$\Phi_{m,p}(x) = H_m\left[\int_0^1 \alpha_p(t)dx(t)\right], \qquad m = 0, 1, \cdots ; p = 1, 2, \cdots ,$$

and

$$\Psi_{m_1,\cdots,m_p}(x) \equiv \Psi_{m_1,\cdots,m_p,0,\cdots,0}(x) = \Phi_{m_1,1}(x) \cdots \Phi_{m_p,p}(x).$$

This is called the Fourier-Hermite set of functionals. As the above notation indicates, any number of zero subscripts may be added at the end of the already given list of subscripts without changing Ψ, because $H_0(u) \equiv \Phi_{0,p}(x) \equiv 1$. Thus the entire set of the $\Psi_{m_1,\cdots,m_p}$ without repetition consists of

Ψ_0;
$\{\Psi_{m_1}\}$, $m_1 = 1, 2, \cdots$;
$\{\Psi_{m_1,m_2}\}$, $m_1 = 0, 1, 2, \cdots , m_2 = 1, 2, \cdots$;
. . . . ;
$\{\Psi_{m_1,\cdots,m_p}\}$, $m_i = 0, 1, 2, \cdots$ for $i = 1, 2, \cdots , p-1, m_p = 1, 2, \cdots$;
. . . .

Despite our complicated set of subscripts, this is still only a countable set of functionals. It has been shown[2] that this is a C.O.N. set over $L_2(C)$, Thus, if $F(x) \in L_2(C)$, its Fourier-Hermite coefficients are defined to be

$$A_{m_1,\cdots,m_N} = \int_C^{w} F(x)\Psi_{m_1,\cdots,m_N}(x)d_w x,$$

and we have

$$\lim_{N\to\infty} \int_C^{w} \left| F(x) - \sum_{m_1,\cdots,m_N=0}^{N} A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x)\right|^2 d_w x = 0.$$

Thus the Fourier-Hermite development of F converges in the sense of a limit in the mean for $L_2(C)$ to the original functional. However, this does not tell us that it converges in any sense at all for any given fixed function x, just as convergence in the mean for a Fourier series does not provide any information at all about the behavior of the series at a specific point. The convergence of the Fourier series at a fixed point requires more specific information about the generating function at that point, and it is reasonable to expect that much more would be required in the case of the functional. It is not at all clear what the analogue of the notion of bounded variation should be in order for us to carry through a pointwise convergence theorem in $L_2(C)$. However, if we are content to establish summability of the Fourier series of a function at the specified point, then we do not need bounded variation at the point, but only continuity; and continuity does have a satisfactory analogue for functionals. Since our orthogonal functionals are given in terms of Hermite functions, our summability will depend on a formula for sums of products of Hermite functions. Such a formula has been given by Wiener,[3] for $|t|<1$, $-\infty<x<\infty$, $-\infty<y<\infty$, namely,

$$(1.2)\quad \sum_{n=0}^{\infty} t^n\psi_n(x)\psi_n(y) \equiv K(x, y, t)$$

$$= [\pi(1-t^2)]^{-1/2} \exp\left[\frac{4xyt-(1+t^2)(x^2+y^2)}{2(1-t^2)}\right],$$

where

$$(1.3)\qquad \psi_n(x) = \pi^{-1/4}H_n(x)e^{-x^2/2}, \qquad n = 0, 1, \cdots .$$

Wiener showed that this series converges absolutely when $|t|<1$ and that for fixed t such that $|t|<1$, the series converges in the mean in x and y to its value. It also converges in the mean as a function of x for almost all fixed y, and vice versa.

The fact that (1.2) involves powers of t led us to investigate infinite-dimensional Abel summability for the Fourier-Hermite series. We now state our main theorem.

THEOREM. *Let C be the space of all continuous functions $x(t)$ on $0\leqq t\leqq 1$ such that $x(0)=0$. Let $F(x)=F(x(\cdot))$ be a bounded Wiener measurable functional (linear or nonlinear) for $x\in C$, and let $F(x)$ be continuous in the Hilbert topology at $x_0\in C$, that is,*

$$|F(x)-F(x_0)| < \epsilon \quad \textit{when} \quad \int_0^1 [x(t)-x_0(t)]^2dt < \delta^2(\epsilon).$$

Let

---

[3] N. Wiener, *The Fourier integral*, pp. 55–67.

$$\Psi_{m_1,\cdots,m_N}(x) = H_{m_1}\left(\int_0^1 2^{1/2}\cos\frac{\pi t}{2}\,dx(t)\right)$$
$$\cdot H_{m_2}\left(\int_0^1 2^{1/2}\cos\frac{3\pi t}{2}\,dx(t)\right)$$
$$\cdots H_{m_N}\left(\int_0^1 2^{1/2}\cos\frac{(2N-1)\pi t}{2}\,dx(t)\right),$$

and let

$$A_{m_1,\cdots,m_N} = \int_C^w F(x)\Psi_{m_1,\cdots,m_N}(x)d_wx;$$

then

$$F(x_0) = \lim_{N\to\infty,\lambda\to 1-0} \sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\cdots,m_N}\lambda^{m_1+\cdots+m_N}\Psi_{m_1,\cdots,m_N}(x_0).$$

(Remark. The reader will note that we have specialized the functional Ψ by taking the C.O.N. set $\{\alpha_j(t)\}$ to be

(1.4) $$\alpha_j(t) = 2^{1/2}\cos\frac{(2j-1)\pi t}{2}, \qquad j = 1, 2, \cdots.$$

It was found necessary to take advantage of the special properties of the cos $(j-1/2)\pi t$. In particular, these functions satisfy the boundary conditions $\alpha_j(1)=\alpha_j'(0)=0$. It is hoped that later investigations will make it possible to use less specialized $\alpha_j(t)$.)

Using the fact that the Abel mean is regular for one-dimensional series we shall prove also the following corollary on pointwise convergence:

COROLLARY. *If we assume, in addition to the hypotheses of the theorem, that for each positive integer N*

(1.5) $$\sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x_0)$$

converges absolutely, then

(1.6) $$F(x_0) = \lim_{N\to\infty} \sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x_0).$$

2. An integral expression for the difference $D(x_0)$ between a functional and its Fourier-Hermite partial sum. In order to prove our main theorem stated above we proceed to sum the Fourier-Hermite series with appropriate powers of λ as the weight factors. Putting $x=y$ and $t=\lambda^2$ in (1.2) we get

$$K(x, x, t) = \sum_{n=0}^{\infty} \lambda^{2n} \psi_n^2(x) = \sum_{n=0}^{\infty} \lambda^{2n} \pi^{-1/2} e^{-x^2} H_n^2(x),$$

and hence the series

$$\sum_{n=0}^{\infty} \lambda^{2n} H_n^2(x) = \pi^{1/2} e^{x^2} K(x, x, t)$$

converges absolutely when $|\lambda| < 1$. If we replace x by $\int_0^1 \alpha_j(t) dx_0(t)$ and understand by $\alpha_j(t)$ the specialized functions of (1.4), then when $x \in C$,

$$\begin{aligned} \prod_{j=1}^{N} \left[\sum_{m_j=0}^{\infty} \lambda^{2m_j} H_{m_j}^2 \left(\int_0^1 \alpha_j(t) dx_0(t) \right) \right] &= \sum_{m_1, \cdots, m_N=0}^{\infty} \lambda^{2(m_1+\cdots+m_N)} H_{m_1}^2 \left(\int_0^1 \alpha_1(t) dx_0(t) \right) \\ &\qquad \cdots H_{m_N}^2 \left(\int_0^1 \alpha_N(t) dx_0(t) \right) \\ &= \sum_{m_1, \cdots, m_N=0}^{\infty} \lambda^{2(m_1+\cdots+m_N)} \Psi_{m_1, \cdots, m_N}^2(x_0) \end{aligned} \tag{2.1}$$

converges absolutely for $|\lambda| < 1$. Since the functionals $\{\Psi_{m_1, \cdots, m_N}(x)\}$ form a C.O.N. set in the space $L_2(C)$, we may use the Riesz-Fischer theorem for that space. Thus

$$\underset{M\to\infty}{L.I.M.} \sum_{m_1, \cdots, m_N=0}^{M} \lambda^{m_1+\cdots+m_N} \Psi_{m_1, \cdots, m_N}(x_0) \Psi_{m_1, \cdots, m_N}(x) \tag{2.2}$$

exists, since (2.1) shows that the sum of the squares of the coefficients of $\Psi_{m_1, \cdots, m_N}(x)$ in (2.2) converges. Since $F(x)$ is bounded and Wiener measurable, $F(x) \subset L_2(C)$, and this with the existence of (2.2) justifies the interchange of order of integration and summation in:

$$\begin{aligned} &\int_C^w F(x) \sum_{m_1, \cdots, m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} \Psi_{m_1, \cdots, m_N}(x) \Psi_{m_1, \cdots, m_N}(x_0) d_w x \\ &= \sum_{m_1, \cdots, m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} \int_C^w F(x) \Psi_{m_1, \cdots, m_N}(x) d_w x \Psi_{m_1, \cdots, m_N}(x_0) \\ &= \sum_{m_1, \cdots, m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} A_{m_1, \cdots, m_N} \Psi_{m_1, \cdots, m_N}(x_0). \end{aligned} \tag{2.3}$$

Thus the convergence of the last member of (2.3) for all x_0 and for $|\lambda| < 1$ is established. In the same way that we proved the absolute convergence of the last member of (2.1) we can show that the first member (as well as the second and third) of the following expression converges absolutely:

$$
\begin{aligned}
&\sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N}\Psi_{m_1,\cdots,m_N}(x)\Psi_{m_1,\cdots,m_N}(x_0) \\
&\quad = \sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} H_{m_1}\left(\int_0^1 \alpha_1(t)dx(t)\right) \\
&\qquad \cdots H_{m_N}\left(\int_0^1 \alpha_N(t)dx(t)\right) \\
&\qquad \cdot H_{m_1}\left(\int_0^1 \alpha_1(t)dx_0(t)\right) \cdots H_{m_N}\left(\int_0^1 \alpha_N(t)dx_0(t)\right) \\
&\quad = \prod_{j=1}^{N}\left[\sum_{m_j=0}^{\infty} \lambda^{m_j} H_{m_j}\left(\int_0^1 \alpha_j(t)dx(t)\right)\right. \\
&\qquad \left. \cdot H_{m_j}\left(\int_0^1 \alpha_j(t)dx_0(t)\right)\right].
\end{aligned}
\tag{2.4}
$$

In view of (1.2) and (1.3) we have

$$
K(x, y, t) = \sum_{n=0}^{\infty} t^n \pi^{-1/2} e^{-(x^2+y^2)/2} H_n(x) H_n(y),
$$

so that

$$
\begin{aligned}
\sum_{n=0}^{\infty} t^n H_n(x) H_n(y) &= \pi^{1/2} e^{(x^2+y^2)/2} K(x, y, t) \\
&= (1 - t^2)^{-1/2} \exp\left[\frac{2xyt - t^2(x^2 + y^2)}{1 - t^2}\right].
\end{aligned}
\tag{2.5}
$$

Thus replacing t, x, y by λ, $\int_0^1 \alpha_j(t)dx_0(t)$, $\int_0^1 \alpha_j(t)dx(t)$ respectively in (2.5) and substituting in (2.4) we get

$$
\begin{aligned}
&\sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N}\Psi_{m_1,\cdots,m_N}(x)\Psi_{m_1,\cdots,m_N}(x_0) \\
&\qquad = \prod_{i=1}^{N}\left[(1 - \lambda^2)^{-1/2} \exp\left\{\frac{2\lambda u_i \xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1 - \lambda^2}\right\}\right],
\end{aligned}
\tag{2.6}
$$

where

$$(2.7) \qquad u_i = \int_0^1 \alpha_i(t)dx(t), \qquad \xi_i = \int_0^1 \alpha_i(t)dx_0(t), \qquad i = 1, 2, \cdots .$$

We now substitute (2.6) and (2.3) and obtain

$$(2.8) \qquad \begin{aligned} &\sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} A_{m_1,\cdots,m_N} \Psi_{m_1,\cdots,m_N}(x_0) \\ &\quad = (1-\lambda^2)^{-N/2} \int_C^w F(x) \exp\left[\sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \lambda^2 (u_i^2 + \xi_i^2)}{1-\lambda^2} \right] d_w x. \end{aligned}$$

If we specialize $F(x)$ for the moment by making $F(x) \equiv 1$, then since $\Psi_{0,0,\cdots,0} = \Psi_0 = H_0 \equiv 1$, and all the other Ψ's are orthogonal to Ψ_0, we have $A_{0,0,\cdots,0} = 1$, and all the other A's are zero. Hence

$$(2.9) \qquad (1-\lambda^2)^{-N/2} \int_C^w 1 \cdot \exp\left[\sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \lambda^2 (u_i^2 + \xi_i^2)}{1-\lambda^2} \right] d_w x = 1$$

with u_i given by (2.7). Then we may multiply through by $F(x_0)$ in (2.9), subtract from (2.8), and obtain

$$(2.10) \qquad \begin{aligned} D(x_0) &\equiv \sum_{m_1,\cdots,m_N=0}^{\infty} \lambda^{m_1+\cdots+m_N} A_{m_1,\cdots,m_N} \Psi_{m_1,\cdots,m_N}(x_0) - F(x_0) \\ &= c_\lambda \int_C^w [F(x) - F(x_0)] \exp\left[\sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \lambda^2 (u_i^2 + \xi_i^2)}{1-\lambda^2} \right] d_w x, \end{aligned}$$

where

$$(2.11) \qquad c_\lambda = (1-\lambda^2)^{-N/2}.$$

We have now obtained an integral expression for the difference $D(x_0)$ between a functional and its Fourier-Hermite partial sum, and we shall establish in the following sections that $D(x_0) \to 0$ as $\lambda \to 1-0$ and $N \to \infty$.

3. **Domination of $D(x_0)$ by a sum of integrals I.** To prove that $D(x_0) \to 0$ as $\lambda \to 1-0$ and $N \to \infty$, we first choose an arbitrary $\epsilon > 0$, then break up our space C and express it as a sum of sets S_j characterized by certain inequalities, and finally dominate $D(x_0)$ by a sum of integrals I_j taken over the sets S_j.

Choose $\epsilon > 0$. Then there exists a $\delta > 0$ such that if

$$(3.1) \qquad \int_0^1 [x(t) - x_0(t)]^2 dt \leq \delta^2, \qquad |F(x) - F(x_0)| < \epsilon/3.$$

We let S_0 denote the set of functions $x(t)$ satisfying this first inequality:

$$S_0 = \mathop{\mathrm{E}}_{x(t)} \left\{ \int_0^1 [x(t) - x_0(t)]^2 dt \leqq \delta^2 \right\}.$$

The complement S_0^{-1} of S_0 will then consist of all $x(t)$ satisfying the reverse inequality:

$$S_0^{-1} = \mathop{\mathrm{E}}_{x(t)} \left\{ \int_0^1 [x(t) - x_0(t)]^2 dt > \delta^2 \right\}. \tag{3.2}$$

Let

$$\mu = \frac{\delta}{2}\left(1 + \int_0^1 x_0^2(t)dt\right)^{-1/2},$$

so that

$$\mu^2 \int_0^1 x_0^2(t)dt < \frac{\delta^2}{4}. \tag{3.3}$$

Choose N_0 so great that

$$\sum_{j=N_0+1}^{\infty}\left(\int_0^1 \beta_j(t)x_0(t)dt\right)^2 < \frac{\delta^2}{16}, \tag{3.4}$$

where

$$\beta_j(t) = 2^{1/2}\sin(j - 1/2)\pi t, \qquad j = 1, 2, \cdots. \tag{3.5}$$

This can be done since this sum is just the latter part of the (necessarily convergent) Parseval sum for $\int_0^1 x_0^2(t)dt$. Now let $x_1(t)$ be the finite-sum

$$x_1(t) = \sum_{j=1}^{N_0} \beta_j(t) \int_0^1 \beta_j(s)x_0(s)ds,$$

and $x_2(t) = x_0(t) - x_1(t)$. Thus

$$\int_0^1 \beta_j(t)x_2(t)dt = \begin{cases} 0 & \text{if} \quad j \leqq N_0 \\ \displaystyle\int_0^1 \beta_j(t)x_0(t)dt & \text{if} \quad j > N_0, \end{cases} \tag{3.6}$$

$$\int_0^1 x_2^2(t)dt = \sum_{j=N_0+1}^{\infty}\left[\int_0^1 \beta_j(t)x_0(t)dt\right]^2 < \frac{\delta^2}{16}. \tag{3.7}$$

We now have from (3.2), (3.3), and (3.7) that when $x(t) \in S_0^{-1}$,

$$\int_0^1 [x(t) - x_0(t)]^2 dt - \mu^2 \int_0^1 x_0^2(t)dt - 4\int_0^1 x_2^2(t)dt > \frac{\delta^2}{2},$$

or by the Parseval equation

$$\sum_{j=1}^{\infty}\left[\left\{\int_0^1 \beta_j(t)[x(t) - x_0(t)]dt\right\}^2 - \mu^2\left\{\int_0^1 \beta_j(t)x_0(t)dt\right\}^2 - 4\left\{\int_0^1 \beta_j(t)x_2(t)dt\right\}^2\right] > \frac{\delta^2}{2},$$

and by integration by parts (using (1.4) and (3.5)),

$$\sum_{j=1}^{\infty} \frac{\{\int_0^1\alpha_j(t)d[x(t)-x_0(t)]\}^2-\mu^2\{\int_0^1\alpha_j(t)dx_0(t)\}^2-4\{\int_0^1\alpha_j(t)dx_2(t)\}^2}{\pi^2(j-1/2)^2} > \frac{\delta^2}{2}.$$

Note that the sum on the left would still converge if the numerator were independent of j and the denominator involved $(j-1/2)$ to the 3/2 power only. Let us therefore investigate the least upper bound of the product of the numerator and $(j-1/2)^{-1/2}$: that is, let

$$\eta = \sup_j \left[\frac{\{\int_0^1\alpha_j(t)d[x(t)-x_0(t)]\}^2-\mu^2\{\int_0^1\alpha_j(t)dx_0(t)\}-4\{\int_0^1\alpha_j(t)dx_2(t)\}^2}{(j-1/2)^{1/2}}\right].$$

Then we have (whether η is finite or $+\infty$)

$$\sum_{j=1}^{\infty} \frac{\eta}{\pi^2(j - 1/2)^{3/2}} > \frac{\delta^2}{2},$$

and

$$\eta > \frac{\pi^2\delta^2}{2\sum_{j=1}^{\infty}\left(j - \frac{1}{2}\right)^{-3/2}}.$$

But

$$\begin{aligned}\sum_{j=1}^{\infty}\left(j - \frac{1}{2}\right)^{-3/2} &= \left(\frac{1}{2}\right)^{-3/2} + \sum_{j=2}^{\infty}\left(j - \frac{1}{2}\right)^{-3/2} \\ &< 2^{3/2} + \int_1^{\infty}\left(s - \frac{1}{2}\right)^{-3/2} ds \\ &= 2^{3/2} + 2^{3/2} < 6,\end{aligned}$$

$$\eta > \pi^2\delta^2/12 > \delta^2/2.$$

By definition of η, whether η is finite or infinite, there must exist some value of i, say $i=i_0$, for which

$$\frac{\delta^2}{2} < \frac{\{\int_0^1\alpha_i(t)d[x(t)-x_0(t)]\}^2-\mu^2\{\int_0^1\alpha_i(t)dx_0(t)\}^2-4\{\int_0^1\alpha_i(t)dx_2(t)\}^2}{(i-1/2)^{1/2}},$$

so that if

$$(3.8)\quad u_i = \int_0^1 \alpha_i(t)dx(t), \quad \xi_i = \int_0^1 \alpha_i(t)dx_0(t), \quad \xi_i' = \int_0^1 \alpha_i(t)dx_2(t),$$

we have for $x\in S_0^{-1}$ and some corresponding value of i (namely, the i_0 chosen above),

$$(u_i - \xi_i)^2 > \frac{\delta^2}{2}\left(i - \frac{1}{2}\right)^{1/2} + \mu^2\xi_i^2 + 4\xi_i'^2.$$

Since

$$(a + b + c)^2 \leqq 3(a^2 + b^2 + c^2),$$

we have

$$\begin{aligned} | u_i - \xi_i | &> \frac{1}{3^{1/2}}\left[\frac{\delta}{2^{1/2}}\left(i - \frac{1}{2}\right)^{1/4} + \mu | \xi_i | + 2 | \xi_i' |\right] \\ &> \frac{\delta}{3}\left(i - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu | \xi_i | + | \xi_i' |. \end{aligned}$$

Thus if we denote by S_j the set

$$(3.9)\quad \begin{aligned} S_j = \underset{x(t)}{\mathrm{E}} \Bigg\{ \left| \int_0^1 \alpha_j(t)dx(t) - \xi_j \right| \\ > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu | \xi_j | + | \xi_j' | \Bigg\} \end{aligned}$$

for $j=1, 2, 3, \cdots$, we have

$$S_0^{-1} \subset \sum_{j=1}^{\infty} S_j,$$

and

$$(3.10)\qquad C = S_0 + \sum_{j=1}^{\infty} S_j = \sum_{j=0}^{\infty} S_j.$$

Now if we let

$$I_j = c_\lambda \int_{S_j}^{w} |F(x) - F(x_0)| \cdot \exp\left\{\sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right\} d_w x, \tag{3.11}$$

then we have by (2.10) and (3.10),

$$|D(x_0)| \leqq \sum_{j=0}^{\infty} I_j. \tag{3.12}$$

In the following sections we shall obtain suitable estimates for the integrals I_j.

4. One-dimensional integral expressions for the Wiener integrals I_j.

It is easy to see from (3.1), (2.9), and (2.11) that for I_0 we shall have

$$\begin{aligned} I_0 &< \frac{\epsilon}{3} c_\lambda \int_{S_0}^{w} \exp\left\{\sum_{i=1}^{N} \frac{2u_i\xi_i\lambda - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right\} d_w x \\ &\leqq \frac{\epsilon}{3} c_\lambda \int_{C}^{w} \exp\left\{\sum_{i=1}^{N} \frac{2u_i\xi_i\lambda - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right\} d_w x = \frac{\epsilon}{3}\cdot \end{aligned} \tag{4.1}$$

Consider now I_j, for $j > 0$. Since $F(x)$ is bounded by hypothesis, there exists a number B such that $|F(x)| \leqq B$. Then from (3.11)

$$\begin{aligned} I_j &\leqq 2c_\lambda B \int_{S_j}^{w} \exp\left\{\sum_{i=1}^{N} \frac{2\lambda u_i\xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right\} d_w x \\ &= 2c_\lambda B \int_{S_j}^{w} \exp\left\{\sum_{i=1}^{N} u_i^2\right\} \exp\left\{-\sum_{i=1}^{N} \frac{(\lambda\xi_i - u_i)^2}{1-\lambda^2}\right\} d_w x. \end{aligned}$$

The Wiener integral on the right depends on the u_i as $i = 1, 2, \cdots, N$ and on u_j; that is, it depends on N linear Stieltjes integrals

$$u_k = \int_0^1 \alpha_k(t)dx(t), \qquad k = 1, 2, \cdots, N,$$

if $j \leqq N$, while if $j > N$, it depends also on the additional

$$u_j = \int_0^1 \alpha_j(t)dx(t).$$

These Wiener integrals can be evaluated by (1.1), and present correspondingly an N- or $(N+1)$-fold Lebesgue integral. Let now

$$\chi_j(v) = \begin{cases} 1, & \text{if } |v - \xi_j| > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu|\xi_j| + |\xi_j'|, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Then we have by (3.9), (3.8), and (4.2)

$$\begin{aligned} I_j \leqq 2Bc_\lambda \pi^{-(N+1)/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_j(v_j) e^{-v_j^2} \\ \cdot \exp\left\{-\sum_{i=1}^{N} \frac{(\lambda\xi_i - v_i)^2}{1-\lambda^2}\right\} dv_1 \cdots dv_N dv_j \qquad (j > N), \end{aligned} \tag{4.3}$$

$$\begin{aligned} I_j \leqq 2Bc_\lambda \pi^{-N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_j(v_j) \\ \cdot \exp\left\{-\sum_{i=1}^{N} \frac{(\lambda\xi_i - v_i)^2}{1-\lambda^2}\right\} dv_1 \cdots dv_N \qquad (j \leqq N). \end{aligned} \tag{4.4}$$

In the case of the $(N+1)$-fold integral of (4.3), we have N factors of the form

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{(\lambda\xi_i - v)^2}{1-\lambda^2}\right\} dv \tag{4.5}$$

and one of the form

$$\int_{-\infty}^{\infty} \chi_j(v) e^{-v^2} dv.$$

For the N-fold integral of (4.4), there are $N-1$ factors of the type (4.5) and the other one (for v_j) is

$$\int_{-\infty}^{\infty} \chi_j(v) \exp\left\{-\frac{(\lambda\xi_j - v)^2}{1-\lambda^2}\right\} dv.$$

The integrals of type (4.5) are easily evaluated, and we get for each one $(\pi(1-\lambda^2))^{1/2}$. Thus we have (remembering (2.11)),

$$I_j \leqq \frac{2B}{\pi^{1/2}} \int_{-\infty}^{\infty} \chi_j(v) e^{-v^2} dv \qquad (j > N), \tag{4.6}$$

$$I_j \leqq \frac{2B}{(\pi(1-\lambda^2))^{1/2}} \int_{-\infty}^{\infty} \chi_j(v) \exp\left\{-\frac{(\lambda\xi_j - v)^2}{1-\lambda^2}\right\} dv \qquad (j \leqq N). \tag{4.7}$$

5. Estimation of I_j for the case $j > N$. In this section we shall refine our estimate for I_j for $j > N$. We take now $N > N_0$ (see (3.4)) and note that since $j > N > N_0$, we have, by (3.6) and (3.8), $\xi_j' = \xi_j$, and hence by (4.2) if $\chi_j(v) = 1$,

$$|v - \xi_j| > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu|\xi_j| + |\xi_j|$$
$$\geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + |\xi_j|.$$

Thus, $\chi_j(v) = 1$ implies that

$$|v| \geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4}, \tag{5.1}$$

so that we don't decrease the integral in (4.6) when we drop $\chi_j(v)$ and integrate over the v defined by (5.1). Thus

$$I_j \leqq \frac{2B}{\pi^{1/2}}\cdot 2\int_{\delta(j-1/2)^{1/4}/3}^{\infty} e^{-v^2}dv = \frac{4B}{\pi^{1/2}} \operatorname{erfc}\left\{\frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4}\right\}, \tag{5.2}$$

where

$$\operatorname{erfc}(M) = \int_M^{\infty} e^{-t^2}dt.$$

We note that this estimate of I_j is *independent* of λ.

6. Estimation of I_j for the case $j \leqq N$. In order to refine our estimate of I_j in this case we make a change of variable in (4.7):

$$s = \frac{v - \lambda\xi_j}{(1 - \lambda^2)^{1/2}},$$

so that the integral of (4.7) becomes

$$\begin{aligned}\int_{-\infty}^{\infty} \chi_j(v) \exp\left\{-\frac{(v - \lambda\xi_j)^2}{1 - \lambda^2}\right\} dv \\ = (1 - \lambda^2)^{1/2}\int_{-\infty}^{\infty} \chi_j(s(1 - \lambda^2)^{1/2} + \lambda\xi_j)e^{-s^2}ds.\end{aligned} \tag{6.1}$$

If $1 > \lambda > 1 - \mu/2$, then

$$(1 - \lambda)|\xi_j| \leqq 2^{-1}\mu|\xi_j|,$$

and if $\chi_j(s(1 - \lambda^2)^{1/2} + \lambda\xi_j) = 1$, we have from (4.2)

$$\begin{aligned}\left| s(1-\lambda^2)^{1/2} + (\lambda - 1)\xi_j \right| &> \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{\mu}{2}\left|\xi_j\right| + \left|\xi_j'\right| \\ &\geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{\mu}{2}\left|\xi_j\right|.\end{aligned}$$

Thus

$$\left| s(1-\lambda^2)^{1/2} \right| \geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4}, \tag{6.2}$$

that is,

$$|s| \geqq \frac{\delta(j - 1/2)^{1/4}}{3(1-\lambda^2)^{1/2}},$$

so that we don't decrease the integral in (6.1) when we drop χ, and integrate over the set of s defined by (6.2). Hence we obtain for $j \leqq N$ and $1 > \lambda > 1 - \mu/2$

$$I_j \leqq \frac{4B}{\pi^{1/2}} \int_{\delta(j-1/2)^{1/4}/3(1-\lambda^2)^{1/2}}^{\infty} e^{-s^2} ds = \frac{4B}{\pi^{1/2}} \operatorname{erfc} \left\{ \frac{\delta(j - 1/2)^{1/4}}{3(1-\lambda^2)^{1/2}} \right\}. \tag{6.3}$$

7. **The final estimate for $D(x_0)$.** We now combine our estimates for I_j, making both assumptions of §§5 and 6; in particular, that $N > N_0$ and $1 > \lambda > 1 - \mu/2$. Then (5.2) holds for $j > N$ and (6.3) for $j \leqq N$. We have then

$$\begin{aligned}\sum_{j=1}^{\infty} I_j \leqq \frac{4B}{\pi^{1/2}} \Bigg[& \sum_{j=1}^{N} \operatorname{erfc} \left\{ \frac{\delta(j - 1/2)^{1/4}}{3(1-\lambda^2)^{1/2}} \right\} \\ & + \sum_{j=N+1}^{\infty} \operatorname{erfc} \left\{ \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} \right\} \Bigg].\end{aligned}$$

Since

$$\operatorname{erfc}(M) = \int_M^{\infty} e^{-t^2} dt < \frac{e^{-M^2}}{2M} \qquad (M > 0)$$

it follows that

$$\begin{aligned}\sum_{j=1}^{\infty} I_j < & \frac{2B}{\pi^{1/2}} \sum_{j=1}^{\infty} \frac{3(1-\lambda^2)^{1/2}}{\delta(j - 1/2)^{1/4}} \exp \left\{ \frac{-\delta^2(j - 1/2)^{1/2}}{9(1-\lambda^2)} \right\} \\ & + \frac{2B}{\pi^{1/2}} \sum_{j=N+1}^{\infty} \frac{3}{\delta(j - 1/2)^{1/4}} \exp \left\{ \frac{-\delta^2(j - 1/2)^{1/2}}{9} \right\}.\end{aligned} \tag{7.1}$$

But the sum in the second term on the right above converges, so that we may take N_ϵ ($\geqq N_0$) large enough to make this term less than $\epsilon/3$ for $N>N_\epsilon$. Moreover, the series of the first term on the right converges for all λ, $0<\lambda<1$, and each term of this sum is decreasing in λ as $\lambda\to 1-0$, and has the limit zero. Thus the first term on the right of (7.1) has the limit zero. We may therefore choose λ_ϵ in the interval $(1-\mu/2,\ 1)$ so that the first term is less than $\epsilon/3$ when $\lambda_\epsilon<\lambda<1$. Thus when $N>N_\epsilon$ and $\lambda_\epsilon<\lambda<1$ we have from (3.12) and (4.1),

$$\left| D(x_0) \right| \leqq \sum_{j=0}^{\infty} I_j < \epsilon.$$

Referring to the theorem of §1 and the definition of $D(x_0)$ in (2.10), we see that the theorem is established.

8. **Proof of the corollary.** The regularity of the Abel mean for the one-dimensional case, together with the absolute convergence of (1.5) assumed in the hypothesis of the corollary, at once permit us to write

$$\begin{aligned}\lim_{\lambda\to 1\to 0} \sum_{n=0}^{\infty} \lambda^n \sum_{m_1+\cdots+m_N=n} & A_{m_1,\ldots,m_N}\Psi_{m_1,\ldots,m_N}(x_0) \\ &= \sum_{n=0}^{\infty} \sum_{m_1+\cdots+m_N=n} A_{m_1,\ldots,m_N}\Psi_{m_1,\ldots,m_N}(x_0) \\ &= \sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\cdots,m_N}\Psi_{m_1,\ldots,m_N}(x_0).\end{aligned}$$

Now let $\epsilon>0$. Then by the theorem there exists an $N_0>0$ and a λ_0 in $(0,\ 1)$ such that when $N>N_0$ and $\lambda_0<\lambda<1$,

$$\text{(8.1)} \qquad \left| F(x_0) - \sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\ldots,m_N}\lambda^{m_1+\cdots+m_N}\Psi_{m_1,\ldots,m_N}(x_0) \right| < \epsilon.$$

Taking the limit with respect to λ in (8.1) we get

$$\left| F(x_0) - \sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\ldots,m_N}\Psi_{m_1,\ldots,m_N}(x_0) \right| \leqq \epsilon.$$

But this establishes (1.6), and hence the corollary.

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SUMMABILITY WITH A GOVERNOR OF INTEGRAL ORDER[1]

G. MILTON WING

1. Introduction. In this paper the method of summability with a governor, introduced by G. Piranian [5],[2] is generalized, and the effectiveness of the new methods is compared with that of the Cesàro methods of integral order.

2. Preliminary theorems. We shall consider throughout this paper an arbitrary formal series

$$\sum_{j=0}^{\infty} a_j. \tag{A}$$

Using the notation $s_n = \sum_{j=0}^{n} a_j$, we consider the standard Toeplitz transformation

$$\sigma_n = \sum_{j=0}^{\infty} b_{nj} s_j. \tag{1}$$

The following theorem is well known.

THEOREM 1 (SILVERMAN-TOEPLITZ [8, 9]). *A necessary and sufficient condition that the transformation* (1) *be regular is that the following hold*:

$$\sum_{j=0}^{\infty} |b_{nj}| < M, \quad M \text{ independent of } n, \tag{α}$$

$$\lim_{n\to\infty} b_{nj} = 0 \quad \text{for all } j, \tag{β}$$

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} b_{nj} = 1. \tag{γ}$$

Consider now an arbitrary sequence $\{\lambda_n\}$ of nonnegative terms, not all zero, and set $\Lambda_n = \sum_{j=0}^{n} \lambda_j$. Then $\Lambda_n > 0$ for all sufficiently large n. If, for these n, we let $b_{nj} = \lambda_{n-j}/\Lambda_n$ for $j \leq n$ and $b_{nj} = 0$ for $j > n$, then (1) becomes

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[1] The material of this paper forms a part of a thesis, prepared under Professor W. Seidel and presented in June, 1947 to the Graduate School of The University of Rochester in partial fulfillment of the requirements for the degree Master of Science.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$\sigma_n = \frac{1}{\Lambda_n} \sum_{j=0}^{n} s_j \lambda_{n-j}. \tag{2}$$

(For the finite number of cases in which $\Lambda_n = 0$ we define σ_n to be zero.)

If $\lim_{n\to\infty} \sigma_n$ exists we say that (A) is Nörlund summable by means of the sequence $\{\lambda_n\}$, or summable (N, λ) [4]. It is easily seen from Theorem 1 that (N, λ) is regular if and only if $\lambda_n = o(\Lambda_n)$. We shall consider only regular Nörlund methods.

It is known that any two regular Nörlund methods, (N, λ) and (N, ω), are consistent. If, in particular, every series summable (N, ω) is summable (N, λ) we shall write $(N, \omega) \subset (N, \lambda)$.

THEOREM 2 (M. RIESZ [7]). *A necessary and sufficient condition that* $(N, \omega) \subset (N, \lambda)$, *where* $\omega_0 > 0$, *is that*

$$\sum_{j=0}^{n} \Omega_{n-j} | q_j | = O(\Lambda_n),$$

and $q_n = o(\Lambda_n)$, *where* $\Omega_n = \sum_{j=0}^{n} \omega_j$ *and* $\{q_j\}$ *is defined by the formal relation*

$$\sum_{j=0}^{\infty} q_j x^j = \sum_{j=0}^{\infty} \lambda_j x^j \Big/ \sum_{j=0}^{\infty} w_j x^j.$$

If we choose $\omega_n = C_{n+k-1,k-1}$ (k a positive integer) the method (N, ω) becomes the classical Cesàro mean of order k, or (C, k) [1].

Defining $\Delta\lambda_n = \lambda_n - \lambda_{n-1}$, and, inductively, $\Delta^k\lambda_n = \Delta^{k-1}\lambda_n - \Delta^{k-1}\lambda_{n-1}$ with $\lambda_n = 0$ for $n < 0$, we obtain the following results:

COROLLARY 1. *A necessary and sufficient condition that* $(C, k) \subset (N, \lambda)$ *is that*

$$\sum_{j=0}^{n} C_{n-j+k,k} | \Delta^k \lambda_j | = O(\Lambda_n). \tag{3}$$

To see this we observe that

$$\sum_{j=0}^{\infty} w_j x^j = \sum_{j=0}^{\infty} C_{j+k-1,k-1} x^j = (1 - x)^{-k},$$

so that

$$\sum_{j=0}^{\infty} q_j x^j = (1 - x)^k \sum_{j=0}^{\infty} \lambda_j x^j = \sum_{j=0}^{\infty} (\Delta^k \lambda_j) x^j.$$

Hence $q_j = \Delta^k \lambda_j$. Further, $\Omega_n = \sum_{j=0}^{n} C_{j+k-1,k-1} = C_{n+k,k}$. Finally, since

$\lambda_n = o(\Lambda_n)$, it is clear that $\Delta^k\lambda_n = o(\Lambda_n)$, and Theorem 2 yields the result.

COROLLARY 2. *A sufficient condition that* $(C, k) \subset (N, \lambda)$ *is that* $\Delta^k\lambda_n \geqq 0$ *for all* n.

Here we have

$$\begin{aligned}\sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^k\lambda_j \right| &= \sum_{j=0}^{n} C_{n-j+k,k} \left\{ \Delta^{k-1}\lambda_j - \Delta^{k-1}\lambda_{j-1} \right\} \\ &= \sum_{j=0}^{n-1} \left\{ C_{n-j+k,k} - C_{n-j-1+k,k} \right\} \Delta^{k-1}\lambda_j + \Delta^{k-1}\lambda_n \\ &= \sum_{j=0}^{n} C_{n-j+k-1,k-1} \Delta^{k-1}\lambda_j.\end{aligned}$$

Repeating the argument k times we find

$$\sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^k\lambda_j \right| = \sum_{j=0}^{n} \lambda_j = \Lambda_n,$$

so that (3) is satisfied.

One more useful corollary is now obtainable.

COROLLARY 3.[3] *If the sequence* $\{\lambda_n\}$ *is such that the sequence* $\{\Delta^k\lambda_n\}$ $(n = 0, 1, \cdots)$ *has at least one but only a finite number of negative elements, a necessary and sufficient condition that* $(C, k) \subset (N, \lambda)$ *is that* $n^k = O(\Lambda_n)$.

Let $\Delta^k\lambda_n \geqq 0$ for all $n > N_0$. We write

$$\begin{aligned}\sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^k\lambda_j \right| &= \sum_{j=0}^{n} C_{n-j+k,k} \Delta^k\lambda_j \\ &\quad + \sum_{j=0}^{N_0} C_{n-j+k,k} \left\{ \left| \Delta^k\lambda_j \right| - \Delta^k\lambda_j \right\} \\ &= \Lambda_n + \sum_{j=0}^{N_0} C_{n-j+k,k} \left\{ \left| \Delta^k\lambda_j \right| - \Delta^k\lambda_j \right\}.\end{aligned}$$

Since $C_{n,k} \sim n^k/k!$ the result is obvious.

3. **The method** (G, k). We shall henceforth consider the series (A) under the additional stipulation that not all $a_n = 0$. We define

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[3] A weaker result was originally stated. The author is indebted to the referee for the improved formulation.

$p_n(0)=|a_n|$, and, by induction, $p_n(k)=\sum_{j=0}^{n} p_j(k-1)$, and set

$$G_n(k) = \frac{1}{p_n(k+1)} \sum_{j=0}^{n} s_j p_{n-j}(k) \qquad (k \geqq 1). \tag{4}$$

(For the finite number of cases for which $p_n(k+1)$ may be zero we define $G_n(k)$ to be zero.)

Definition. If $\lim_{n\to\infty} G_n(k)=\sigma$ exists and if $\lim_{n\to\infty} p_n(k)/p_n(k+1)=0$, the series (A) is summable to σ by means of a governor of order k, or (G, k) summable to σ.[4]

The method (G, k) reduces to the method (G) of Piranian when $k=1$. It is easily shown to be a regular Nörlund method. A stronger result is the following:

Theorem 3. *For every positive integer* k, $(G, k)\subset(G, k+1)$.

Let (A) be (G, k) summable to σ. Since

$$\begin{aligned} G_n(k+1) &= \frac{1}{p_n(k+2)} \sum_{j=0}^{n} s_j p_{n-j}(k+1) \\ &= \frac{1}{p_n(k+2)} \sum_{j=0}^{n} s_j \sum_{i=0}^{n-j} p_i(k) \\ &= \frac{1}{p_n(k+2)} \sum_{m=0}^{n} \sum_{r=0}^{m} s_r p_{m-r}(k) \\ &= \sum_{m=0}^{n} G_m(k) p_m(k+1) \Big/ \sum_{m=0}^{n} p_m(k+1), \end{aligned}$$

the sequence $\{G_n(k+1)\}$ is obtained from the sequence $\{G_n(k)\}$ by a Riesz transformation (R, q) with $q_n=p_n(k+1)$ [2, 6]. Further, $\lim_{n\to\infty} p_n(k+1)=+\infty$ so that the transformation is regular and $\lim_{n\to\infty} G_n(k+1)=\sigma$.

That $\lim_{n\to\infty} p_n(k+1)/p_n(k+2)=0$ follows from a result of Piranian [5, Lemma]. Thus (A) is $(G, k+1)$ summable.

All of the results known for summability (G) are readily extended to the case (G, k), although the details are frequently tedious. Since the topic of greatest interest here is the relation between Cesàro summability and the new method we shall not investigate these extensions. The next theorem will, however, prove useful in our work. When two or more series are under consideration we shall use the notations $p_n(k;\, a)$, $s_n(a)$, $G_n(k;\, a)$, and so on, to refer to the series (A).

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[4] For the sake of completeness we may agree that a series with all zero terms is (G, k) summable to the value zero for all positive integers k. This will be entirely consistent with all further work.

THEOREM 4. *Let* (A) *be such that* $\lim_{n\to\infty} G_n(k;\, a)=\sigma$ *and*

$$\lim_{n\to\infty} \sum_{j=0}^{n} C_{n-j+k-1,k-1}s_j(a)/p_n(k+1;\, a) = 0. \tag{5}$$

Let

$$\sum_{j=0} b_j \tag{B}$$

be such that $b_j=a_j$ *for* $j\neq i$ (*i fixed*) *and* $b_i=a_i+h$. *Then, in order that* $\lim_{n\to\infty}G_n(k;\, b)=\sigma+h$, *it is necessary and sufficient that the condition* $\lim_{n\to\infty}p_n(k;\, a)/p_n(k+1;\, a)=0$ *hold.*

We first observe that

$$\begin{aligned} p_n(r) &= \sum_{j=0}^{n} C_{n-j+r-2,r-2}p_j(1) \\ &= \sum_{j=0}^{n} C_{n-j+r-1,r-1}\,|\, a_j\,|, \end{aligned} \tag{6}$$

as may easily be verified by induction. Thus

$$\begin{aligned} &s_n(b) = s_n(a); \qquad p_n(r;\, b) = p_n(r;\, a) \qquad\qquad (n<i;\, r\geqq 0); \\ &s_n(b) = s_n(a)+h; \\ &p_n(r;\, b) = p_n(r;\, a) + \{\,|\, a_i+h\,| - |\, a_i\,|\,\}C_{n-i+r-1,r-1} \qquad (n\geqq i;\, r\geqq 1). \end{aligned}$$

Hence, for $n\geqq i$,

$$\frac{p_n(1;\, b)}{p_n(1;\, a)} = 1 + \frac{\{\,|\, a_i+h\,| - |\, a_i\,|\,\}}{p_n(1;\, a)},$$

while for $n\geqq i$, $r\geqq 2$,

$$\frac{p_n(r;\, b)}{p_n(r;\, a)} = 1 + \{\,|\, a_i+h\,| - |\, a_i\,|\,\}\, \frac{C_{n-i+r-1,r-1}}{C_{n+r-1,r-1}} \cdot \left\{ \frac{C_{n+r-1,r-1}}{\sum_{j=0}^{n} C_{n-j+r-2,r-2}p_j(1;\, a)} \right\}.$$

Since only the case of divergent series is of interest we may assume that $\lim_{n\to\infty} p_n(1;\, a)=+\infty$, so that

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} C_{n-j+r-2,r-2}\, p_j(1;\, a)}{C_{n+r-1,r-1}} = +\,\infty. \tag{7}$$

Therefore, $p_n(r;\, b) = \{1+\epsilon_n(r)\} p_n(r;\, a)$, where $\lim_{n\to\infty}\epsilon_n(r) = 0$, $r \geqq 1$. Now, for $n > i$,

$$\begin{aligned}
&\sum_{j=0}^{n} s_j(b) p_{n-j}(k;\, b) \\
&\qquad = \sum_{j=0}^{n-i} s_j(a) \{ p_{n-j}(k;\, a) + [\,|\, a_i + h \,| - |\, a_i \,|\,] C_{n-i-j+k-1,k-1} \} \\
&\qquad\quad + \sum_{j=n-i+1}^{n} s_j(a) p_{n-j}(k;\, a) + h \sum_{j=i}^{n} p_{n-j}(k;\, b).
\end{aligned}$$

Thus

$$\begin{aligned}
G_n(k; b) &= \frac{G_n(k;\, a)}{1 + \epsilon_n(k+1)} + h \left\{ 1 - \frac{1}{p_n(k+1;\, b)} \sum_{j=n-i+1}^{n} p_j(k;\, b) \right\} \\
&\quad + \frac{\{\, |\, a_i + h \,| - |\, a_i \,|\, \}}{1 + \epsilon_n(k+1)} \, \frac{p_{n-i}(k+1;\, a)}{p_n(k+1;\, a)} \, \frac{\sum_{j=0}^{n-i} C_{n-i-j+k-1,k-1} s_j(a)}{p_{n-i}(k+1;\, a)}.
\end{aligned}$$

Since $0 \leqq p_{n-i}(k+1;\, a)/p_n(k+1;\, a) \leqq 1$ the third term approaches zero, by (5). The result now follows readily.

4. Relation between (G, k) and (C, k) summability.

THEOREM 5. *For every positive integer k, $(C, k) \subset (G, k)$.*

Let us assume that (A) is summable (C, k) to σ. That is, we have $\lim_{n\to\infty} T_n(k) = \sigma$, where $T_n(k) = S_n(k)/C_{n+k,k}$ with $S_n(k) = \sum_{j=0}^{n} C_{n-j+k-1,k-1}\, s_j$. From (6) we have

$$\begin{aligned}
\sum_{j=0}^{n} s_j p_{n-j}(k) &= \sum_{j=0}^{n} s_j \sum_{r=0}^{n-j} C_{r+k-1,k-1} \,|\, a_{n-j-r} \,| \\
&= \sum_{j=0}^{n} |\, a_{n-j} \,| \sum_{r=0}^{j} C_{j-r+k-1,k-1} s_r \\
&= \sum_{j=0}^{n} |\, a_{n-j} \,|\, C_{j+k,k} T_j(k).
\end{aligned} \tag{8}$$

Thus

$$G_n(k) = \frac{\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k} T_j(k)}{\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k}}. \tag{9}$$

Now (9) represents a Toeplitz transformation applied to the convergent sequence $\{T_n(k)\}$ where, in the notation of Theorem 1,

$$b_{ni} = \frac{| a_{n-i} | C_{i+k,k}}{\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k}} \qquad (i \leqq n),$$

$$b_{ni} = 0 \qquad (i > n).$$

That conditions (α) and (γ) of Theorem 1 are satisfied is evident. We need investigate only condition (β).

Let $a_m \neq 0$. Then, for $n \geqq m$,

$$\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k} \geqq | a_m | C_{n-m+k,k}.$$

Hence

$$b_{ni} \leqq \frac{| a_{n-i} | C_{i+k,k}}{| a_m | C_{n-m+k,k}} \qquad (n \geqq \max (i, m)).$$

But, since (A) is summable (C, k), $a_n = o(n^k)$ [3, p. 484]. Therefore, $\lim_{n\to\infty} b_{ni} = 0$ for all i. Thus (9) is regular and $\lim_{n\to\infty} G_n(k) = \sigma$.

The condition $\lim_{n\to\infty} p_n(k)/p_n(k+1) = 0$ must still be established. We consider the series (B). Since (A) is (C, k) summable to σ, (B) is (C, k) summable to $\sigma + h$. Further,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{p_n(k+1; a)} \sum_{j=0}^{n} C_{n-j+k-1,k-1} s_j(a) &= \lim_{n\to\infty} T_n(k; a) \left\{ \frac{C_{n+k,k}}{p_n(k+1; a)} \right\} \\ &= 0, \end{aligned}$$

since the bracketed expression approaches zero (see (7)). Therefore Theorem 4 applies and $\lim_{n\to\infty} p_n(k; a)/p_n(k+1; a) = 0$.

THEOREM 6. *For every integer $k > 1$ there exist series (G, k), but not $(G, k-1)$, summable.*[5]

We establish this result by example. Choosing the integer m such that $m > k+1$ we consider the series $\sum_{j=0}^{\infty} d_j$ for which

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[5] That there are divergent series which are $(G, 1)$ summable is evident from Theorem 5.

$$S_{mn}(k-1) = C_{n+k-1,k-1} \qquad (n = 0, 1, 2, \cdots),$$
$$S_p(k-1) = 0 \qquad (p \neq mn).$$

Recalling that $S_n(1) = \sum_{j=0}^{n} s_j$, $S_n(r) = \sum_{j=0}^{n} S_j(r-1)$ for $r > 1$ [3, p. 466], we compute

$$S_{mn}(k-2) = C_{n+k-1,k-1} \qquad (n = 0, 1, 2, \cdots),$$
$$S_{mn+1}(k-2) = -C_{n+k-1,k-1},$$
$$S_p(k-2) = 0 \qquad (p \neq mn \text{ or } mn+1).$$

Repeating the argument we eventually find

$$s_{mn+j} = (-1)^j C_{k-1,j} C_{n+k-1,k-1} \quad (n = 0, 1, 2, \cdots; 0 \leqq j \leqq k-1),$$
$$s_p = 0 \qquad (p \neq mn + j),$$

with

$$d_{mn+j} = (-1)^j C_{k,j} C_{n+k-1,k-1} \qquad (n = 0, 1, 2, \cdots; 0 \leqq j \leqq k),$$
$$d_p = 0 \qquad (p \neq mn + j).$$

We notice in particular that $d_{mn-1} = 0$ $(n = 1, 2, 3, \cdots)$. Therefore, $\sum_{j=0}^{mn-1} |d_j| S_{mn-j-1}(k-1) = 0$. To see this we observe that $S_{mn-j-1}(k-1) \neq 0$ only when $mn - j - 1 = mp$, p an integer, and here $j = m(n-p) - 1$, so that $d_j = 0$. Therefore, by (8), $G_{mn-1}(k-1) = 0$ $(n = 1, 2, 3, \cdots)$. Hence, if $\lim_{n\to\infty} G_n(k-1) = \sigma$ exists, then $\sigma = 0$.

But, for r such that $mn \leqq r < m(n+1)$, $S_r(k) = C_{n+k,k}$, and thus

$$\frac{C_{n+k,k}}{C_{m(n+1)+k,k}} < \frac{S_r(k)}{C_{r+k,k}} = T_r(k) \leqq \frac{C_{n+k,k}}{C_{mn+k,k}}.$$

Therefore $\lim_{n\to\infty} T_n(k) = 1/m^k$, so that $\sum_{j=0}^{\infty} d_j$ is (C, k) summable. By Theorem 5, then, the series is also (G, k) summable and $\lim_{n\to\infty} G_n(k) = 1/m^k$. Finally, if $\sum_{j=0}^{\infty} d_j$ were also $(G, k-1)$ summable to σ, then σ would be $1/m^k$. This cannot be the case and our series therefore has the desired properties.

We have seen that the method (G, k) is at least as effective as the method (C, k). We shall now determine conditions under which (G, k) is stronger than the corresponding Cesàro mean.

THEOREM 7. *Let* (A) *be* (C, k) *summable. Then, in order that* (A) *be* (G, r) *summable,* $r < k$, *it suffices that* $\lim_{n\to\infty} p_n(r)/p_n(r+1) = 0$ *and that any one of the following conditions holds:*

$$\text{(I)} \qquad \sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^{k-r}(|a_j|) \right| = O(p_n(r+1));$$

$$\text{(II)} \qquad \Delta^{k-r}(|a_n|) \geqq 0 \quad \textit{for all } n;$$

(III) *There exists an* N_0 *such that* $\Delta^{k-r}(|a_n|) \geqq 0$ *for all* $n > N_0$ *and* $n^k = O(p_n(r+1))$.

To obtain these results from the corollaries of Theorem 2 we merely notice that

$$\begin{aligned} \Delta^k p_n(r) &= \Delta^{k-1}\{p_n(r) - p_{n-1}(r)\} = \Delta^{k-1}p_n(r-1) \\ &= \cdots = \Delta^{k-r+1}p_n(1) = \Delta^{k-r}(|a_n|). \end{aligned}$$

It is interesting to observe that Theorem 5 could be obtained from this result if a direct proof that (C, r) summability implies $\lim_{n\to\infty} p_n(r)/p_n(r+1) = 0$ were available. The author has been unable to find such a proof.

We conclude by giving a new proof of a theorem due to Piranian [5, Theorem 8].

THEOREM 8 (PIRANIAN). *Let* $f(x)$ *be a polynomial with real coefficients. Then*

$$\text{(10)} \qquad \sum_{j=0}^{\infty} (-1)^j f(j)$$

is $(G, 1)$ *summable.*

Let $f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$, where $a_0 > 0$. It is known that (10) is $(C, k+1)$ summable [3, p. 479], and that $\Delta^k f(n) = a_0 k!$ for $n > k$. Further, there exists an N_0 such that $f(n) > 0$ for all $n > N_0$. Hence, for $n > \max\ (k, N_0)$, $\Delta^k(|f(n)|) > 0$. Finally, we have $p_n(2) \sim a_0 n^{k+2}/(k+1)(k+2)$ and Theorem 7 (III) yields the result.

For $a_0 < 0$ we need only consider $g(x) = -f(x)$.

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ON THE REPRESENTATION OF A FUNCTION AS A HELLINGER INTEGRAL

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We derive in this note a necessary and sufficient condition that a nondecreasing, continuous function h of a single variable x be representable as a Hellinger integral of the form $\int_0^x (df)^2/dg$. This condition was first proved by Hellinger in his dissertation [1].[1] Other proofs have been given by Hahn [2] and Hobson [3], who transform to Lebesgue integrals and make use of Lebesgue theory. Hellinger's proof and the less complicated proof given here have a certain simplicity in that they avoid reliance on these notions and even remain entirely within the range of monotone functions.

We consider nondecreasing functions of a real variable x on the interval $0 \leqq x \leqq 1$ (henceforth denoted as $[0, 1]$). For such a function $f(x)$ and a closed interval Δ with end points x_1 and x_2 ($x_1 \leqq x_2$), we define a new function $f_\Delta(x)$ to denote the length of the interval on the f-axis determined by the interval on the x-axis common to Δ and $(0, x)$. More precisely, denoting

$$f(x \pm 0) = \lim_{h \to 0} f(x \pm |h|) \qquad \text{if } 0 < x < 1,$$

$$f(0 - 0) = f(0); \qquad f(1 + 0) = f(1),$$

we define

$$f_\Delta(x) = \begin{cases} 0 & \text{if } 0 \leqq x < x_1, \\ f(x + 0) - f(x_1 - 0) & \text{if } x_1 \leqq x \leqq x_2, \\ f(x_2 + 0) & \text{if } x_2 < x \leqq 1. \end{cases} \tag{1}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

The interval $[f(x_1-0), f(x_2+0)]$ on the f-axis will be denoted as $f(\Delta)$. We retain the notation Δf for the proper difference $f(x_2)-f(x_1)$.

Given sets $\{E_i\} \subset [0, 1]$ for which the symbols $f_{E_i}(x)$ and $f(E_i)$ have been defined, we define

$$\left.\begin{aligned} f_{\sum E_i}(x) &= \sum f_{E_i}(x) \\ f\left(\sum E_i\right) &= \sum f(E_i) \end{aligned}\right\} \quad \text{if } E_i \cdot E_j = 0 \text{ when } i \neq j. \tag{2}$$

$$\left.\begin{aligned} f_{E_2-E_1}(x) &= f_{E_2}(x) - f_{E_1}(x) \\ f(E_2 - E_1) &= f(E_2) - f(E_1) \end{aligned}\right\} \quad \text{if } E_1 \subset E_2.$$

The sets which we shall consider will be constructed from sets of intervals in a manner such that (1) and (2) will assign to each such set E a function $f_E(x)$ called the measure function of E on the f-axis and a set $f(E)$ of points on the f-axis. Our $f_E(1)$ is just the measure with respect to $f(x)$ of the set E as defined in [4; p. 277]. It is immediate that if $f_{E_D}(x)$ denotes the measure on the $f_E(x)$ axis of the set D on the x-axis, then

$$f_{E_D}(x) = f_{E \cdot D}(x). \tag{3}$$

We now consider functions $f(x)$, $g(x)$, and $h(x)$ which are nondecreasing and continuous in $[0, 1]$ and satisfy the inequality

$$(\Delta f)^2 \leqq \Delta g \Delta h \qquad \text{on every subinterval } \Delta \text{ of } [0, 1]. \tag{4}$$

Let M be an arbitrary set on the x-axis. For arbitrary $\epsilon > 0$, the set $g(M)$ on the g-axis may be enclosed in a set I_ϵ of countably many mutually disjoint intervals d_i such that the measures on the g-axis of the sets $g(M)$ and I_ϵ differ by less than ϵ; that is, the inverse function $x(g)$ defines a corresponding set $x(I_\epsilon)$ of the x-axis which we denote for brevity as X_ϵ such that

$$0 \leqq g_{X_\epsilon}(1) - g_M(1) \leqq \epsilon. \tag{5}$$

Now for Δ an arbitrary interval of the x-axis and Δ_i the intersection of Δ with $x(d_i)$, we have:

$$\Delta f_M \leqq \Delta f_{X_\epsilon} = \sum_{i=1}^{\infty} \Delta_i f. \tag{6}$$

But $(\Delta_i f)^2 \leqq \Delta_i g \Delta_i h$ on every Δ_i so that from the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{\infty} \Delta_i f\right)^2 \leqq \sum_{i=1}^{\infty} \Delta_i g \sum_{i=1}^{\infty} \Delta_i h \leqq \Delta g_{X_\epsilon} \Delta h. \tag{7}$$

With use of (5) and (6), this gives

$$(\Delta f_M)^2 \leqq \Delta g_M \Delta h. \tag{8}$$

Replacing (4) by (8) and carrying out an analogous argument for a set N and the h-axis, we have

$$(\Delta f_{M\cdot N})^2 \leqq \Delta g_M \Delta h_N \qquad \text{if (4) holds.} \tag{9}$$

THEOREM. *To a given pair of nondecreasing, continuous functions* $g(x)$ *and* $h(x)$ *for which* $g(0)=0=h(0)$ *there corresponds an* $f(x)$ *such that*

$$h(x) = \int_0^x \frac{(df)^2}{dg}$$

if and only if for every set E *of the x-axis such that* $g_E(1)=g(1)$, *we have* $h_E(1)=h(1)$.

For the proof, we use the following elementary properties of Hellinger integrals [1] which hold for arbitrary functions $g(x)$ and $h(x)$ that are nondecreasing and continuous in $[0, 1]$:

(10)
- a. Existence of $t(x)=\int_0^x (du)^2/dg$ implies $t(x)=\int_0^x \left(d\int_0^x (dg\,dt)^{1/2}\right)^2/dg$.
- b. $f(x)=\int_0^x (dg\,dh)^{1/2}$ exists and $(\Delta f)^2 \leqq \Delta g \Delta h$.
- c. The inequality $(\Delta f)^2 \leqq \Delta g \Delta h$ implies that $s(x)=\int_0^x (df)^2/dg$ exists and $(\Delta f)^2/\Delta g \leqq \Delta s \leqq \Delta h$ if $\Delta g \neq 0$.

It follows from (10a) that the desired representation of $h(x)$ exists only if it is given by $\int_0^x (df)^2/dg$ where $f(x)=\int_0^x (dg\,dh)^{1/2}$. By (10b), $(\Delta f)^2 \leqq \Delta g \Delta h$. Let E be any set such that $g_E(1)=g(1)$. Then from (8), we have on replacing M by the complement $\overline{E}$ of E with respect to $[0, 1]$ and taking for Δ the interval $[0, 1]$ that

$$(f_{\overline{E}}(1))^2 \leqq g_{\overline{E}}(1)h(1) = 0 \tag{11}$$

and consequently $f_E(1)=f(1)$. It follows from application of (11) to (9) that

$$(\Delta f)^2 \leqq \Delta g \Delta h_E. \tag{12}$$

Property (10c) with (12) gives that $\Delta s \leqq \Delta h_E$ so that the function $a(x)=h_E(x)-s(x)$ is nondecreasing and if $\Delta g \neq 0$

$$\begin{aligned} 0 \leqq \Delta a = \Delta h_E - \Delta s &\leqq \Delta h - (\Delta f)^2/\Delta g \\ &= ((\Delta g \Delta h)^{1/2} - \Delta f)((\Delta h/\Delta g)^{1/2} + \Delta f/\Delta g). \end{aligned}$$

Consequently

$$(13)\qquad 0 \leqq \Delta a = \Delta h_E - \Delta s \leqq 2(\Delta h/\Delta g)^{1/2}((\Delta g\Delta h)^{1/2} - \Delta f),$$
$$\text{if } g_E(1) = g(1) \text{ and } \Delta g \neq 0.$$

We next choose a sequence of divisions D_n of the interval (0, 1) into finitely many nonoverlapping intervals (that is, no two intervals of D_n have more than an end point in common) such that each D_n is formed from D_{n-1} by addition of finitely many division points and

$$(14)\qquad \sum_{D_n} ((\Delta g\Delta h)^{1/2} - \Delta f) \leqq 4^{-n}.$$

That such a choice is possible is implied by (10a).

For each division D_n, we distinguish two types of intervals:

(15) a. The set G_n of intervals such that $\Delta g \geqq 4^{-n}\Delta h$;
b. The set S_n of intervals such that $\Delta g < 4^{-n}\Delta h$.

Then from (13), (14), (15), we have

$$(16)\qquad a_{G_n}(1) = \sum_{G_n} \Delta a \leqq 2^{-n+1},$$

$$(17)\qquad g_{S_n}(1) = \sum_{S_n} \Delta g \leqq 4^{-n}h(1).$$

We define $R_n = \prod_{m=n}^{\infty} G_m$ and $R = \lim_{n\to\infty} R_n$. It is immediate that

$$(18)\qquad a_R(1) = \lim_{n\to\infty} a_{R_n}(1) \leqq \lim_{n\to\infty} a_{G_n}(1) = 0 \qquad \text{(see (16))},$$

and since $\bar{R} \subset \sum_{m=n}^{\infty} S_m$,

$$g_{\bar{R}}(1) \leqq \lim_{n\to\infty} \sum_{m=n}^{\infty} g_{S_m}(1) = 0 \qquad \text{(see (17))}.$$

Thus R is a set of the type E assumed in (13). Consequently, $0 \leqq \Delta a \leqq \Delta h_R$, and it follows from (9) and (3) that $a_{\bar{R}}(1) \leqq h_{R\cdot\bar{R}}(1) = 0$. Hence

$$(19)\qquad a_{\bar{R}}(1) = 0.$$

The function $a(x)$ is nondecreasing with $a(0) = 0$ and by (18) and (19) has $a(1) = 0$. Thus $a(x) \equiv 0$, that is,

$$(20)\qquad s(x) = h_R(x).$$

Therefore,

$$s(x) = \int_0^x \frac{\left(d\int_0^x (dgdh)^{1/2}\right)^2}{dg}$$

is the measure function on the h-axis of a set R of the x-axis. This integral is actually equal to $h(x)$ if and only if $h_B(1)=h(1)$. The condition that $g_B(1)=g(1)$ implies $h_B(1)=h(1)$ is surely necessary to provide that $h(x)=s(x)$, for $g_B(1)=g(1)$ implies $s_B(1)=s(1)$. On the other hand, if $g_B(1)=g(1)$ does imply $h_B(1)=h(1)$, the condition that $h_B(1)=h(1)$ is fulfilled and $h(x)\equiv s(x)$. This completes the proof.

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ON THE EXTENSION OF A TRANSFORMATION

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0. Introduction. In a problem on surface area the writer and Helsel[1] were confronted with the following question. Can a Lipschitzian transformation from a set in a Euclidean three-space into a Euclidean three-space be extended to a Lipschitzian transformation defined on the whole space? The affirmative answer to this question has been given by Kirszbraun.[2] In fact, Kirszbraun shows this result for any Euclidean spaces (see also Valentine).[3] In studying these papers the writer noted that a more general extension problem could be formulated and a different method of proof to the problem could be obtained. To formulate the more general problem we first give some definitions.

Let M be a metric space, the distance between two points $p_1, p_2 \in M$ being denoted by p_1p_2. Let $\mathcal{P}(M)$ be the class of real-valued continuous functions $g(t)$, $0 \leqq t < \infty$, which satisfy the conditions: (a) $g(0)=0$, (b) $g(t)>0$ for $t>0$, (c) for any finite number of points $p_0, p_1, \cdots, p_m$ in M the real quadratic form $\sum_{i,j=1}^{m} [g(p_0p_i)^2+g(p_0p_j)^2-g(p_ip_j)^2]\xi_i\xi_j$ is positive. Let $g(t) \in \mathcal{P}(M)$. A transformation $p^*=\phi(p)$ from a set E in M into a metric space M^* will be said to satisfy the condition $C(g)$ on E if, for every pair of points $p_1, p_2 \in E$, $p_1^*p_2^* \leqq g(p_1p_2)$, where $p_i^*=\phi(p_i)$, $i=1, 2$. We shall say that $\phi(p)$ can be extended to a set E', $E \subset E' \subset M$, preserving the condition $C(g)$ if there exists a transformation $p^*=\Phi(p)$ from E' into M^* which satisfies the condition $C(g)$ on E' and is equal to $\phi(p)$ on E.

In this paper we prove the following result. Let M be a separable metric space and let $g(t) \in \mathcal{P}(M)$. Then any transformation from a set E in M into a Euclidean space which satisfies the condition $C(g)$ on E can be extended to M preserving the condition $C(g)$.

We give two examples to illustrate this result. We shall use the vector notation x to represent a point in a Euclidean n-space E_n, and we shall denote by $|x_1-x_2|$ the distance between two points x_1, x_2. Let $x_0, x_1, \cdots, x_m$ be $m+1$ points in E_n and let $\xi_1, \cdots, \xi_m$ be m real numbers. From the relation $(x_i-x_j)^2=(x_0-x_i)^2+(x_0-x_j)^2-2(x_0-x_i)(x_0-x_j)$, the square of the vector $x=L\xi_1(x_0-x_1)+\cdots$

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[1] Helsel and Mickle, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 235–238.

[2] Kirszbraun, Fund. Math. vol. 22 (1934) pp. 77–108.

[3] Valentine, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 100–108.

$+L\xi_m(x_0-x_m)$, $L>0$, is given by

$$x^2 = L^2 \sum_{i,j=1}^{m} (x_0 - x_i)(x_0 - x_j)\xi_i\xi_j$$
$$= \frac{L^2}{2} \sum_{i,j=1}^{m} [\,|x_0 - x_i|^2 + |x_0 - x_j|^2 - |x_i - x_j|^2]\xi_i\xi_j \geqq 0.$$

Thus the function $g(t)=Lt$, $L>0$, is in $\Phi(E_n)$ and the results of Kirszbraun follow as a special case of the results of this paper. Schoenberg[4] has shown that the function $g(t)=Lt^\alpha$, $L>0$, $0<\alpha\leqq 1$, is in $\Phi(E_n)$. Thus a transformation from a set in a Euclidean space into a Euclidean space which satisfies a Lipschitz-Hölder condition ($L>0$, $0<\alpha\leqq 1$) can be extended to the whole space preserving this same Lipschitz-Hölder condition.

1. **Preliminary remarks.** In this section we give some well known concepts and lemmas for a Euclidean n-space E_n. A set is called convex if the line segment joining any two points of the set is in the set. If E is a closed convex set and $x \notin E$, then there is a unique point $x^* \in E$ which is closest to x. (Since E is closed there is one such point. If there were two, the midpoint of the line segment joining them would be in E and closer to x than either of them.) For a finite set of points $x_1, \cdots, x_m$, we denote by $V(x_1, \cdots, x_m)$ the smallest convex set containing them. $V(x_1, \cdots, x_m)$ is a closed set consisting of those points given by the relation $x=c_1x_1+\cdots+c_mx_m$, where the c_i's are non-negative and $c_1+\cdots+c_m=1$.

LEMMA 1.1. *Let E be a closed convex set, x_0 a point not in E, x_0^* the unique point of E closest to x_0, and $y=tx_0+(1-t)x_0^*$, $0\leqq t<1$. Then $|y-x|<|x_0-x|$ for every point $x\in E$.*

PROOF. Since $|y-x|\leqq t|x_0-x|+(1-t)|x_0^*-x|$, $t\neq 1$, it is sufficient to prove that $|x_0^*-x|<|x_0-x|$ for all $x\in E$. Assume there is a point $x_1\in E$ for which $|x_0^*-x_1|\geqq|x_0-x_1|$. Then the numbers $a=|x_0-x_0^*|$, $b=|x_0-x_1|$, $c=|x_0^*-x_1|$ satisfy the inequalities $a<b\leqq c$ and the number $t^*=(a^2+c^2-b^2)/2c^2$ satisfies the inequalities $0<t^*<1/2$. Thus the point $x=t^*x_1+(1-t^*)x_0^*$ is in E and $|x_0-x|^2=|x_0-x_0^*|^2-t^{*2}c^2<a^2$, contradicting the assumption that x_0^* is the point of E closest to x_0.

LEMMA 1.2. *Let Σ be a set of closed spheres in E_n such that there is no point in common to all of them. Then there is a finite set of spheres in Σ*

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[4] Schoenberg, Ann. of Math. vol. 38 (1937) pp. 787–793; Amer. J. Math. vol. 67 (1945) pp. 83–93.

which have no point in common.[5]

PROOF. Let S_0 be one of the spheres in Σ. Then the sum of the complements of the remaining spheres cover S_0. Since this is an open covering of S_0, there is a finite number of spheres $S_1, \cdots, S_m$ in Σ the sum of whose complements covers S_0. Hence, $S_0, S_1, \cdots, S_m$ have no point in common.

2. Lemmas. For a given integer m, let $a_1, \cdots . a_m$ be a given set of m positive numbers and let $x_1, \cdots, x_m$ be a given set of (not necessarily distinct) m points in E_n. For each point x let

$$f(x) = \max(|x - x_i|/a_i), \qquad i = 1, \cdots, m. \tag{2.1}$$

$f(x)$ is obviously a continuous function of x and there exists a point x_0 for which $f(x_0) = \min f(x)$. We have the following result concerning the location of a point x_0 at which $f(x)$ assumes a minimum value.

LEMMA 2.1. *Let x_0 be a point such that $f(x_0) = \min f(x)$ and let $x_{m_1}, \cdots x_{m_k}$ be all the points in the set of points $x_1, \cdots, x_m$ for which the equality*

$$f(x_0) = |x_0 - x_i|/a_i \tag{2.2}$$

holds. Then $x_0 \in V(x_{m_1}, \cdots, x_{m_k})$.

PROOF. Assume $x_0 \notin V(x_{m_1}, \cdots, x_{m_k})$. Let x_0^* be the unique point of $V(x_{m_1}, \cdots, x_{m_k})$ closest to x_0. For each integer j let $y_j = t_j x_0 + (1 - t_j)x_0^*$, $0 \leqq t_j < 1$, $t_j \to 1$, for $j \to \infty$. By Lemma 1.1, $|y_j - x| < |x_0 - x|$ for every point $x \in V(x_{m_1}, \cdots, x_{m_k})$. Thus, since $f(x_0) \leqq f(y_j)$, $f(y_j) = |y_j - x_i|/a_i$, $1 \leqq i \leqq m$, for some $x_i \notin V(x_{m_1}, \cdots, x_{m_k})$. There are an infinite number of the x_i's corresponding to the points y_j which are the same and we can assume without loss of generality that all of them are the same point $x_{m_{k+1}}$, $1 \leqq m_{k+1} \leqq m$. Since $f(y_j) \to f(x_0)$ for $j \to \infty$, $f(x_0) = \lim |y_j - x_{m_{k+1}}|/a_{m_{k+1}} = |x_0 - x_{m_{k+1}}|/a_{m_{k+1}}$, $x_{m_{k+1}} \notin V(x_{m_1}, \cdots, x_{m_k})$. Thus (2.2) holds for $i = m_{k+1}$. This contradicts the fact that (2.2) holds only for the points $x_{m_1}, \cdots, x_{m_k}$. Hence $x_0 \in V(x_{m_1}, \cdots, x_{m_k})$.

We now prove the fundamental lemma of the paper.

LEMMA 2.2. *Let M be a metric space, let $g(t) \in \mathfrak{P}(M)$ and let $p_1, \cdots, p_m$ be m distinct points and $x_1, \cdots, x_m$ be m points in M and E_n respectively for which the inequalities $|x_i - x_j| \leqq g(p_i p_j)$, $i, j = 1, \cdots, m$, hold. Then, for any point $p_0 \in M$, there exists a point*

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[5] We use the term closed sphere to mean the set of points x which satisfy the inequality $|x - x_0| \leqq r$ for fixed x_0 and $r > 0$.

$x_0 \in E_n$ for which the inequalities $|x_0 - x_i| \leqq g(p_0p_i)$, $i=1, \cdots, m$, hold.

PROOF. If $p_0 = p_i$, $1 \leqq i \leqq m$, let $x_0 = x_i$. Assume $p_0 \neq p_1, \cdots, p_m$. Set $a_i = g(p_0p_i)$, $i=1, \cdots, m$. Since $g(t) \in \mathfrak{P}(M)$, each $a_i > 0$. If x_0 is a point for which $f(x_0) = \min f(x)$ (see (2.1)), we assert that $\lambda = f(x_0) \leqq 1$. That is to say, $|x_0 - x_i| \leqq g(p_0p_i)$, $i=1, \cdots, m$. If $\lambda = 0$, then $\lambda \leqq 1$. Assume $\lambda > 0$. Set $a_{ij} = g(p_ip_j)$, $b_{ij} = |x_i - x_j|$ and $b_i = |x_0 - x_i|$, $i, j = 1, \cdots, m$. By renumbering if necessary, let $x_1, \cdots, x_k$ be the points for which the equality $f(x_0) = b_i/a_i = \lambda$ holds. By Lemma 2.1, $x_0 \in V(x_1, \cdots, x_k)$. Thus we have non-negative numbers $c_1, \cdots, c_k$, $c_1 + \cdots + c_k = 1$ such that $x_0 = c_1x_1 + \cdots + c_kx_k$ or $c_1(x_0 - x_1) + \cdots + c_k(x_0 - x_k) = 0$. By squaring this expression and using the relation $(x_i - x_j)^2 = (x_0 - x_i)^2 + (x_0 - x_j)^2 - 2(x_0 - x_i)(x_0 - x_j)$ we obtain

$$(2.3)\quad \sum_{i,j=1}^{k} (x_0 - x_i)(x_0 - x_j)c_ic_j = \frac{1}{2} \sum_{i,j=1}^{k} (b_i^2 + b_j^2 - b_{ij}^2)c_ic_j = 0.$$

Since $\lambda > 0$, $x_0 \neq x_1, \cdots, x_k$, and hence at least two of the c_i's are different from zero. Since $g(t) \in \mathfrak{P}(M)$, the quadratic form $\sum_{i,j=1}^{k} (a_i^2 + a_j^2 - a_{ij}^2)\xi_i\xi_j$ is positive. Setting $\xi_i = \lambda c_i$, $i = 1, \cdots, k$, and using the fact that $\lambda a_i = b_i$, $i = 1, \cdots, k$, we obtain

$$(2.4)\quad \frac{1}{2} \sum_{i,j=1}^{k} (a_i^2 + a_j^2 - a_{ij}^2)\lambda^2 c_ic_j = \frac{1}{2} \sum_{i,j=1}^{k} (b_i^2 + b_j^2 - \lambda^2 a_{ij}^2)c_ic_j \geqq 0.$$

Subtracting (2.3) from (2.4) gives

$$(2.5)\qquad \frac{1}{2} \sum_{i,j=1}^{k} (b_{ij}^2 - \lambda^2 a_{ij}^2)c_ic_j \geqq 0.$$

Since $a_{ij} = b_{ij} = 0$ for $i = j$, the c_i's are non-negative and at least two of the c_i's are different from zero, it follows from (2.5) that $b_{ij}^2 - \lambda^2 a_{ij}^2 \geqq 0$ for some pair of integers i, j with $i \neq j$. For this pair of integers $1 \leqq i, j \leqq k$, $i \neq j$, we have $g(p_ip_j)^2 \geqq |x_i - x_j|^2 = b_{ij}^2 \geqq \lambda^2 a_{ij}^2 = \lambda^2 g(p_ip_j)^2$. Hence $\lambda \leqq 1$. Thus a point x_0 at which $f(x)$ assumes a minimum satisfies the conditions of the lemma.

LEMMA 2.3. *Let M be a metric space, let $g(t) \in \mathfrak{P}(M)$, let $x = \phi(p)$ be a transformation from a set E in M into E_n and let p_0 be any point in M. Then, if $\phi(p)$ satisfies the condition $C(g)$ on E, $\phi(p)$ can be extended to $E + p_0$ preserving the condition $C(g)$.*

PROOF. If $p_0 \in E$ the extension is immediate. Assume $p_0 \notin E$. For each $p \in E$, let S_p be the set of points $x \in E_n$ which satisfy the inequality $|x-\phi(p)| \leqq g(p_0p)$. Since $g(t) \in \mathfrak{P}(M)$, each S_p is a closed sphere in E_n. Assume that there is no point in common to all the spheres. By Lemma 1.2 there is a finite number of these spheres which have no point in common. This contradicts Lemma 2.2. Hence, there is at least one point x_0 in all the spheres S_p, $p \in E$. Then $\Phi(p_0) = x_0$, $\Phi(p) = \phi(p)$, $p \in E$, is an extension of $\phi(p)$ to $E+p_0$ preserving the condition $C(g)$.

3. The main result.

We now state and prove the main result of this paper.

THEOREM. *Let M be a separable metric space, let $g(t) \in \mathfrak{P}(M)$ and let $x = \phi(p)$ be a transformation from a set E in M into a Euclidean space E_n. Then if ϕ satisfies the condition $C(g)$ on E, ϕ can be extended to M preserving the condition $C(g)$.*

PROOF. Let D be a finite or denumerable set which is dense in M. By Lemma 2.3, $\phi(p)$ can be extended to E plus any point of D and by induction to $E+D$ preserving the condition $C(g)$. Let $x = \Phi(p)$, $p \in E+D$ be the extended transformation. Since E_n is complete and $g(t) \in \mathfrak{P}(M)$, a convergent sequence of points $p_m \in E+D$, $m = 1, 2, \cdots$, determines a convergent sequence of points $\Phi(p_m)$ in E_n. Since $E+D$ is dense in M, $\Phi(p)$ can be extended to M preserving the condition $C(g)$ in one and only one way.

4. Additional remarks.

The writer is indebted to the referee for pointing out the following facts. Any finite set of points in a unitary space is isometrically equivalent to a set of points in some Euclidean space. Hence Lemma 2.2 is valid in any unitary space. Lemma 1.2 is valid in any complete unitary space (see Murray[6] for the case where the space is separable and Alaoglu[7] for the general case). Hence Lemma 2.3 is valid if E_n is replaced by a complete unitary space. Then the theorem in §3 with E_n replaced by a complete unitary space U and with M not assumed to be separable follows from Lemma 2.3 (with E_n replaced by U) by applying Zorn's lemma or transfinite induction.

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[7] Alaoglu, Ann. of Math. vol. 41 (1940) pp. 252–267.

AN INITIAL VALUE PROBLEM FOR A CLASS OF EQUATIONS OF MIXED TYPE[1]

STEFAN BERGMAN

1. Introduction. In connection with the consideration of second order linear partial differential equations of mixed type (that is, equations which are elliptic for some values of the independent variables and hyperbolic for others) there arises an initial value problem, namely that of expressing a solution of the equation in terms of the values of the solution and of its normal derivative on a portion of the boundary between the regions of elliptic and of hyperbolic behavior of the equation. Of particular interest are expressions in the form of single or multiple integrals representing the solution in terms of the prescribed data. These may enable us to deduce relationships between the initial data and various properties of the solution.[2]

In the present paper such a formula is derived for solutions of the following equation of mixed type:

$$\frac{\partial^2 \psi}{\partial x^2} + (-x)^s \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{for } x < 0, \tag{1a}$$

$$\frac{\partial^2 \psi}{\partial x^2} - x^s \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{for } x > 0, \tag{1b}$$

where s is any constant greater than -1. For $s=1$ our equation coincides with one which has been studied in an important investigation by Tricomi [6], see also [5], and which is met in the theory of two-dimensional flows of a compressible fluid when we investigate the streamfunction ψ in the so-called hodograph plane and make a certain simplifying assumption concerning the equation of state of the fluid. See [1, 2, 4]. We note that the equation is elliptic for $x<0$, hyperbolic for $x>0$.

We shall derive here an integral representation for the solution of (1a), (1b) in terms of the prescribed values $\psi(0, y) = T(y)$ and

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[2] As has been found in the study of equations of elliptic type and those of hyperbolic type, it is possible to exploit various integral representations of their solutions for the investigation of their properties. For various possibilities in this direction see §3 and [3, p. 140 ff.]. Numbers in brackets refer to the bibliography at the end of the paper.

$\psi_x(0, y) \equiv (\partial\psi(x, y)/\partial x)_{x=0} = S(y)$, under the assumption that $T(y)$ and $S(y)$ are real analytic functions of y and therefore can be analytically continued into a domain B of the complex η-plane ($\eta = y + iy_1$) which contains the portion of the real axis ($y_1 = 0$) along which $T(y)$ and $S(y)$ are initially defined. The solution obtained will, for each fixed value of y, be analytic in the real variable $(-x)^{2+s}$ for $x < 0$ and in the real variable x^{2+s} for $x > 0$.[3]

I should like to take this opportunity to thank A. Zeichner for his valuable assistance in connection with the present paper.

2. **An integral representation for the solutions of equation** (1). The solution ψ will be obtained as the sum of two solutions, $\psi^{(1)}$ and $\psi^{(2)}$, of (1). These solutions will satisfy along the line $x = 0$ the conditions:

$$\psi^{(1)}(0, y) = T(y), \qquad \psi_x^{(1)}(0, y) = 0, \tag{2}$$

$$\psi^{(2)}(0, y) = 0, \qquad \psi_x^{(2)}(0, y) = S(y). \tag{3}$$

Since (1) is linear and homogeneous, the desired solution ψ may be obtained by superposition of $\psi^{(1)}$ and $\psi^{(2)}$.

THEOREM 1. *Let*

$$\psi^{(1)}(x, y) = \frac{1}{\pi^2 i}\int_{t=0}^{\pi/2}\int_{X=0}^{1}\oint_C \frac{T(\eta)(1-X)^{-(2+s)^{-1}}[1+s-v^2(\eta-y)^{-2}]}{(2+s)(\eta-y)[1+v^2(\eta-y)^{-2}]^2}\, dt\, dX\, d\eta \tag{4}$$

where

$$v = RX^{1/2}\sin t, \qquad R \equiv R(x) \equiv (1+s/2)^{-1}|x|^{1+s/2} \tag{5}$$

and where C is a simple closed curve in the plane of the complex variable $\eta = y + iy_1$, which curve encloses the circle $|\eta - y| = R$. If $|x|$ is so small that the curve C lies in the regularity domain of $T(\eta)$, then $\psi^{(1)}(x, y)$ represents a solution of (1a) *which satisfies the initial conditions* (2).

PROOF. In order to prove our assertions, let us first write the required solution in the form of a series:

$$\psi^{(1)}(x, y) = T^{(0)}(y) + T^{(2)}(y)(-x)^{2+s} + T^{(4)}(y)(-x)^{2(2+s)} + \cdots$$

$$= \sum_{n=0}^{\infty} T^{(2n)}(y)(-x)^{n(2+s)}.$$

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[3] An interesting problem which we do not consider here is that of determining whether solutions of (1) exist which do not satisfy all the conditions of analyticity stated above. We note that here and in the following we write "(1)" instead of "(1a)" and "(1b)."

Proceeding formally we obtain:

$$(-x)^s\psi_{yy}^{(1)} = \sum_{n=0}^{\infty} T_{yy}^{(2n)}(y)(-x)^{n(2+s)+s},$$

$$\psi_{xx}^{(1)} = \sum_{n=0}^{\infty} T^{(2n+2)}(y)[(n+1)(2+s)][(n+1)(2+s)-1](-x)^{n(2+s)+s}.$$

Adding these last two equations and taking account of (1a), we find that the functions $T^{(0)}$, $T^{(2)}$, $T^{(4)}$, $\cdots$ must satisfy

$$T^{(2n)}(y) = -\frac{T_{yy}^{(2n-2)}(y)}{[n(2+s)][n(2+s)-1]}, \qquad n = 1, 2, 3, \cdots.$$

By repeatedly applying this recursion formula, we can express $T^{(2n)}(y)$ in terms of $d^{2n}T(y)/dy^{2n}$, namely:

$$T^{(2n)}(y) = \frac{(-1)^n d^{2n}T(y)/dy^{2n}}{\prod_{k=1}^{n}[k(2+s)][k(2+s)-1]}, \qquad T^{(0)}(y) \equiv T(y).$$

Now let C be a curve satisfying the conditions stated in the theorem. Then the Cauchy integral formula enables us to rewrite the above equation in the following manner:

$$\tag{6} \begin{aligned} 2\pi i T^{(2n)}(y) &= \frac{(-1)^n(2n)!\oint_C \frac{T(\eta)d\eta}{(\eta-y)^{2n+1}}}{\prod_{k=1}^{n}[k(2+s)][k(2+s)-1]} \\ &= \frac{(-1)^n(2n)!\oint_C \frac{T(\eta)d\eta}{(\eta-y)^{2n+1}}}{n!(2+s)^{2n}\prod_{k=1}^{n}[k-(2+s)^{-1}]}. \end{aligned}$$

Now, by applying the well known identities: $\Gamma(n+1)=n!$, $\Gamma(2n+1)=(2n)!$, and $p\Gamma(p)=\Gamma(p+1)$, we find that (6) may be rewritten

$$\tag{7} \begin{aligned} 2\pi i T^{(2n)}(y) = {} & \frac{(-1)^n[n+1-(2+s)^{-1}]\Gamma(n+1)\Gamma(2n+1)\Gamma(1-(2+s)^{-1})}{(2+s)^{2n}[\Gamma(n+1)]^2\Gamma(n+2-(2+s)^{-1})} \\ & \times \oint_C \frac{T(\eta)d\eta}{(\eta-y)^{2n+1}}. \end{aligned}$$

Now, since

$$\frac{\Gamma(2n+1)}{[\Gamma(n+1)]^2} = \frac{2^{2n+1}}{\pi}\int_0^{\pi/2} \sin^{2n} t dt,$$

$$\frac{\Gamma(n+1)\Gamma(1-(2+s)^{-1})}{\Gamma(n+2-(2+s)^{-1})} = \int_0^1 \mathrm{X}^n(1-\mathrm{X})^{-(2+s)^{-1}} d\mathrm{X},$$

we obtain finally:

$$\begin{aligned} &2\pi^2 i T^{(2n)}(y) = (-1)^n[n+1-(2+s)^{-1}](2+s)^{-2n}2^{2n+1} \\ (8)\qquad &\times \int_{t=0}^{\pi/2} \sin^{2n} t dt \int_{\mathrm{X}=0}^{1} \mathrm{X}^n(1-\mathrm{X})^{-(2+s)^{-1}} d\mathrm{X} \oint_C \frac{T(\eta)d\eta}{(\eta-y)^{2n+1}}. \end{aligned}$$

If we multiply both sides of (8) by $(-x)^{n(2+s)}$ and sum for $n=0, 1, 2, \cdots$, we obtain:

$$\begin{aligned} &2\pi^2 i\psi^{(1)}(x, y) = \sum_{n=0}^{\infty}\int_{t=0}^{\pi/2}\int_{\mathrm{X}=0}^{1}\oint_C \frac{2T(\eta)(1-\mathrm{X})^{-(2+s)^{-1}}}{(\eta-y)} \\ (9)\qquad &\qquad\times [n+1-(2+s)^{-1}]\left\{\frac{-v^2}{(\eta-y)^2}\right\}^n dt d\mathrm{X} d\eta. \end{aligned}$$

Let the minimum distance of the point y, considered temporarily fixed, from the curve C be $\geqq R+\delta$ (see (5)), δ any small positive constant. Then, since X and $\sin^2 t$ never exceed one in absolute value,

$$\sum_{n=0}^{\infty} [n+1-(2+s)^{-1}]\left[\frac{-4\mathrm{X}(-x)^{2+s}\sin^2 t}{(2+s)^2(\eta-y)^2}\right]^n$$

will converge absolutely and uniformly since $4(-x)^{2+s}/(2+s)^2|\eta-y|^2 \leqq (R/(R+\delta))^2 \leqq 1-\epsilon$, and the sum of the above series will be $(u+1+s)/(2+s)(1-u)^2$ where $u = -4\mathrm{X}(-x)^{2+s}\sin^2 t/(2+s)^2(\eta-y)^2 = -v^2/(\eta-y)^2$. Since we may interchange the order of summation and integration, we thus obtain from (9):

$$\begin{aligned} &\psi^{(1)}(x, y) \\ (10)\qquad &= \frac{1}{\pi^2 i}\int_{t=0}^{\pi/2}\int_{\mathrm{X}=0}^{1}\oint_C \frac{T(\eta)(1-\mathrm{X})^{-(2+s)^{-1}}[1+s-v^2(\eta-y)^{-2}]}{(2+s)(\eta-y)[1+v^2(\eta-y)^{-2}]^2} dt d\mathrm{X} d\eta. \end{aligned}$$

COROLLARY 1. *The function*

$$\begin{aligned} &\psi^{(1)}(x, y) \\ (11)\qquad &= \frac{1}{\pi^2 i}\int_{t=0}^{\pi/2}\int_{\mathrm{X}=0}^{1}\oint_C \frac{T(\eta)(1-\mathrm{X})^{-(2+s)^{-1}}[1+s+v^2(\eta-y)^{-2}]}{(2+s)(\eta-y)[1-v^2(\eta-y)^{-2}]^2} dt d\mathrm{X} d\eta \end{aligned}$$

represents, for sufficiently small positive values of x, a solution of (1b) *which satisfies the initial conditions* (2).

Proof. The derivation of formula (11) is entirely parallel to that of (4). We have only to take into account that $(-x)^s$ has to be replaced by $-x^s$.

Theorem 2. *The expression*

$$\psi^{(2)}(x, y) = \frac{x}{\pi^2 i}\int_{t=0}^{\pi/2}\int_{X=0}^{1}\oint_{C}\frac{S(\eta)(1-X)^{(2+s)^{-1}}[3+s+v^2(\eta-y)^{-2}]}{(2+s)(\eta-y)[1+v^2(\eta-y)^{-2}]^2}\,dt\,dX\,d\eta, \tag{12}$$

where C and η have the meaning explained in Theorem 1, *and $S(y)$ is an arbitrary real analytic function of one real variable, represents a solution of* (1a) *which satisfies the initial conditions* (3).

Proof. The proof proceeds exactly as in Theorem 1. We write:

$$\psi^{(2)}(x, y) = -\sum_{n=0}^{\infty} T^{(2n+1)}(y)(-x)^{1+n(2+s)}, \qquad T^{(1)}(y) \equiv S(y), \tag{13}$$

which formally satisfies conditions (3). Proceeding as in the proof of Theorem 1, we find that the $T^{(2n+1)}$ can be expressed as derivatives of $T^{(1)}$, namely:

$$T^{(2n+1)}(y) = \frac{(-1)^n d^{2n}T^{(1)}(y)/dy^{2n}}{n!(2+s)^{2n}\prod_{k=1}^{n}(k+(s+2)^{-1})}, \qquad n = 1, 2, 3, \cdots, \tag{14}$$

or:

$$\begin{aligned}&-2\pi^2 i T^{(2n+1)}(y)(-x)^{1+n(2+s)}\\ &\quad= x\frac{2^{2n+1}[n+1+(2+s)^{-1}](-1)^n(-x)^{n(s+2)}}{(2+s)^{2n}}\\ &\qquad\cdot\int_{t=0}^{\pi/2}\int_{X=0}^{1}\oint_{C}\frac{S(\eta)}{(\eta-y)^{2n+1}}\cdot\sin^{2n}t\cdot X^n(1-X)^{(s+2)^{-1}}dt\,dX\,d\eta.\end{aligned} \tag{15}$$

Summing for $n=0, 1, 2, 3, \cdots$, we get, for values of x sufficiently close to zero:

$$\begin{aligned}&2\pi^2 i\psi^{(1)}(x, y)\\ &\quad= -x\int_{t=0}^{\pi/2}\int_{X=0}^{1}\oint_{C}\frac{2S(\eta)(1-X)^{(2+s)^{-1}}[3+s+v^2(\eta-y)^{-2}]}{(2+s)(\eta-y)[1+v^2(\eta-y)^{-2}]^2}dt\,dX\,d\eta\end{aligned} \tag{16}$$

which yields the statement of Theorem 2.

Finally, by repeating the reasoning used in the proof of Corollary 1, we obtain:

COROLLARY 2. *The expression*

$$\psi^{(2)}(x, y) \tag{17}$$
$$= \frac{x}{\pi^2 i} \int_{t=0}^{\pi/2} \int_{\mathrm{X}=0}^{1} \oint_C \frac{S(\eta)(1-\mathrm{X})^{(2+s)^{-1}}[3+s-v^2(\eta-y)^{-2}]}{(2+s)(\eta-y)[1-v^2(\eta-y)^{-2}]^2} \, dt\, d\mathrm{X}\, d\eta$$

represents, for sufficiently small positive values of x, a solution of (1b) *which satisfies the initial conditions* (3).

3. **Concluding remarks.** In §2 we did not indicate explicitly the domain of the xy-plane in which the representations (4), (11), (12), (17) hold, but an almost obvious consideration shows that representations (4) and (11) are valid in the domain[4] $\sum_{Y\in J}[y=Y, R(x)<r(Y)]$ (see (5)), where J denotes the intersection of the (real) y-axis with the regularity domain B of $T(\eta)$, $\eta=y+iy_1$, in the (complex) η-plane and $r(Y)$ the distance of Y from the boundary of B. (It should be mentioned that using, in addition, certain summation methods, one might obtain representations which are valid in larger domains of the xy-plane.)

If $T(y)$ and $S(y)$ become, upon continuation into the complex η-plane, *entire* functions, then the above representation can be used to define $\psi(x, y)$ for *all* values of x and y, for in this case the curve C can always be chosen in such a way as to provide the determination $\psi(x, y)$ at any preassigned point.[5]

As has been shown in another place, integral formulas transforming analytic functions of one complex variable into solutions of a linear partial differential equation can be used in order to derive various results from theorems in the theory of functions pertaining to solutions of these equations. As an example for such a procedure, we shall obtain upper bounds for the growth of $\psi^{(1)}$ and $\psi^{(2)}$ from the derivatives, $n!a_n$ of $T(y)$ and $n!b_n$ of $S(y)$ respectively, at some fixed point, say at the origin.

In order to obtain this relation, we shall rewrite the formulas (4), (11), (12), and (17) in a certain modified form, which form holds in the case where $T(\eta)$ or $S(\eta)$ are entire functions.

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[4] $\sum_{Y\in J}$ denotes the sum of all points x, y where for every Y, x ranges over the interval indicated in the bracket, and Y ranges over J.

[5] In order to apply the method of §2 in the case where $T(y)$ and $S(y)$ are not analytic functions of one real variable y, we have to approximate $T(y)$ and $S(y)$ by analytic functions, say by polynomials.

LEMMA 3.1. *Under the assumption that $T(\eta)$ is an entire function of a complex variable, the representation* (4) *can be written in the form*

$$\begin{aligned}\psi^{(1)}(x, y) = \frac{2}{\pi}\,\mathrm{B}\left(\frac{1}{2}, \frac{1+s}{2+s}\right)\int_{q=0}^{1}\Bigg[&\frac{1+s}{2+s}\left\{\mathrm{Re}\, T(y+iRq)\right\} \\ &- \frac{Rq}{2}\left\{\mathrm{Im}\, T'(y+iRq)\right\}\Bigg](1-q^2)^{s/(4+2s)}dq\end{aligned} \tag{18}$$

(Re = *Real part of* $\cdots$, Im = *Imaginary part of* $\cdots$).

PROOF. The proof of representation (18) follows immediately by the use of the Cauchy integral formula. We may write (4) in the form

$$\begin{aligned}\psi^{(1)}(x, y) = &\frac{1}{\pi^2 i}\int_{t=0}^{\pi/2}\int_{X=0}^{1} \\ &\cdot\left\{\oint_C \frac{T(\eta)(\eta-y)\left[(1+s)(\eta-y)^2-v^2\right]d\eta}{(2+s)\left[(\eta-y)^2+v^2\right]^2}\right\}(1-X)^{-(2+s)^{-1}}dt\,dX.\end{aligned} \tag{19}$$

Since we assume that $x<0$, v can vanish only when $X=0$, or $t=0$.

On the other hand, it is easy to show that (19) may also be written in the form

$$\begin{aligned}\psi^{(1)}(x, y) = &\lim_{\epsilon\to 0,\,\delta\to 0}\frac{1}{\pi^2 i}\int_{t=\epsilon}^{\pi/2}\int_{X=\delta}^{1} \\ &\times\left\{\oint_C \frac{T(\eta)(\eta-y)\left[(1+s)(\eta-y)^2-v^2\right]d\eta}{(2+s)\left[(\eta-y)^2+v^2\right]^2}\right\}(1-X)^{-(2+s)^{-1}}dt\,dX.\end{aligned} \tag{20}$$

In order to evaluate the integral in the braces, we now apply the Cauchy theorem.

Since v never becomes 0, the only poles of the integrand are $\eta = y \pm iv$. In order to determine the value of the integral in the braces, we have to determine the residues of the integrand considered as a function of η at the points $\eta = y \pm iv$. A formal computation yields

$$\begin{aligned}&\frac{T(\eta)(\eta-y)\left[(1+s)(\eta-y)^2-v^2\right]}{(2+s)\left[(\eta-y)^2+v^2\right]^2} \\ &\qquad = \left(\frac{1+s}{2+s}\right)\frac{1}{2}\left[\frac{T(\eta)}{\eta-y+iv}+\frac{T(\eta)}{\eta-y-iv}\right] \\ &\qquad\quad - \frac{iv}{4}\left[\frac{T(\eta)}{(\eta-y+iv)^2}-\frac{T(\eta)}{(\eta-y-iv)^2}\right].\end{aligned} \tag{21}$$

Using (20), (21) and Cauchy's integral formula, we obtain

$$\psi^{(1)}(x, y) = \frac{2}{\pi}\int_{t=0}^{\pi/2}\int_{X=0}^{1}\left[\frac{1+s}{2+s}\{\operatorname{Re} T(y+iv)\}\right.$$
$$\left. - \frac{v}{2}\{\operatorname{Im} T'(y+iv)\}\right](1-X)^{-(2+s)^{-1}}dtdX. \tag{22}$$

Equation (22) may also be written in the form

$$\psi^{(1)}(x, y) = \lim_{\delta\to 0,\epsilon\to 0}\frac{2}{\pi}\int_{t=0}^{(\pi/2)-\delta}\int_{X=0}^{1-\epsilon}\left[\frac{1+s}{2+s}\operatorname{Re} T(y+iv)\right.$$
$$\left. - \frac{v}{2}\operatorname{Im} T'(y+iv)\right](1-X)^{-(2+s)^{-1}}dtdX. \tag{23}$$

If now we introduce the variables [see (5)]

$$q = X^{1/2}\sin t \equiv \frac{v}{R}, \qquad \tau = \frac{X^{1/2}\cos t}{[1 - X\sin^2 t]^{1/2}},$$

that is,

$$X = \tau^2 + q^2 - \tau^2 q^2, \qquad t = \arctan\left[q\tau^{-1}(1-q^2)^{-1/2}\right]$$

and pass to the limit, we obtain the representation

$$\psi^{(1)}(x, y) = \frac{4}{\pi}\int_{q=0}^{1}\int_{\tau=0}^{1}\left[\frac{1+s}{2+s}\operatorname{Re} T(y+iRq)\right.$$
$$\left. - \frac{Rq}{2}\operatorname{Im} T'(y+iRq)\right](1-q^2)^{s/(4+2s)}(1-\tau^2)^{-(2+s)^{-1}}d\tau dq. \tag{24}$$

Integration of the right-hand side with respect to τ yields the desired result.

Exactly the same considerations can be repeated in the case of representations (11), (12), and (17).

THEOREM 3.1. *Let $\psi^{(1)}$ be an analytic solution of* (1a) *satisfying condition* (2). *If $T(y) = \sum_{n=0}^{\infty} a_n y^n$ is an entire function (when continued to the complex values of the argument), then $\psi^{(1)}(x, y)$ can be extended to all values of y and x, $x<0$. $\psi^{(1)}$ satisfies the inequality*

$$|\psi^{(1)}(x, y)| \leqq Me^{\sigma r^{\rho}}[1 + 2(1+s)^{-1}(-x)^{(2+s)/2}],$$
$$r = [y^2 + 4(2+s)^{-2}(-x)^{2+s}]^{1/2} \tag{25}$$

where M is a suitably chosen constant and ρ and σ are given by

$$\rho = \limsup_{n\to\infty}\frac{n\log n}{\log(1/|a_n|)}, \qquad (\sigma\rho e)^{1/\rho} = \limsup_{n\to\infty} n^{1/\rho}(|a_n|)^{1/n}. \tag{26}$$

The proof of this inequality follows immediately from the representation (18) and the fact that by classical results of the theory of entire functions of one complex variable, the inequality $|T(\eta)| \leqq Me^{\sigma|\eta|^\rho}$ holds.

The expression (18) can be simplified further when $s \geqq 0$ by using the fact that $-R[\operatorname{Im} T'(y+iRq)] = d[\operatorname{Re} T(y+iRq)]/dq$ and then integrating by parts. This yields:

$$\tag{27} \begin{aligned} \psi^{(1)}(x, y) &= \frac{2}{\pi} \mathrm{B}\left(\frac{1}{2}, \frac{1+s}{2+s}\right)\frac{s}{4+2s} \\ &\times \int_{q=0}^{1} [\operatorname{Re} T(y+iRq)](1-q^2)^{(-4-s)/(4+2s)}dq \quad \text{for} \quad s > 0, \end{aligned}$$

$$\tag{28} \psi^{(1)}(x, y) = \operatorname{Re}\,[T(y+ix)] \qquad \text{for } s = 0.$$

By noting the fact that

$$\begin{aligned} \int_0^1 (1-q^2)^{-(4+s)/(4+2s)}dq &= \frac{1}{2}\mathrm{B}\left(\frac{1}{2}, \frac{s}{4+2s}\right) \\ &= \left[\frac{2}{\pi}\mathrm{B}\left(\frac{1}{2}, \frac{1+s}{2+s}\right)\frac{s}{4+2s}\right]^{-1} \end{aligned}$$

we finally obtain the formula

$$\tag{29} \psi^{(1)}(x, y) = \frac{\displaystyle\int_{q=0}^{1} [\operatorname{Re} T(y+iRq)](1-q^2)^{(-4-s)/(4+2s)}dq}{\displaystyle\int_{q=0}^{1} (1-q^2)^{(-4-s)/(4+2s)}dq}.$$

Thus for $s \geqq 0$, we obtain the following result.

THEOREM 3.2. *Let $\psi^{(1)}$ be an analytic solution of* (1a) *satisfying condition* (2). *If $T(\eta)$ is entire and $s \geqq 0$, then using the notation of Theorem* 3.1, *we obtain*

$$\tag{30} |\psi^{(1)}(x, y)| \leqq Me^{\sigma r^\rho}.$$

The proof of this inequality follows immediately from (28) and (29).

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HARVARD UNIVERSITY

AN INVERSION FORMULA FOR THE GENERALIZED STIELTJES TRANSFORM

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1. Introduction. The problem of finding formulae to invert the Stieltjes transforms

$$f(x)=\int_0^\infty d\alpha(t)/(x+t), \tag{1}$$

$$f(x)=\int_0^\infty \phi(t)dt/(x+t), \tag{2}$$

and the generalized transforms

$$f(x)=\int_0^\infty d\alpha(t)/(x+t)^\rho, \tag{3}$$

$$f(x)=\int_0^\infty \phi(t)dt/(x+t)^\rho, \tag{4}$$

has been solved by Widder [A, pp. 7–60][2] and by Pollard [F, pp. 14–16]. The function $\phi(t)\in L(0,\infty)$, $\alpha(t)$ is a normalized function of

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[2] Roman letters in brackets refer to the bibliography at the end of the paper.

bounded variation in $(0, R)$ for all positive R, the integrals converge for some complex x which is neither zero nor negative, and ρ is positive. Widder showed that if

$$L_{kt}(f) = \frac{(-)^{k-1}}{k!(k-2)!}\left[t^{2k-1}f^{(k-1)}(t)\right]^{(k)} \tag{5}$$

then for (2), $\phi(t) = \lim_{k\to\infty} L_{kt}(f)$ at all points of the Lebesgue set for $\phi(t)$; and for (1), $\lim_{k\to\infty} \int_0^t L_{ku}(f)du = \alpha(t) - \alpha(0+)$. Pollard defined the operator

$$L^{\rho}_{kt}(f) = \frac{(-)^{k-1}2^{\rho-1}(2k-1)!\,\Gamma(\rho)}{k!(k-2)!\Gamma(2k+\rho-1)}\left[t^{2k+\rho-2}f^{(k-1)}(t)\right]^{(k)} \tag{6}$$

as a generalization of (5); and proved results corresponding to Widder's for (3) and (4).

In [G] the author discussed the properties of the functions defined by (3) and (4); and by arguments based on the method of steepest descents, obtained independently of Pollard the operator

$$M_{kt}(f) = \frac{(-)^{k-1}t^{k+\rho-2}\Gamma(2k+\rho-1)\Gamma(\rho)}{\Gamma(k+\rho)(k-2)!}\sum_{r=0}^{k}\frac{C_{k,r}t^r f^{(k+r-1)}(t)}{\Gamma(k+\rho+r-1)} \tag{7}$$

which has the same effect on (3) and (4) as Pollard's operator L^{ρ}_{kt}. This was to be expected, since as can easily be verified, the operators (6) and (7) are asymptotically the same for large k, and indeed differ only in the outside multipliers.

We shall show in the present work that by certain formal arguments an inversion formula involving a complex integral for (3) and (4) can be obtained from (7). This operator turns out to be a generalization of the one used by Stieltjes [B, pp. 473–476] to invert (2).

2. **Definitions and elementary properties.** It will be noted that when ρ is not an integer, the nucleus $(x+t)^{-\rho}$ in (3) and (4) is many-valued. Moreover, even when ρ is an integer, it may be seen from the example

$$\begin{aligned} \alpha(t) &= 0, & 0 \leqq t \leqq c, \\ \alpha(t) &= 1, & t > c, \end{aligned}$$

that possible singularities of (3) and (4) arise when t is zero or negative. We avoid both these difficulties by considering $f(x)$ in the region D, where D is the x-plane cut from the origin along the negative real axis. It is open to us to choose any arbitrarily fixed determination for $(x+t)^{-\rho}$ (and indeed for any number $z^{-\rho}$ which occurs in our argument), when the nucleus $(x+t)^{-\rho}$ or the function $z^{-\rho}$ will be one-valued and analytic in D. For convenience we shall take that determination of $z^{-\rho}$ for which arg $[z^{-\rho}]$ has its principal value. But it can be seen from subsequent arguments that any fixed determination could have been chosen, and that our results would then be equally valid.

It is convenient at this stage to give some elementary properties of the functions defined by (3) and (4). Proofs have been given by Widder [A] for the case $\rho=1$. The proof in each case is an easy generalization of his methods. We combine them in the following theorem.

THEOREM 2. *If the integral in* (3) *or* (4), *with the determination of* $(x+t)^{-\rho}$ *given above, converges for any* x *in* D, *then it converges for all such* x, *and converges uniformly in any closed subregion of* D. *The functions defined by* (3) *and* (4) *are then analytic in* D, *and for* (3)

$$f^{(k)}(x) = \frac{(-)^k\Gamma(k+\rho)}{\Gamma(\rho)}\int_0^\infty \frac{d\alpha(t)}{(x+t)^{k+\rho}}$$

with a similar result for (4). *When* (3) *converges*, $\alpha(t)=o(t^\rho)$, *and when* (4) *converges* $\int_1^t\phi(u)du=o(t^\rho)$, *as* $t\to\infty$.

3. **Some formal arguments.** Widder has shown [H, p. 381] that the operator (5), with variable t, can be expressed symbolically as

$$-\frac{1}{\pi}\sin \pi tD = \frac{1}{2i\pi}\left[e^{-i\pi tD} - e^{i\pi tD}\right]$$

where $D=d/dt$, and uses this result to obtain the inversion formula of Stieltjes [B] for the transform (2). We proceed from (7) and write

$$M_{kt}(f) = \frac{(-)^{k-1}t^{\rho-1}\Gamma(2k+\rho-1)\Gamma(\rho)}{\Gamma(k+\rho)(k-2)!}\sum_{r=0}^{k} C_{k,r}\frac{t^{k+r-1}D^{k+r-1}}{\Gamma(k+\rho+r-1)}\cdot f(t).$$

Using the usual notation $\Theta=t\cdot d/dt$, we see that

$$\begin{aligned} M_{kt}(f) &= \frac{t^{\rho-1}\Gamma(2k+\rho-1)\Gamma(\rho)}{\Gamma(k+\rho)(k-2)!}\sum_0^k C_{k,r}\frac{(-)^r(-\Theta)_{k+r-1}}{\Gamma(k+\rho+r-1)}f(t) \\ &= \frac{t^{\rho-1}\Gamma(2k+\rho-1)\Gamma(\rho)\Gamma(k-1-\Theta)}{\Gamma(k+\rho)(k-2)!\Gamma(-\Theta)\Gamma(k+\rho-1)}\sum_0^k \frac{(-k)_r(k-1-\Theta)_r}{r!(k+\rho-1)_r}f(t) \\ &= \frac{t^{\rho-1}\Gamma(k-1-\Theta)\Gamma(k+\rho+\Theta)\Gamma(\rho)}{\Gamma(-\Theta)\Gamma(\rho+\Theta)\Gamma(k+\rho)(k-2)!}\cdot f(t), \end{aligned}$$

on summing the hypergeometric series by Gauss' theorem. Then by Stirling's formula

$$\lim_{k\to\infty} M_{kt} = \frac{t^{\rho-1}\Gamma(\rho)}{\Gamma(-\Theta)\Gamma(\rho+\Theta)};$$

so that we have the formal result

$$\lim_{k\to\infty} M_{kt}(f) = \frac{t^{\rho-1}\Gamma(\rho)}{\Gamma(-\Theta)\Gamma(\rho+\Theta)} f(t). \tag{8}$$

We express this operation as an integral by the well known formula

$$\int_0^{\pi/2} \cos^{p+q-2}\omega\cdot\cos(p-q)\omega d\omega = \pi\Gamma(p+q-1)/2^{p+q-1}\Gamma(p)\Gamma(q)$$

[C, p. 40], which is true when $p+q>1$. On setting $p+q=1+\rho$, $p-q=\rho-1+2\Theta$, the operator in (8) assumes the form

$$\begin{aligned} &\frac{-(2t)^{\rho-1}2\Theta}{\pi}\int_0^{\pi/2}\cos^{\rho-1}\omega\cos(\rho-1+2\Theta)\omega d\omega \\ &\qquad = \frac{-(2t)^{\rho-1}tD}{\pi}\int_{-\pi/2}^{\pi/2}\cos^{\rho-1}\omega e^{i\omega(\rho-1)}\cdot e^{2i\Theta\omega}d\omega. \end{aligned} \tag{9}$$

Now applying (9) to $f(t)=g(u)$, where $t=e^u$, and bearing in mind that $e^{a\Theta}g(u)=g(u+a)$ formally, we get

$$\begin{aligned} &\frac{-(2t)^{\rho}D}{2\pi}\int_{-\pi/2}^{\pi/2}\cos^{\rho-1}\omega\cdot e^{i(\rho-1)\omega}g(u+2i\omega)d\omega \\ &\qquad = \frac{-(2t)^{\rho}D}{2\pi}\int_{-\pi/2}^{\pi/2}\cos^{\rho-1}\omega\cdot e^{i(\rho-1)\omega}\cdot f(te^{2i\omega})d\omega \\ &\qquad = \frac{-(2t)^{\rho}}{2\pi}\int_{-\pi/2}^{\pi/2}\cos^{\rho-1}\omega\cdot e^{i(\rho+1)\omega}f'(te^{2i\omega})d\omega \\ &\qquad = \frac{-1}{2i\pi}\int_C (z+t)^{\rho-1}f'(z)dz, \end{aligned}$$

on making the substitution $z=te^{2i\omega}$. C is the circle $|z|=t$, cut at the point $-t$.

We now define as our complex inversion operator

$$M_{\eta t}(f) = \frac{-1}{2i\pi}\int_{C_{\eta t}}(z+t)^{\rho-1}f'(z)dz, \tag{10}$$

where $C_{\eta t}$ is the contour which starts at the point $-t-i\eta$, proceeds along the straight line $\operatorname{Im}(z)=-\eta$ to the point $-i\eta$, then along the semi-circle $|z|=\eta$, $\operatorname{Re}(z)\geqq 0$, to the point $i\eta$, and finally along the line $\operatorname{Im}(z)=\eta$ to the point $-t+i\eta$. Since the integrand in (10) is analytic and one-valued in D, the integral along C is equal to the limit as η tends to zero of the integral along $C_{\eta t}$.

Before discussing the relation of (10) to (3) and (4), it is of interest to observe that on setting $\rho=1$,

$$\lim_{\eta\to 0+} M_{\eta t}(f)=(1/2i\pi)\left[f(te^{-i\pi})-f(te^{i\pi})\right],$$

which is the inversion operator given by Stieltjes [B, pp. 473–476] for (2).

4. **Complex inversion formula.** In order to show that (10) will invert (3) and (4) it is desirable to prove three lemmas, which play a decisive part in the inversion theorems. It will be noted that the third of these lemmas gives the value of a generalized form of "Cauchy's singular integral" [D, p. 133 and E, p. 30]. We state the following lemma.

LEMMA 4a.

$$\lim_{\eta\to 0+}\frac{\eta^{\rho}}{2i\pi}\int_{0}^{R}\left[\frac{1}{(\eta-iu)^{\rho}}-\frac{1}{(\eta+iu)^{\rho}}\right]\frac{du}{u}=\frac{1}{2}.$$

For on setting $u=\eta\tan\phi$, and then making η tend to zero through positive values, the expression on the left becomes

$$\frac{1}{\pi}\int_{0}^{\pi/2}\frac{\cos^{\rho-1}\phi\cdot\sin\rho\phi d\phi}{\sin\phi};$$

and this integral, which clearly converges, is the Cauchy principal value of

$$\frac{1}{2i\pi}\int_{-\pi/2}^{\pi/2}\frac{e^{i\rho\phi}\cos^{\rho-1}\phi d\phi}{\sin\phi}.$$

On setting $z=e^{2i\phi}$, this integral is seen to be equal to

$$\frac{1}{2i\pi}\,\frac{1}{2^{\rho-1}}\int_{C}\frac{(z+1)^{\rho-1}dz}{z-1}$$

where C is the unit circle indented at $z=\pm 1$, to allow for the pole at $z=1$ and the possible branch point at $z=-1$. Since

$\lim_{z\to-1}(z+1)^\rho/(z-1)=0$, and $\lim_{z\to1}(z+1)^{\rho-1}=2^{\rho-1}$, it follows that the integral has the value 1/2.

LEMMA 4b. *If* $g(u)\in L\ (0\leqq u\leqq R)$ *and* $g(0+)$ *exists, then*

$$\lim_{\eta\to0+}\frac{\eta^\rho}{2i\pi}\int_0^R\frac{g(u)}{u}\left[\frac{1}{(\eta-iu)^\rho}-\frac{1}{(\eta+iu)^\rho}\right]du=\frac{1}{2}g(0+).$$

For consider

$$I\equiv\frac{\eta^\rho}{2i\pi}\int_0^R\frac{g(u)-g(0+)}{u}\left[\frac{1}{(\eta-iu)^\rho}-\frac{1}{(\eta+iu)^\rho}\right]du;$$

and let a positive δ be chosen so that $|g(u)-g(0+)|<\epsilon$ for $0<u\leqq\delta$. If the integral in I is taken in two parts, over $(0,\delta)$ and (δ,R), then

$$|I(0,\delta)|\leqq\frac{\epsilon}{\pi}\int_0^{\operatorname{arc}\tan\delta/\eta}\frac{\cos^{\rho-1}\phi\,|\sin\rho\phi|\,d\phi}{\sin\phi}\quad(\text{on setting } u=\eta\tan\phi)$$

$$\to\frac{\epsilon}{\pi}\int_0^{\pi/2}\frac{\cos^{\rho-1}\phi\,|\sin\rho\phi|\,d\phi}{\sin\phi}=\epsilon K;$$

since ρ is positive and $|\sin\rho\phi/\sin\phi|$ is bounded. On the other hand

$$|I(\delta,R)|\leqq\frac{\eta^\rho}{\pi}\int_\delta^R\frac{|g(u)-g(0+)|}{u}\frac{du}{(\eta^2+u^2)^{\rho/2}}$$

$$\leqq\frac{\eta^\rho}{\pi}\int_\delta^R\frac{|g(u)-g(0+)|\,du}{u^{\rho+1}}.$$

Hence $\limsup_{\eta\to0+}|I|=0$; and this means the required limit has the value

$$\lim_{\eta\to0+}\frac{g(0+)\cdot\eta^\rho}{2i\pi}\int_0^R\left[\frac{1}{(\eta-iu)^\rho}-\frac{1}{(\eta+iu)^\rho}\right]\frac{du}{u}=\frac{1}{2}g(0+),$$

by Lemma 4a.

LEMMA 4c. *If* $g(u)\in L\ (0\leqq u\leqq R)$, *if* $0<t<R$, *and if* $g(t+)$ *and* $g(t-)$ *exist, then*

$$\lim_{\eta\to0+}\frac{\eta^\rho}{2i\pi}\int_0^R\frac{g(u)}{t-u}\left[\frac{1}{(\eta-i\cdot(t-u))^\rho}-\frac{1}{(\eta+i\cdot(t-u))^\rho}\right]du$$

$$=\frac{1}{2}[g(t+)+g(t-)].$$

This is proved by splitting the integral into two parts corresponding to the intervals $(0, t)$ and (t, R), and applying Lemma 4b to each part.

We now state and prove the main inversion theorems:

THEOREM 4a. *If $\phi(u) \in L$ $(0 \leqq u \leqq R)$ for all positive R, and is such that* (4) *converges, then $\lim_{\eta \to 0+} M_{\eta t}(f) = [\phi(t+) + \phi(t-)]/2$ for any positive t at which both $\phi(t+)$ and $\phi(t-)$ exist.*

For we have

$$M_{\eta t}(f) = \frac{-1}{2i\pi}\int_{C_{\eta t}} (z+t)^{\rho-1} f'(z)dz$$

$$= \frac{\rho}{2i\pi}\int_{C_{\eta t}} (z+t)^{\rho-1}dz \int_0^\infty \frac{\phi(u)du}{(z+u)^{\rho+1}}$$

$$= \frac{\rho}{2i\pi}\int_0^\infty \phi(u)du \int_{C_{\eta t}} \frac{(z+t)^{\rho-1}}{(z+u)^{\rho-1}}\,dz \text{ (by uniform convergence)}$$

$$= \frac{\eta^\rho}{2i\pi}\int_0^\infty \frac{\phi(u)}{t-u}\left[\frac{1}{(\eta - i\cdot(t-u))^\rho} - \frac{1}{(\eta + i\cdot(t-u))^\rho}\right]du.$$

Now let $R > t$, and let the integral be split into two parts corresponding to the intervals $(0, R)$ and (R, ∞). Then by Lemma 3c,

$$I(0, R) \to [\phi(t+) + \phi(t-)]/2$$

as $\eta \to 0+$.

Since $I(R, \infty)$ need not converge absolutely, we define $p(t) = \int_1^t \phi(u)du$. Then by Theorem 2, $p(t) = o(t^\rho)$ as $t \to \infty$; and

$$I(R, \infty) = \frac{-\eta^\rho \psi(t, R)p(R)}{2i\pi\cdot(t-R)} - \frac{\eta^\rho}{2i\pi}\int_R^\infty p(u)\frac{\partial}{\partial u}\left[\frac{\psi(t, u)}{t-u}\right]du,$$

where $\psi(t, u) \equiv [\eta - i(t-u)]^{-\rho} - [\eta + i(t-u)]^{-\rho}$. The first term clearly tends to zero with η. If the second be denoted by $I'(\eta)$, we have

$$|I'(\eta)| \leqq \frac{\eta^\rho}{2\pi}\int_R^\infty \frac{|p(u)|\,|\psi(t, u)|\,du}{(t-u)^2}$$

$$+ \frac{\rho\eta^\rho}{2\pi}\int_R^\infty \frac{|p(u)|}{u-t}\left|\frac{1}{(\eta - i\cdot(t-u))^{\rho+1}} - \frac{1}{(\eta + i\cdot(t-u))^{\rho+1}}\right|du.$$

Since $p(u) = o(u^\rho)$ as $u \to \infty$, there exists a constant M such that for

$u \geqq R$, $|p(u)| \leqq Mu^\rho$. Then

$$
\begin{aligned}
|I'(\eta)| &\leqq \frac{M\eta^\rho}{\pi}\int_R^\infty \frac{u^\rho du}{(t-u)^2\{\eta^2+(t-u)^2\}^{\rho/2}} \\
&\quad + \frac{M\rho\eta^\rho}{\pi}\int_R^\infty \frac{u^\rho du}{(u-t)\{\eta^2+(t-u)^2\}^{(\rho+1)/2}} \\
&\leqq \frac{M\eta^\rho(1+\rho)}{\pi}\int_R^\infty \frac{u^\rho du}{(u-t)^{\rho+2}}.
\end{aligned}
$$

But the integral converges, since $R>t$ and $\rho>0$, so that $|I'(\eta)|\to 0$. This completes the proof of our theorem.

THEOREM 4b. *If* (3) *converges, then for any positive t,*

$$
\lim_{\eta\to 0+}\int_0^t M_{\eta u}(f)du = \frac{1}{2}\,[\alpha(t+)+\alpha(t-)-\alpha(0+)].
$$

For

$$
\begin{aligned}
&\int_0^t M_{\eta u}(f)du \\
&\quad = \frac{-1}{2i\pi}\int_0^t du\Big[(z+u)^{\rho-1}f(z)dz-(\rho-1)\int(z+u)^{\rho-2}f(z)dz\Big]_{C_{\eta u}} \\
&\quad = \frac{-1}{2i\pi}\int_0^t [(i\eta)^{\rho-1}f(-u+i\eta)-(-i\eta)^{\rho-1}f(-u-i\eta)]du \\
&\qquad\qquad + \frac{\rho-1}{2i\pi}\int_0^t du\int_{C_{\eta u}}(z+u)^{\rho-2}f(z)dz.
\end{aligned}
$$

Writing the last term as $I_1+I_2+I_3$, where I_1 is the integral from $-t-i\eta$ to $-i\eta$, I_2 the integral around the semi-circle from $-i\eta$ to $i\eta$, and I_3 the integral along the straight line from $i\eta$ to $-t+i\eta$, we have

$$
\begin{aligned}
I_1 &= \frac{\rho-1}{2i\pi}\int_0^t du\int_0^u (u-\xi-i\eta)^{\rho-2}f(-\xi-i\eta)d\xi \\
&= \frac{\rho-1}{2i\pi}\int_0^t f(-\xi-i\eta)d\xi\int_\xi^t (u-\xi-i\eta)^{\rho-2}du \\
&\qquad\qquad\qquad\qquad \text{(by uniform convergence)} \\
&= \frac{1}{2i\pi}\int_0^t f(-\xi-i\eta)[(t-\xi-i\eta)^{\rho-1}-(-i\eta)^{\rho-1}]d\xi,
\end{aligned}
$$

$$I_3 = \frac{-1}{2i\pi}\int_0^t f(-\xi + i\eta)\left[(t - \xi + i\eta)^{\rho-1} - (i\eta)^{\rho-1}\right]d\xi$$

(by a similar argument),

and

$$\begin{aligned} I_2 &= \frac{\rho - 1}{2i\pi}\int_0^t du\int_{C_{\eta 0}} (z + u)^{\rho-2}f(z)dz \\ &= \frac{1}{2i\pi}\int_{C_{\eta 0}} \left[(z + t)^{\rho-1} - z^{\rho-1}\right]f(z)dz. \end{aligned}$$

On adding we have

$$\int_0^t M_{\eta u}(f)du = \frac{1}{2i\pi}\int_{C_{\eta t}} (z + t)^{\rho-1}f(z)dz - \frac{1}{2i\pi}\int_{C_{\eta 0}} z^{\rho-1}f(z)dz. \tag{11}$$

Now since $\alpha(t)$ is normalized, $f(z) = \rho\int_0^\infty \alpha(u)du/(z+u)^{\rho+1}$, and the integral converges uniformly for z on $C_{\eta t}$ and $C_{\eta 0}$. Then the first term in (11) is equal to

$$\begin{aligned} &\frac{\rho}{2i\pi}\int_{C_{\eta t}} (z + t)^{\rho-1}dz\int_0^\infty \frac{\alpha(u)du}{(z + u)^{\rho+1}} \\ &\quad= \frac{\rho}{2i\pi}\int_0^\infty \alpha(u)du\int_{C_{\eta t}}\left(\frac{z + t}{z + u}\right)^{\rho-1}\cdot\frac{dz}{(z + u)^2} \\ &\quad= \frac{1}{2i\pi}\int_0^\infty \frac{\alpha(u)du}{u - t}\left[\left(\frac{i\eta}{u - t + i\eta}\right)^{\rho} - \left(\frac{-i\eta}{u - t - i\eta}\right)^{\rho}\right] \\ &\quad= \frac{\eta^\rho}{2i\pi}\int_0^\infty \frac{\alpha(u)du}{t - u}\left[\frac{1}{(\eta - i\cdot(t - u))^\rho} - \frac{1}{(\eta + i\cdot(t - u))^\rho}\right]; \end{aligned} \tag{12}$$

while the second term in (11) is equal to

$$\begin{aligned} &\frac{-\rho}{2i\pi}\int_{C_{\eta 0}} z^{\rho-1}dz\int_0^\infty \frac{\alpha(u)du}{(z + u)^{\rho+1}} \\ &\quad= \frac{-\rho}{2i\pi}\int_0^\infty \alpha(u)du\int_{C_{\eta 0}} \frac{z^{\rho-1}dz}{(z + u)^{\rho+1}} \qquad \text{(by uniform convergence)} \\ &\quad= \frac{-\eta^\rho}{2i\pi}\int_0^\infty \frac{\alpha(u)du}{u}\left[\frac{1}{(\eta - iu)^\rho} - \frac{1}{(\eta + iu)^\rho}\right]. \end{aligned} \tag{13}$$

It follows now from Theorem 4a that (12)→$[\alpha(t+)+\alpha(t-)]/2$ as $\eta\to 0+$. While if (13) be written as $I(0, R)+I(R, \infty)$, R being positive, it follows from Lemma 4b that $I(0, R) \to -\alpha(0+)/2$, and by an argument similar to that used in the proof of Theorem 4a, it can be seen that $I(R, \infty) \to 0$ with η. This completes the proof of the second inversion theorem.

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ON THE MEAN MODULUS OF AN ANALYTIC FUNCTION

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Throughout this paper $f=f(z)$ will denote an analytic function of the complex variable z in the open unit circle $|z|<1$. The circle $C(r)$, on which $|z|=r$, of radius $r\geqq 0$ about the origin $z=0$ lies in the region of analyticity of f provided $r<1$. For every positive real parameter t $(0<t<\infty)$ the mean of order t of the modulus of f on the circle $C(r)$ is defined as

$$(1)\qquad M_t(r;f)=\left[\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^t d\theta\right]^{1/t}.$$

For fixed f and r this mean modulus $M_t(r;f)$ as a function of t is continuous, nonnegative, nondecreasing, and is bounded above by the maximum modulus of f on $C(r)$ [1, 2].[1] Therefore the limit of $M_t(r;f)$ exists as $t\rightarrow 0$ and $t\rightarrow\infty$. This limit is defined to be the mean modulus of f on $C(r)$ of order 0 and of order ∞ respectively. It may be shown that the mean modulus of order 0 is the geometric mean of the modulus of f on $C(r)$, which is simply evaluated by Jensen's formula, and that the mean modulus of order ∞ is the maximum modulus of f on $C(r)$ [1, 2]. Thus $M_t(r;f)$ is defined for all parameters t in the compact infinite interval $0\leqq t\leqq\infty$.

For fixed f and t $(0\leqq t\leqq\infty)$ the mean modulus $M_t(r;f)$ as a function of r in the interval $0\leqq r<1$ is continuous, nonnegative, nondecreasing, and, except for the limiting parameters 0 and ∞, possesses a continuous derivative with respect to r [1, 3]. Moreover, its logarithm is a nondecreasing convex function of $\log r$ (for $t=\infty$ this is the Hadamard three-circle theorem) [1, 3].

We shall be concerned here with the convexity of the mean modulus $M_t(r;f)$ as a function of r. Let $T(f)$ be the set of all parameters t in the compact infinite interval $0\leqq t\leqq\infty$ such that $M_t(r;f)$ is a convex function of r in the interval $0\leqq r<1$. Since $M_t(r;f)$ is continuous with respect to the parameter t and since any function which is the limit of convex functions is also convex, the set $T(f)$ is a closed and hence compact subset of the parameter interval $0\leqq t\leqq\infty$. The set $T(f)$ need not, however, coincide with the entire parameter interval and indeed

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[1] Numbers in brackets refer to the references cited at the end of the paper.

may be a bounded subset of this interval as shown by the following example. Let

$$f(z) = (z + \epsilon)/(1 + \epsilon z), \qquad |z| < 1,\ 0 < \epsilon < 1.$$

This linear fractional function maps the open unit circle onto itself and maps the circle $C(r)$ into a circle whose maximal distance from the origin is attained at the image point $f(r)$. Therefore

$$M_\infty(r; f) = f(r),$$

which is easily seen to be a strictly concave function of r. Thus the set $T(f)$ for this function f does not contain the parameter ∞ and hence, being closed, is bounded.

THEOREM. *The set $T(f)$ contains the values $t = 2/k$ $(k = 1, 2, 3, \cdots)$ and their limit value $t = 0$. Furthermore, if f has at most k zeros counting multiplicities $(k = 1, 2, 3, \cdots)$, then $T(f)$ contains the closed interval $0 \leqq t \leqq 2/k$.*

PROOF. Evidently the theorem is true for a constant function f, so we may assume that f is nonconstant. We shall first investigate the convexity of the mean modulus $M = M_t(r; f)$ in any open interval $\alpha < r < \beta$ such that the associated open annulus $\alpha < |z| < \beta$ contains no zeros of f.

Let the integer $m \geqq 0$ be the number of zeros, counting multiplicities, which lie in the closed circular interior, $|z| \leqq \alpha$, of the nonconstant function $f(z)$. If $m > 0$, we denote these m zeros, counting multiplicities, by z_h $(h = 1, \cdots, m)$. The function

$$A(z) = \prod_1^m{}_h \left(1 - \frac{z_h}{z}\right) \tag{2}$$

(where we understand any such product to have the value 1 if $m = 0$) is then analytic and has no zeros in the open simply-connected circular exterior $|z| > \alpha$ including $z = \infty$. A single-valued function $a(z)$, analytic in this open circular exterior, then exists such that

$$A(z) = a(z)^{2/t}. \tag{3}$$

Since $f(z)$ has no zeros in the open annulus $\alpha < |z| < \beta$, the only zeros of $f(z)$ in the open circular interior $|z| < \beta$ are the zeros z_h $(h = 1, \cdots, m)$. Consequently the function

$$B(z) = f(z) \Big/ \prod_1^m{}_h (z - z_h) \tag{4}$$

is analytic and has no zeros in the open simply-connected circular interior $|z| < \beta$. A single-valued function $b(z)$, analytic in this open circular interior, then exists such that

$$B(z) = b(z)^{2/t}. \tag{5}$$

From (2), (3), (4), and (5) we see that

$$f(z) = z^m A(z) B(z) = z^m a(z)^{2/t} b(z)^{2/t} = z^m c(z)^{2/t} \tag{6}$$

where the function $c(z) = a(z)b(z)$ is single-valued and analytic in the open annulus $\alpha < |z| < \beta$ and hence admits a Laurent expansion

$$c(z) = \sum_{-\infty}^{\infty}{}_{p}\, c_p z^p \tag{7}$$

in this annulus. In the associated interval $\alpha < r < \beta$ we define

$$s(r) = \frac{1}{2\pi}\int_0^{2\pi} |c(re^{i\theta})|^2 d\theta. \tag{8}$$

This integral (8) may be evaluated in terms of the coefficients of the series (7) according to the well known formula

$$s(r) = \sum_{-\infty}^{\infty}{}_{p}\, |c_p|^2 r^{2p}. \tag{9}$$

From (1), (6), and (8) we see that

$$M = r^m s^{1/t}.$$

Differentiating this expression twice with respect to r we obtain the formula

$$\begin{aligned} \mu &= t^2 r^2 s^2 M''/M \\ &= t^2 m(m-1)s^2 + 2tmsrs' + (1-t)r^2 s'^2 + tsr^2 s'', \end{aligned} \tag{10}$$

where primes denote differentiation with respect to r.

Now consider the function

$$N = r^n s^{1/2} \tag{11}$$

where n is an as yet undetermined integer. Differentiating this expression twice with respect to r we obtain the formula

$$\nu = 4r^2 s^2 N''/N = 4n(n-1)s^2 + 4nsrs' - r^2 s'^2 + 2sr^2 s''. \tag{12}$$

Substitution of the power series (9) into (12) gives

$$\nu = \sum_{-\infty}^{\infty}{}_{pq}\, \nu_{pq} |c_p c_q|^2 r^{2p+2q},$$

where the symmetrized coefficient $\nu_{pq}=\nu_{qp}$ has the value

$$\nu_{pq} = (2n + p + q - 1)^2 + 3(p - q)^2 - 1.$$

By examining the two cases $p=q$ and $p\neq q$ it is easy to see, since n, p, q are integers, that $\nu_{pq}\geqq 0$. Consequently $\nu\geqq 0$ and $N''\geqq 0$.

If $t=2$ and $n=m$, then $N=M$, whence $M''\geqq 0$. Therefore $M_t(r;f)$ is a convex function of r in $\alpha<r<\beta$ if $t=2$.

We now show that $M_t(r;f)$ is a convex function of r in $\alpha<r<\beta$ if $0<t<2$ and if the open interval

$$I_t(m):\quad tm - t < x < tm$$

contains no even integer. Evidently the interval

$$tm - t < x \leqq tm - t + 2$$

contains exactly one even integer $2n$. Since $2n$ does not lie in the interval $I_t(m)$ we conclude that

$$tm \leqq 2n \leqq tm - t + 2,$$

whence

$$tm - 2n \leqq 0 \leqq tm - 2n + 2 - t. \tag{13}$$

Now let this integer n be the n of (11). Consider the following expression quadratic in the variables s and

$$rs':\phi = 2\mu - t\nu = 2t(tm(m-1) - 2n(n-1))s^2 + 4t(m-n)srs' + (2-t)r^2s'^2.$$

The discriminant of this quadratic form is

$$\Delta = 8t(tm - 2n)(tm - 2n + 2 - t).$$

Since $0<t<2$ the coefficient $2-t$ of $r^2s'^2$ in ϕ is positive, and from (13) we see that $\Delta\leqq 0$. Therefore $\phi\geqq 0$ and, as we have already shown, $\nu\geqq 0$, so we conclude that

$$2\mu = \phi + t\nu \geqq 0,$$

whence $M''\geqq 0$ in the interval $\alpha<r<\beta$.

The open interval $I_t(m)$ contains no even integer $2h$, if t is of the form $2/k$ ($k=2, 3, \cdots$); else we would have the inequality

$$m - 1 < hk < m,$$

which is impossible since h, k, m are integers. Moreover, it is evident that the open interval $I_t(m)$ contains no even integer if $0<t<2$ and $m=0$ or if $0<t<2/m$ and $m\geqq 1$. Therefore $M_t(r;f)$ is a convex function of r in $\alpha<r<\beta$ if $t=2/k$ $(k=1, 2, 3, \cdots)$ or if $0<t\leqq 2/k$ $(k=1, 2, 3, \cdots)$ and $m\leqq k$.

It is easy to see how the theorem follows from this result. As we have already mentioned, the mean modulus M possesses a continuous derivative M' with respect to r in the interval $0\leqq r<1$ provided $0<t<\infty$. We have shown that under certain conditions M is convex in any open subinterval of $0\leqq r<1$ which contains no moduli of zeros of f; therefore under these conditions M' is nondecreasing in any subinterval closed in $0\leqq r<1$ which contains no moduli of zeros of f in its interior. Since the zeros of the nonconstant analytic function f are isolated, the collection of all such closed subintervals covers the interval $0\leqq r<1$. Thus M' is a nondecreasing function of r and hence M is a convex function of r in this interval.

The theorem is proved except for inclusion of the value $t=0$. However, since $T(f)$ is closed and contains the values $t=2/k$ $(k=1, 2, 3, \cdots)$ it also contains the limit value $t=0$. That $T(f)$ contains the value $t=0$ may also be seen directly from the Jensen formula for the geometric mean.

The following corollary concerns the length $l(r; f)$ of the map under f of the circle $C(r)$.

COROLLARY. *Both the length $l(r; f)$ of the map under f of the circle $C(r)$ and the circular expansion ratio $l(r; f)/2\pi r$ are nondecreasing convex functions of r in the interval $0\leqq r<1$.*

PROOF. It suffices merely to exhibit the following formula for the length of the map of $C(r)$:

$$l(r; f) = \int_0^{2\pi} \left| f'(re^{i\theta}) \right| r d\theta = 2\pi r \cdot M_1(r; f'),$$

where f' is the derivative of f and hence is analytic in the open unit circle.

THEOREM. *If f vanishes at the origin, then $T(f)$ contains the entire parameter interval $0\leqq t\leqq \infty$.*

PROOF. The theorem is evidently true if f is of the form cz where c is a constant. If f is not of this simple form, then, since f vanishes at $z=0$, there exists a nonconstant function g, analytic in the open unit circle, such that

$$f(z) = zg(z).$$

For a fixed parameter t in the interval $0<t<\infty$ let $F=M_t(r; f)$ and $G=M_t(r; g)$, so that

$$F = rG. \tag{14}$$

We shall denote successive differentiations of a function with respect to r by primes, and successive differentiations of the logarithm of a function with respect to log r by asterisks. In any interval $\alpha<r<\beta$ containing no moduli of zeros of the nonconstant analytic function g the first and second prime and asterisk derivatives of G exist and are connected by the following relations:

$$rG' = GG^*,$$
$$r^2G'' = G(G^{**} + G^{*2} - G^*).$$

Differentiating (14) twice with respect to r and using these relations we obtain the formula

$$rF'' = r^2G'' + 2rG' = G(G^{**} + G^{*2} + G^*).$$

As we have already mentioned $G^* \geqq 0$ and $G^{**} \geqq 0$, so that $F'' \geqq 0$ in $\alpha<r<\beta$. The extension of convexity of F to $0 \leqq r < 1$ proceeds as before. Therefore $T(f)$ contains the interval $0<t<\infty$ and being closed also includes the limiting parameters $t=0$ and $t=\infty$.

According to Schwarz's lemma the maximum modulus on the circle $C(r)$ of an analytic function f, which maps the origin into itself and the open unit circle into itself, is not greater than r. The above theorem shows that *this maximum modulus is also a convex function of* r.

Although the mean modulus $M_t(r; f)$ may not be convex in the entire interval $0 \leqq r < 1$, it may be convex in some subinterval containing $r=0$. We define $\rho(t; f)$ to be the length of the maximal such subinterval if one exists and to be 0 if no such subinterval exists. Since the limit function of convex functions is convex, it is clear that

$$\rho(t; f) \geqq \limsup_{t' \to t} \rho(t'; f),$$

whence $\rho(t; f)$ is an upper semicontinuous function of t.

We now show that $\rho(t; f) > 0$ if $0 \leqq t < \infty$. We have already seen by example that $\rho(\infty; f)$ may be 0. Since $\rho(t; f) = 1$ if $t=0$, if f is constant, or if f vanishes at $z=0$, we may suppose that $0<t<\infty$ and that f is a nonconstant analytic function which does not vanish at $z=0$. A neighborhood $|z| < \beta$ of $z=0$ then exists in which f does not vanish. Thus, in the notation of our first theorem we have $m=0$ and $A(z)=1$. The expansion (7) is then a Taylor expansion with $c_0 \neq 0$. Moreover, since f is nonconstant, not all the coefficients $c_1, c_2, c_3, \cdots$ vanish. Let $c_q \neq 0$ be the first such nonvanishing coefficient. Substitution of the power series (9) into (10) gives

$$\mu = 2tq(2q-1)\,|\,c_0c_q\,|^2 r^{2q} + O(r^{2q+2}).$$

Since q is a positive integer we infer that $\mu > 0$ and hence $M'' > 0$ in some neighborhood of $r = 0$. Consequently $\rho(t; f) > 0$.

We conclude with the following question: Is $\rho(t; f)$ a continuous, non-increasing function of t?

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AN INTEGRATION SCHEME OF MARÉCHAL

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The French physicist Maréchal [1][1] has invented a mechanical integrator for studying the distribution of light in an optical image. This integrator approximates a double integral $\int_0^{2\pi}\int_0^R f(r, \phi) r dr d\phi$ by a line integral $2\pi a \int_C f(r, \phi) ds$ extended over that portion of an archimedean spiral

$$C: \qquad r = a\phi$$

which lies inside the circle $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$. The validity of this procedure when $f(r, \phi)$ is continuous (as it always is in the case of the integrals determining distribution of light in an optical image) was taken for granted by Maréchal when a is small. It is the purpose of this note to justify Maréchal's approximation by proving the following theorem.

THEOREM. *If $f(r, \phi)$ is continuous on $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$ and is periodic with period 2π in ϕ, then*

$$(1) \qquad \lim_{a=0} 2\pi \int_0^R f(r, r/a)(a^2 + r^2)^{1/2} dr = \int_0^{2\pi} \int_0^R f(r, \phi) r dr d\phi.$$

Let us define

$$(2) \qquad \begin{aligned} P_N(r, \phi) &= \frac{1}{2N\pi} \int_0^{\pi} \{f(r, \phi + u) \\ &\qquad + f(r, \phi - u)\} \sin^2 (Nu/2) \csc^2 (u/2) du, \\ a_{nN}(r) &= \frac{1}{2\pi} (1 - |n|/N) \int_0^{2\pi} f(r, \phi) e^{-in\phi} d\phi. \end{aligned}$$

Then it is known from the theory of $(C, 1)$ summability of Fourier series that

$$(3) \qquad \begin{aligned} P_N(r, \phi) &= \sum_{n=-(N-1)}^{n=N-1} a_{nN}(r) e^{in\phi}, \\ \lim_{N=\infty} P_N(r, \phi) &= f(r, \phi) \end{aligned}$$

uniformly on $0 \leqq r \leqq R$, $0 \leqq \phi < 2\pi$. For each positive ϵ we can therefore

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[1] Numbers in brackets refer to the reference cited at the end of the paper.

pick an integer N such that

$$|P_N(r, \phi) - f(r, \phi)| < \epsilon/3\pi R^2. \tag{4}$$

Let M be the maximum of $|f(r, \phi)|$. Then

$$\left|2\pi\int_0^R f(r, r/a)(a^2 + r^2)^{1/2}dr - 2\pi\int_0^R f(r, r/a)rdr\right|$$
$$\leqq 2\pi M\int_0^R \{(a^2 + r^2)^{1/2} - r\}dr < \epsilon/3$$

if a is less than a suitably chosen $A_1(\epsilon)$. Moreover, since (4) holds

$$\left|2\pi\int_0^R f(r, r/a)rdr - 2\pi\int_0^R P_N(r, r/a)rdr\right| < \epsilon/3.$$

By virtue of (3),

$$\left|2\pi\int_0^R P_N(r, r/a)rdr - 2\pi\int_0^R a_{0N}(r)rdr\right|$$
$$\leqq 2\pi\sum_{|n|=1}^{N-1}\left|\int_0^R a_{nN}(r)e^{inr/a}dr\right|.$$

Since N is fixed when ϵ is chosen we infer from the Riemann-Lebesgue lemma (which is surely applicable since we see from (2) that $a_{nN}(r)$ is continuous) that if a is less than a suitably chosen $A_2(\epsilon)$, then

$$\left|2\pi\int_0^R P_N(r, r/a)rdr - 2\pi\int_0^R a_{0N}(r)rdr\right| < \epsilon/3.$$

Since it follows from (2) that $2\pi\int_0^R a_{0N}(r)rdr = \int_0^{2\pi}\int_0^R f(r, \phi)rdrd\phi$, we can now conclude from the above inequalities that when $a < A_1(\epsilon)$, $a < A_2(\epsilon)$,

$$\left|2\pi\int_0^R f(r, r/a)(a^2 + r^2)^{1/2}dr - \int_0^{2\pi}\int_0^R f(r, \phi)rdrd\phi\right| < \epsilon,$$

and the theorem is an immediate consequence of this inequality.

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A METHOD OF ANALYTIC CONTINUATION SUGGESTED BY HEURISTIC PRINCIPLES

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Suppose we are given an analytic function $f(z)$ represented by a power series (supposed convergent for some $z\neq 0$)

$$f(z)=\sum_{n=0}^{\infty}\frac{f_n}{n!}z^n \tag{1}$$

where f_n is the value of the nth derivative of $f(z)$ at the origin. For small values of δ we can approximate $f(\delta)$ in the following manner:

$$f(\delta)\simeq f_0+\delta f_1. \tag{2}$$

Refinement of this approximation leads to Taylor's theorem and back to the power series (1). It is possible, however, to use the linear approximation in a different way: we can use such approximations to go from one point to another along a chain of points $z=\delta$, 2δ, 3δ, $\cdots$, $n\delta$. Thus we shall say

$$f(\delta)\simeq f_0+\delta f_1, \qquad f'(z)\big|_{z=\delta}\simeq f_1+\delta f_2,$$

and so on, and

$$f(2\delta)\simeq f(\delta)+\delta f'(z)\big|_{z=\delta}\simeq f_0+2\delta f_1+\delta^2 f_2.$$

In general

$$f(n\delta)\simeq\sum_{m=0}^{n}a_{n,m}f_m\delta^m. \tag{3}$$

It is easily verified that $a_{n,m}=a_{n-1,m}+a_{n-1,m-1}$ and hence that $a_{n,m}$ is the binomial coefficient $C_{n,m}$. If we now define the following:

$$\sigma_n(z)=\sum_{m=0}^{n}f_mC_{n,m}\left(\frac{z}{n}\right)^m \tag{4}$$

and if $n\delta=z$, then (3) is equivalent to

$$f(z)\simeq\sigma_n(z). \tag{5}$$

The question now presents itself: for what values of z does the sequence of polynomials $\sigma_n(z)$ converge to the function $f(z)$? It is evident that if z is inside the circle of convergence of the series (1), then

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[1] The authors are indebted to Prof. R. Salem for help and advice.

we may say directly

$$\begin{aligned}\lim_{n\to\infty}\sigma_n(z) &= \lim_{n\to\infty}\sum_{m=0}^{n} f_m C_{n,m}\left(\frac{z}{n}\right)^m \\ &= \lim_{n\to\infty}\sum_{m=0}^{n}\frac{f_m z^m}{m!}\,\frac{n!}{(n-m)!n^m} \qquad (6)\\ &= \sum_{m=0}^{\infty}\frac{f_m z^m}{m!} = f(z).\end{aligned}$$

However, this is not the complete answer.

It was shown by Borel[2] that we can without loss of generality restrict ourselves to the special case $f(z)=1/(1-z)$. Here

$$f_n = n!,$$

$$\sigma_n(z) = \sum_{m=0}^{n}\frac{n!}{(n-m)!}\left(\frac{z}{n}\right)^m. \qquad (7)$$

Obviously if $|z| < 1$

$$\lim_{n\to\infty}\sigma_n(z) = \sum_{m=0}^{\infty} z^m = \frac{1}{1-z} \qquad (8)$$

and the convergence is uniform if $|z| \leqq 1-\epsilon$. It will be deduced later that the same is true if $|z| \leqq 1$ except for a neighborhood of the point $z=1$. Hence we shall confine ourselves to values of z such that $|z| > 1$. From (7)

$$\begin{aligned}\sigma_n(z) &= \sum_{m=0}^{n}\frac{n!}{(n-m)!}\left(\frac{z}{n}\right)^m \\ &= n!\left(\frac{z}{n}\right)^n\sum_{p=0}^{n}\frac{1}{p!}\left(\frac{n}{z}\right)^p \\ &= n!\left(\frac{z}{n}\right)^n e^{n/z} - \sum_{p=n+1}^{\infty}\frac{n!}{p!}\left(\frac{n}{z}\right)^{p-n};\end{aligned}$$

$$\sigma_n(z) = n!\left(\frac{z}{n}\right)^n e^{n/z} - \sum_{r=1}^{\infty} z^{-r}\frac{n!n^r}{(n+r)!}. \qquad (9)$$

If

$$|z| > 1 \qquad (10)$$

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[2] Borel, *Leçons sur les series divergentes*, Paris, 1924, pp. 198 ff.

then

$$\lim_{n\to\infty} -\sum_{r=1}^{\infty} z^{-r}\frac{n!n^r}{(n+r)!} = -\sum_{r=1}^{\infty} z^{-r} = \frac{1}{1-z} \tag{11}$$

and the convergence is uniform if $|z| \geqq 1+\epsilon$. Using Stirling's formula, we have

$$\lim_{n\to\infty} n!\left(\frac{z}{n}\right)^n e^{n/z} = \lim_{n\to\infty} (2\pi n)^{1/2}(ze^{-1+1/z})^n. \tag{12}$$

This is zero if

$$|ze^{-1+1/z}| < 1. \tag{13}$$

If we let $z=\rho e^{i\theta}$ then

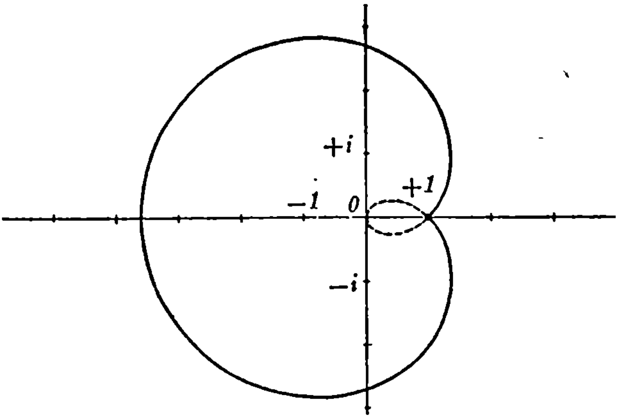
$$|ze^{-1+1/z}| = \rho e^{-1+(\cos\theta)/\rho}$$

and (13) is the same as

$$\rho e^{-1+(\cos\theta)/\rho} < 1$$

or

$$\cos\theta < \rho(1-\log\rho). \tag{14}$$



The curve $\cos\theta=\rho(1-\log\rho)$ and the region S. The region S is bounded by the solid part of the curve.

We shall call the region bounded by the curve $\cos\theta=\rho(1-\log\rho)$, $\rho\geqq 1$, the region S (see figure). In view of (8)–(12) and (14) we see that this is the region in which $\sigma_n(z)\to 1/(1-z)$ uniformly, except possibly for a neighborhood of $|z|=1$ and a neighborhood of the boundary.

For the purpose of applying Borel's theorems it is also necessary to show that in the region $(1-\epsilon\leqq\rho\leqq 1+\epsilon,\ \epsilon'\leqq\theta\leqq 2\pi-\epsilon')$, $\sigma_n(z)$ is bounded uniformly in z. This follows for $\rho\leqq 1$ from (7), by an application of Abel's transformation to the series $\sum_0^\infty z^n$ with the coefficients (obviously monotonically decreasing) $n!/(n-m)!n^m$. Similarly it follows from (9) for $\rho\geqq 1$.

Borel[8] now proceeds as follows: suppose we have a system of polynomials $\sigma_n(z)=c_0^{(n)}+c_1^{(n)}z+c_2^{(n)}z^2+\cdots$ and such that $\sigma_n(z)\to 1/(1-z)$. Suppose we have also a function $f(z)$ represented by the power series

$$\sum_{n=0}^{\infty}\frac{f_n}{n!}z^n.$$

We may write Cauchy's integral thus:

$$f(z)=\frac{1}{2\pi i}\int_c\frac{f(x)}{x}\frac{1}{1-z/x}dx$$

where c is a contour around z not including any singularities of $f(z)$. Let

$$F_n(z)=\frac{1}{2\pi i}\int_c\frac{f(x)}{x}\sigma_n\left(\frac{z}{x}\right)dx.$$

Borel showed that

$$F_n(z)=\sum_{k=0}^{\infty}c_k^{(n)}\frac{f_k}{k!}z^k.$$

In case $\sigma_n(z)\to 1/(1-z)$ uniformly in a certain star-shaped region including the circle of convergence of $\sum_0^\infty z^n$ Borel deduces by appropriate choice of the contour c a region in which $F_n(z)\to f(z)$ which we describe as follows:

Construct for each singularity ζ of $f(z)$ a region $R(\zeta)$ similar to that in which $\sigma_n(z)$ converges to $1/(1-z)$, such that the point corresponding to the origin is at the origin and the point corresponding to $z=1$ is at the singularity ζ. Let $R=\prod_\zeta R(\zeta)$ be the region common to all the $R(\zeta)$. In any finite region R' interior to R

$$F_n(z)\to f(z)$$

and the convergence is uniform if it is uniform in the special case

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[8] Borel, loc. cit. We have changed the notation to agree with our own.

$f(z)=1/(1-z)$, $F_n(z)=\sigma_n(z)$.

In our case, $c_k^{(n)}=n!/(n-k)!n^k$, $\sigma_n(z)\rightarrow 1/(1-z)$ uniformly in any subregion of S interior to S except in a region as small as we please where $|\sigma_n(z)-1/(1-z)|$ is uniformly bounded. The reader may convince himself that this restriction does not alter Borel's results.

If in the identity (9) we replace z by $1/z$ we can derive a more powerful method of continuation. It is not, however, as powerful as those of Borel, Mittag-Leffler, and others, and hence we shall not discuss it.

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ON THE REALITY OF ZEROS OF BESSEL FUNCTIONS

ABRAHAM HILLMAN

We shall present some observations on the reality of zeros of Bessel functions of real order, that is, functions satisfying the differential equation

$$z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2)y = 0 \tag{1}$$

with ν real. The two linearly independent solutions $J_\nu(z)$ and $Y_\nu(z)$ may be defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{r!\Gamma(\nu + r + 1)} \tag{2}$$

and

$$\begin{aligned} Y_\nu(z) &= \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} \quad \text{for } \nu \text{ not an integer,} \\ Y_n(z) &= \lim_{\nu \to n} Y_\nu(z) \quad \text{for integers } n. \end{aligned} \tag{3}$$

J_ν and Y_ν are in general many-valued functions of z. If in (2) we replace z by the positive real variable x and use the principal value of $(x/2)^\nu$, a real valued function, $J_\nu(x)$, is obtained. Substituting $J_\nu(x)$ for $J_\nu(z)$ in (3) gives a real function $Y_\nu(x)$.

All branches of any Bessel function, $B(z)$, can be obtained by analytic continuation of a function

$$B(x) = (a + ib)J_\nu(x) + (h + ik)Y_\nu(x),$$

where a, b, h, and k are real numbers. In particular let $B(x, m)$ stand for the result of continuing $B(x)$ through an angle of $m\pi$ along a circle with center at the origin. Restricting m to be an integer, it can be shown[1] that

$$\begin{aligned} B(x, m) = {} & [(aC - bS - 2kS \cot \nu\pi)J_\nu(x) + (hC + kS)Y_\nu(x)] \\ & + i[(bC + aS + 2hS \cot \nu\pi)J_\nu(x) + (kC - hS)Y_\nu(x)] \end{aligned}$$

where $C = \cos m\nu\pi$ and $S = \sin m\nu\pi$. Each real (positive or negative) zero on any branch of the analytic function $B(z)$ is a zero of $B(x, m)$

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[1] See G. N. Watson, *A treatise on the theory of Bessel functions*. p. 75.

for proper choice of m. Equating the real and imaginary parts of $B(x, m)$ to zero (here we use the hypothesis that ν is real), gives

$$(aC - bS - 2kS \cot \nu\pi)J_\nu(x) + (hC + kS)Y_\nu(x) = 0,$$
$$(bC + aS + 2hS \cot \nu\pi)J_\nu(x) + (kC - hS)Y_\nu(x) = 0.$$

$J_\nu(x)$ and $Y_\nu(x)$ are never simultaneously zero since they are linearly independent solutions of equation (1). Hence a necessary condition for $B(x, m)$ to have zeros is that

$$\begin{aligned}\Delta \equiv {} & (aC - bS - 2kS \cot \nu\pi)(kC - hS) \\ & - (bC + aS + 2hS \cot \nu\pi)(hC + kS)\end{aligned}$$

vanishes. This can be simplified to

$$(4)\quad \Delta \equiv [ak - bh] \cos 2m\nu\pi - [ah + bk + (h^2 + k^2) \cot \nu\pi] \sin 2m\nu\pi.$$

If Δ is equal to zero, $B(x, m)$ is a multiple of a real cylinder function. It is well known[2] that the real cylinder functions have (positive) real zeros. This shows that $\Delta = 0$ is also a sufficient condition for $B(x, m)$ to have zeros. (It is clear that there are two values of m associated with each branch of $B(z)$, an even value for positive zeros and an odd one for negative zeros.)

Before going on to some interesting special cases it should be noted that every Bessel function has, on every branch and in both the left and the right half-planes, an infinite number of zeros which lie on or approach a horizontal straight line. (The line depends on the branch and the half-plane.) This is a consequence of the fact[3] that

$$J_\nu(z) \cos \alpha - Y_\nu(z) \sin \alpha$$

where ν and α need not be real, has zeros whose real parts tend towards plus infinity and which are given by the asymptotic expansion

$$\begin{aligned} z_s \sim \beta - \frac{u - 1}{2^3\beta} - \frac{(u - 1)(7u - 31)}{3 \cdot 2^7 \beta^3} - \frac{(u - 1)(83u^2 - 982u + 3779)}{15 \cdot 2^{10} \beta^5} \\ - \frac{(u - 1)(6949u^3 - 153855u^2 + 1585743u - 6277237)}{105 \cdot 2^{15} \beta^7} - \cdots \end{aligned}$$

where $u = 4\nu^2$ and $\beta = (s + \nu/2 - 1/4)\pi - \alpha$.

This implies that the condition for the non-existence of positive or negative real zeros on a particular branch (that is, for a particular value of m) of a Bessel function is the condition for the existence of non-real complex zeros in that half-plane on that branch. However a Bessel function can have both real and non-real zeros in the same half-plane on the same branch. For example, if r is zero or a positive integer and if ν is between $-(2r+1)$ and $-(2r+2)$, $J_\nu(z)$ has $4r+2$ complex[4] zeros as well or an infinite number of positive and negative real[5] zeros on each branch.

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[2] Ibid. p. 481.

[3] Ibid. pp. 505–506.

Now let us take up some of the special cases. We assume that a, b, h, k are not all zero, that is, that B is not identically zero.

I. Let B be a multiple of J_ν. That is, let $h=k=0$. Then the Δ of (4) is zero for all values of m and ν. In other words, J_ν has positive and negative real zeros on every branch for all real values of ν.

II. Let ν be an integer n. Then Δ becomes $ak-bh-2m(h^2+k^2)$. We see that with the exception of multiples of J_ν (treated under I), Bessel functions of integral order have real zeros if and only if $(ak-bh)/2(h^2+k^2)$ is an integer and then only on the branch for which $m=(ak-bh)/2(h^2+k^2)$.

In particular $Y_n(z)$ has positive real zeros only on the branch for which $m=0$ and has no negative real zeros. Thus $Y_n(z)$ has non-real complex zeros in the left half-plane on all branches and in the right half-plane on all but one branch.

III. Let B be a multiple of Y_ν, that is, let $a=b=0$. Then $\Delta=0$ is equivalent to $\cot \nu\pi \sin 2m\nu\pi=0$. This means that Y_ν always has positive real zeros on the branch for which $m=0$ and has other real zeros only if ν is rational but not an integer. In the latter case Y_ν has real zeros only on the branches for which $2m\nu$ is an integer. In particular $Y_{n+1/2}$ where n is an integer has positive and negative real zeros on all branches.

IV. Let B be a multiple of a function $\lambda J_\nu+\mu Y_\nu$ with λ and μ real, that is, let $ak-bh=0$. Then B has positive and negative real zeros on all branches if B is a multiple of J_ν or of $J_\nu \cos \nu\pi - Y_\nu \sin \nu\pi$. Otherwise B has real zeros if and only if $2m\nu$ is an integer but ν is not.

V. If $2m\nu$ is an integer but ν is not then B has real zeros if and only if it is a multiple of a function $\lambda J_\nu+\mu Y_\nu$ with λ and μ real.

VI. The condition on a, b, h, and k for Δ to be zero for all values of m and ν is that $h=k=0$. (VI is the converse of I.)

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[4] Ibid. p. 483.

[5] Ibid. p. 478.

A PROPERTY OF POWER SERIES WITH POSITIVE COEFFICIENTS

P. ERDÖS, W. FELLER, AND H. POLLARD

The following theorem is suggested by a problem in the theory of probability.[1]

Let p_k *be a sequence of non-negative numbers for which* $\sum_0^\infty p_k=1$, *and let* $m=\sum_1^\infty kp_k \leqq \infty$. *Suppose further that*

$$P(x) = \sum_0^\infty p_k x^k$$

is not a power series in x^t *for any integer* $t>1$. *Then* $1-P(x)$ *has no zeros in the circle* $|x|<1$, *and the series*

$$U(x) = \frac{1}{1-P(x)} = \sum_0^\infty u_k x^k$$

has the property

$$\lim_{n\to\infty} u_n = 1/m.$$

(*If* $m=\infty$, *we define* $1/m$ *to be zero.*)

We shall first give a proof in case $m<\infty$. The method used is not elementary, but yields somewhat more information than stated in the theorem. Later in this paper an elementary proof is given, valid for both $m<\infty$ and $m=\infty$.

We suppose that $m<\infty$. Let

$$(1)\qquad r_n = \sum_{k=n+1}^\infty p_k, \qquad R(x) = \sum_0^\infty r_n x^n.$$

Then $m=\sum_0^\infty r_k$ and

$$(2)\qquad 1-P(x) = (1-x)R(x).$$

Since $m<\infty$ the power series for $R(x)$ converges absolutely and uniformly in $|x|\leqq 1$. We claim that $R(x)$ has no zeros for $|x|\leqq 1$. For $|x|<1$ this is clear from (2), since $P(x)$ has positive coefficients

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[1] To be published elsewhere. The theorem and method of the present paper were extended to the continuous case by D. Blackwell, *A renewal theorem*, Duke Math. J. vol. 15 (1948) pp. 145–150. This research originated from work under an ONR contract at Cornell University.

and therefore $|P(x)|<1$, for $|x|<1$. Then any zeros of $R(x)$ must occur on the circle $|x|=1$. From (1) it follows that $R(1)=m\neq 0$. Hence any zero must be of the form $x_0=e^{i\theta_0}$, $0<\theta_0<2\pi$. If $R(x_0)=0$, then (2) implies $P(e^{i\theta_0})=1$. Since $p_k\geqq 0$ this can happen only if $\cos\theta_0 k=1$ for all k for which $p_k\neq 0$. But this is impossible because $P(x)$ would be a power series in x^t, for some integer $t>1$.

Then the function

$$\frac{1-x}{1-P(x)}=\frac{1}{R(x)}$$

has no singularities in $|x|<1$ and we can expand it in a power series

$$\frac{1}{R(x)}=\sum_{n=0}^{\infty}a_nx^n, \tag{3}$$

where

$$a_n=\frac{1}{2\pi i}\int_{|x|=r}\frac{x^{-n-1}}{R(x)}\,dx, \qquad r<1.$$

Now $[R(x)]^{-1}$ is bounded in $|x|<1$; therefore we can apply Lebesgue's convergence theorem to let $r\to 1$; we obtain

$$a_n=\frac{1}{2\pi i}\int_{|x|=1}\frac{x^{-n-1}}{R(x)}\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{R(e^{i\theta})}\,d\theta, \tag{4}$$

for $n=0, 1, \cdots$.

But we have already seen that

$$R(e^{i\theta})=\sum_{n=0}^{\infty}r_ne^{i\theta n}$$

converges absolutely and has no zeros. It follows by a theorem of Wiener[3] that $[R(e^{i\theta})]^{-1}$ has an absolutely convergent expansion

$$\frac{1}{R(e^{i\theta})}=\sum_{n=-\infty}^{\infty}b_ne^{in\theta},$$

where $\sum|b_n|<\infty$ and

$$b_n=\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{R(e^{i\theta})}\,d\theta, \qquad -\infty<n<\infty.$$

A comparison of this and (4) shows that $\sum|a_n|<\infty$. From (3) we can conclude that $\sum a_n = m^{-1}$. But

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[3] A. Zygmund, *Trigonometrical series*, Warsaw, 1935, p. 140.

$$\sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{k=0}^{\infty} u_k x^k.$$

It follows from Abel's theorem that $u_k \to 1/m$.

This argument does not work in the case $m = \infty$ since then $R(x)$ is not bounded. The following proof is quite elementary and does not distinguish between the two cases.

Clearly

$$u_n = p_0 u_n + p_1 u_{n-1} + \cdots + p_{n-1} u_1 + p_n u_0, \qquad n \geqq 1.$$

Moreover, since $R(x)U(x) = (1-x)^{-1}$, we have

$$r_0 u_n + r_1 u_{n-1} + \cdots + r_{n-1} u_1 + r_n u_0 = 1.$$

Let

$$\lambda = \limsup u_n,$$

and let $\{n_\nu\}$ be a sequence such that $u_{n_\nu} \to \lambda$. We claim that for any fixed $j > 0$ for which $p_j > 0$ we have $u_{n_\nu - j} \to \lambda$. In fact, assume that $u_{n_\nu - j} \to \lambda' < \lambda$. Then for sufficiently large ν

$$\begin{aligned} \lambda - \epsilon &< u_{n_\nu} \\ &< (p_0 + \cdots + p_{j-1} + p_{j+1} + \cdots + p_N)(\lambda + \epsilon) + p_j \lambda' + \epsilon \\ &\leqq (1 - p_j)(\lambda + \epsilon) + p_j \lambda' + \epsilon < \lambda - p_j(\lambda - \lambda') + 2\epsilon \end{aligned}$$

whence $\lambda' = \lambda$.

Repeating the same argument, we see that

$$\lim_{\nu \to \infty} u_{n_\nu - sj} = \lambda \tag{5}$$

for every fixed integer $j \geqq 0$, provided only that $u_{n_\nu} \to \lambda$ and $p_j > 0$. Now consider the set of all integers j for which $p_j > 0$. By hypothesis, their greatest common divisor is 1. We can, therefore, find a *finite* collection $a_1, a_2, \cdots, a_t$ of subscripts such that $p_{a_1} > 0, \cdots, p_{a_t} > 0$ and that their greatest common divisor is 1. Then by (5)

$$\lim_{\nu \to \infty} u_{n_\nu - k} = \lambda \tag{6}$$

for every fixed integer k of the form

$$k = x_1 a_1 + x_2 a_2 + \cdots + x_t a_t. \tag{7}$$

However, *every* integer $k > a_1 a_2 \cdots a_t$ can be put into the form (7) and hence (6) holds for every sufficiently large fixed k. Now put in (5) $n=N_\nu=n_\nu-a_1a_2\cdots a_t$. Then for every fixed M

$$1 \geqq r_0u_{N_\nu}+r_1u_{N_\nu-1}+\cdots+r_Mu_{N_\nu-M}.$$

As $\nu\to\infty$ all terms $u_{N_\nu-k}\to\lambda$ and hence

$$1 \geqq \lambda(r_0+r_1+\cdots+r_M)$$

or $\lambda\leqq 1/m$ (with $\lambda=0$ if $\sum r_n=\infty$).

If $m<\infty$ we can use a similar argument for $\mu=\liminf u_n$ to show that $\mu\geqq 1/m$. This proves the theorem.

SYRACUSE UNIVERSITY AND
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A CONSISTENCY THEOREM

H. D. BRUNK[1]

1. Introduction. Of primary importance in a theory of representation of functions by series which do not necessarily converge is its consistency theorem, which states that if a series which represents a function F converges to a function Φ, then $F\equiv\Phi$. Such a theorem for asymptotic representation in a strip region of a function by Dirichlet series with a certain logarithmic precision, an idea introduced by Mandelbrojt [1],[2] is the subject matter of this note. From it follow similar theorems for less general extensions of the idea of asymptotic series. The method consists in using the proof of the fundamental theorem in [1] to set up a homogeneous linear differential equation of infinite order with constant coefficients, which must be satisfied by the difference $F-\Phi$; then applying a method of Ritt to show that the only solution is identically zero.

The notation used by Mandelbrojt in [1] will be used here also. Let $\{\lambda_n\}$ be an increasing sequence of positive numbers $(0<\lambda_n\uparrow)$. Denote by $N(\lambda)$, defined for $\lambda>0$, the *distribution function* of $\{\lambda_n\}$; that is, the number of terms in the sequence $\{\lambda_n\}$ less than λ; and by $D(\lambda)$ the *density function* of $\{\lambda_n\}$: $D(\lambda)=N(\lambda)/\lambda$. Let D^* represent the *upper density*: $D^*=\limsup_{\lambda\to\infty}D(\lambda)$; and $D^*(\lambda)$ the *upper density function* of $\{\lambda_n\}$: $D^*(\lambda)=\text{l.u.b.}_{x\geqq\lambda}D(x)$; clearly $D^*(\lambda)$ is continuous and decreases to D^* (unless $D^*(\lambda)\equiv D^*=\infty$).

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[1] The author is indebted to Professor Mandelbrojt for suggesting the problem considered in this note.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

The *mean density function* is $\bar{D}(\lambda)$: $\bar{D}(\lambda) = [\int_0^\lambda D(x)dx]/\lambda$; $\bar{D}^\cdot$, the *upper mean density*: $\bar{D}^\cdot = \lim\sup_{\lambda\to\infty}\bar{D}(\lambda)$; and $\bar{D}^\cdot(\lambda) = \text{l.u.b.}_{x\geqq\lambda}\bar{D}(x)$, the *upper mean density function*. The function $\bar{D}^\cdot(\lambda)$ decreases to $\bar{D}^\cdot$ as λ becomes infinite. Also $\bar{D}^\cdot \leqq D^\cdot$ (cf. [1]); and if either $\bar{D}^\cdot$ or $D^\cdot$ is finite, so is the other [1, p. 353].

Definition. Let Δ be a region of the s-plane ($s=\sigma+it$) containing points with arbitrarily large real part, and $F(s)$ a function holomorphic in Δ. Let n be a positive integer, $\{d_k\}$ a sequence of complex numbers, and $p_n(x)$ a function increasing to infinity such that for s in Δ and for x sufficiently large

$$\text{(1)}\qquad \underset{m\geqq n}{\text{g.l.b.}}\ \underset{\sigma\geqq x}{\text{l.u.b.}}\left|F(s) - \sum_{k=1}^{m} d_k e^{-\lambda_k s}\right| \leqq e^{-p_n(x)}.$$

The sums $\sum_{k=1}^{m} d_k e^{-\lambda_k s}$ $(m\geqq n)$ *are then said to represent* $F(s)$ *in* Δ *with the logarithmic precision* $p_n(\sigma)$. If the series $\sum_{k=1}^{\infty} d_k e^{-\lambda_k s}$ converges to $F(s)$ for σ sufficiently large, the logarithmic precision is identically infinite for each n [1, p. 356].

Theorem I. *If*

(i) *the sequence* $\{\lambda_n\}$ *has finite upper mean density* $\bar{D}^\cdot$;

(ii) *the function* $F(s)$ *is holomorphic in the region* Δ *of the s-plane* ($s=\sigma+it$) *given by*: $\{\sigma > b,\ |t| < \pi g(\sigma)\}$, *where b is a real constant, and* $g(\sigma)$ *is a continuous function of bounded variation defined for* $\sigma > b$ *such that* $\lim_{\sigma\to\infty} g(\sigma) > \bar{D}^\cdot$;

(iii) $\text{g.l.b.}_{n\geqq 1}(\lambda_{n+1}-\lambda_n) > 0$;

(iv) *for infinitely many positive integers n, the sums* $\sum_{k=1}^{m} d_k e^{-\lambda_k s}$ $(m\geqq n)$ *represent* $F(s)$ *in* Δ *with a logarithmic precision* $p_n(\sigma)$ *satisfying*[3]

$$\text{(2)}\qquad \int^{\sigma} p_n(\sigma)\exp\left(-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-\bar{D}^\cdot(p_n(u))}\,d\sigma\right) = \infty;$$

then the series $\sum_{k=1}^{\infty} d_k e^{-\lambda_k s}$ *converges to* $F(s)$ *for* σ *sufficiently large*; *that is, it has a half-plane of convergence, in which it converges to* $F(s)$.

Theorem I gives immediately a corresponding result for the theory of asymptotic representation with respect to a sequence of functions[4]

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[3] The omission of the lower limits on the integrals indicates that the relation is to hold provided these limits are chosen sufficiently large.

[4] The following definition of asymptotic representation with respect to an asymptotic sequence, as well as the discussion of its relation to representation with a certain logarithmic precision which appears in §3, is based on lectures given at The Rice Institute in 1947 by Professor Mandelbrojt.

$\{A_n(\sigma)\}$. Consider a sequence of continuous functions $\{A_n(\sigma)\}$ having the property that

(3) $A_n(\sigma)$ decreases to zero as $\sigma \rightarrow \infty$ $(n \geqq 1)$;

(4) $A_{n+1}(\sigma) = O(A_n(\sigma))$ as $\sigma \rightarrow \infty$ $(n \geqq 1)$.

DEFINITION. The function $F(s)$ is represented asymptotically in a region Δ containing points with arbitrarily large real part by $\sum_{k=1}^{\infty} d_k e^{-\lambda_k s}$ with respect to the *asymptotic sequence* $\{A_n(\sigma)\}$ if there exists a real number σ_0 independent of n such that

$$(5) \qquad \left| F(s) - \sum_{k=1}^{n} d_k e^{-\lambda_k s} \right| < A_n(\sigma) \quad \text{for} \quad \sigma > \sigma_0 \qquad (n \geqq 1).$$

Define $A(\sigma)$, the *lower envelope* of the sequence $\{A_n(\sigma)\}$, as $\text{g.l.b.}_{n \geqq 1} A_n(\sigma)$, and set $p(\sigma) = -\log A(\sigma)$. The following theorem is a corollary of Theorem I.

THEOREM II. *If* (i), (ii), (iii), *and*

(iv′) *$F(s)$ is represented asymptotically in Δ by $\sum d_k e^{-\lambda_k s}$ with respect to an asymptotic sequence $\{A_n(\sigma)\}$ such that*

$$(2') \qquad \int^{\infty} p(\sigma) \exp\left(-\frac{1}{2} \int^{\sigma} \frac{du}{g(u) - \overline{D}^{\cdot}(p(u))} \right) d\sigma = \infty$$

then the series converges to $F(s)$ for σ sufficiently large.

In both theorems the upper mean density $\overline{D}^{\cdot}$ may be replaced by the upper density $D^{\cdot}$, the upper mean density function $\overline{D}^{\cdot}(\lambda)$ being replaced by the upper density function $D^{\cdot}(\lambda)$. Moreover (2) and (2′) may be replaced by hypotheses corresponding to those in forms C, D, and E of the statement of the fundamental theorem in [1]. In particular (2) may be replaced by:

(2a) There exist continuous functions $h_n(\sigma)$ decreasing to $\overline{D}^{\cdot}$, and increasing functions $C_n(\sigma)$ such that

$$2\nu(h_n(\sigma)) - p_n(\sigma) < -C_n(\sigma),$$

$$(n \geqq 1)$$

$$\int^{\infty} C_n(\sigma) \exp\left(-\frac{1}{2} \int^{\sigma} \frac{du}{g(u) - h_n(u)} \right) d\sigma = \infty;$$

where $\nu(D) = \text{l.u.b.}_{\lambda > 0}\, \lambda(\overline{D}(\lambda) - D) = \text{l.u.b.}_{\lambda > 0} \int_0^{\lambda} [D(x) - D]\,dx$. This is a non-negative, nonincreasing continuous function of D for $D > \overline{D}^{\cdot}$, *called the excess function of the sequence* $\{\lambda_n\}$.

If the $\{A_n(\sigma)\}$ have the special form $A_n(\sigma) = M_n e^{-\lambda_n \sigma}$, Theorem II reduces to a theorem proved by Mandelbrojt by a different method, mentioned briefly in §4.

2. **Proof of I.** The proof of Theorem I falls naturally into two parts, Lemmas 1 and 2 below.

LEMMA 1. *Under the hypotheses of Theorem* I, *the series* $\sum d_k e^{-\lambda_k s}$ *converges for* σ *sufficiently large.*

This lemma was proved by Mandelbrojt in his lectures at The Rice Institute in 1947.

One may assume that $p_n(\sigma)$ is a decreasing function of n for each σ. For if (1) is satisfied with right-hand member $e^{-p_n^*(x)}$, it is also satisfied if $p_n^*(x)$ is replaced by $p_n(x) = \text{l.u.b.}_{j \geqq n} p_j^*(x)$, since for each $j \geqq n$ (writing $\phi_m(x) = \text{l.u.b.}_{\sigma \geqq x} \left| F(s) - \sum_{k=1}^{m} d_k e^{-\lambda_k s} \right|$) $\text{g.l.b.}_{m \geqq n} \phi_m(x) \leqq \text{g.l.b.}_{m \geqq j} \phi_m(x) \leqq e^{-p_j^*(x)}$ hence $\text{g.l.b.}_{m \geqq n} \phi_m(x) \leqq \text{g.l.b.}_{j \geqq n} e^{-p_j^*(x)} = e^{-p_n(x)}$. Moreover, if the $p_n^*(\sigma)$ satisfy (2) (or 2a) so do the $p_n(\sigma)$.

Hence if (2) is satisfied for any integer n, it is satisfied also for any smaller integer ($\overline{D}^{\cdot}(\lambda)$ is a decreasing function of λ). Thus (iv) is satisfied for all $n \geqq 1$. The hypotheses of Theorem I are then sufficient in order that the conclusion of Mandelbrojt's fundamental theorem [1, p. 357] may hold for $n \geqq 1$:

$$|d_n| \leqq A \lambda_n \Delta_n^* e^{\lambda_n \sigma_0} \qquad \text{for some } \sigma_0 > b;$$

where A is independent of n, depending on the sequence $\{\lambda_n\}$ and an upper bound on F in Δ (it is clear from (1) that F is bounded in Δ), and where

$$\Delta_n^* = \frac{1}{|\Lambda_n(i\lambda_n)|}, \qquad \Lambda_n(iz) = \prod_{k \neq n} \left(1 - \frac{z^2}{\lambda_k^2}\right).$$

It follows from Mandelbrojt's inequalities on Δ_n^* [1, p. 353] that if $\lim \inf_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h > 0$, there exists a number $B = B(h, D^{\cdot})$ such that $\log \Delta_n^* \leqq B \lambda_n$ for n sufficiently large. Hence

$$\begin{aligned} \log |d_n| &\leqq \log (A\lambda_n) + B\lambda_n + \lambda_n \sigma_0, \\ (\log |d_n|)/\lambda_n &\leqq B + \sigma_0 + (\log (A\lambda_n))/\lambda_n. \end{aligned} \tag{6}$$

But since the upper density $D^{\cdot}$ is finite, a necessary and sufficient condition that $\sum d_k e^{-\lambda_k s}$ converge for σ sufficiently large is that $\lim \sup_{n \to \infty} (\log |d_n|)/\lambda_n$ be finite (not positively infinite), and indeed this number is the abscissa of convergence.[5] This completes the proof of Lemma 1.

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[5] Cf. V. Bernstein, *Séries de Dirichlet*, Paris, Gauthier-Villars, 1933, p. 4.

LEMMA 2. *If* (i), (ii), (iv), *and*
(v) $\sum d_k e^{-\lambda_k s}$ *converges for* σ *sufficiently large, then it converges to* $F(s)$.

Let $\Phi(s)$ denote the function to which the series converges under (v). Define the coefficients $\{c_k^{(n)}\}$ by

$$\Lambda_n(iz) = \prod_{k\neq n}\left(1 - \frac{z^2}{\lambda_k^2}\right) = \sum_{k=0}^{\infty} (-1)^k c_k^{(n)} z^{2k}. \tag{7}$$

In the proof of his fundamental theorem, Mandelbrojt proves that in a certain closed strip $\bar{\Delta}$ (*not* the closure of Δ) given by $\{\sigma \geqq d, |t| \leqq G(\sigma)\}$ (where d is a certain real number and $G(\sigma)$ a positive function, both independent of n), the series $\sum_{k=0}^{\infty}(-1)^k c_k^{(n)} F^{(2k)}(s)$ converges absolutely and uniformly to the function $d_n \Lambda_n(i\lambda_n) e^{-\lambda_n s}$. Since $\Phi(s)$ satisfies the same hypotheses, $\sum (-1)^k c_k^{(n)} \Phi^{(2k)}(s)$ converges absolutely and uniformly in $\bar{\Delta}$ to the same function; and if $H(s) = F(s) - \Phi(s)$, then for s in $\bar{\Delta}$ and for $n \geqq 1$,

$$\sum (-1)^k c_k^{(n)} H^{(2k)}(s) \equiv 0. \tag{8}$$

From the definition (7) of the $c_k^{(n)}$ it is seen that (8) may be written symbolically as

$$[(1 - D^2/\lambda_1^2)(1 - D^2/\lambda_2^2) \cdots \cdot (1 - D^2/\lambda_{n-1}^2)(1 - D^2/\lambda_{n+1}^2) \cdots]H(s) \equiv 0,$$

where D indicates differentiation with respect to s. A method due to Ritt[6] may then be used to show that $H(s) \equiv 0$, which is the desired result. Define functions $N_m(z)$ and constants $c_{k,m}$ by:

$$N_m(z) = \prod_{k>m}\left(1 - \frac{z^2}{\lambda_k^2}\right) = \sum_{k=0}^{\infty} (-1)^k c_{k,m} z^{2k} \qquad (m \geqq 1). \tag{9}$$

Then for each m,

$$c_{0,m} = 1; \quad \text{and for each } k, \quad c_{k,m} \text{ decreases to zero as } m \to \infty. \tag{10}$$

For

$$c_{k,m} = \sum_{i_1=m+1}^{\infty} \frac{1}{\lambda_{i_1}^2} \sum_{i_k > i_{k-1} > \cdots > i_2 > i_1} \frac{1}{\lambda_{i_2}^2} \cdot \frac{1}{\lambda_{i_3}^2} \cdots \frac{1}{\lambda_{i_k}^2} \qquad (k \geqq 1,\ m \geqq 1),$$

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[6] Ritt, J. F., *On a general class of linear homogeneous differential equations of infinite order with constant coefficients*, Trans. Amer. Math. Soc. vol. 18 (1917) pp. 27–49; cf. in particular Theorem XIV.

the second sum being bounded by a number, $c_{k-1}^{(1)}$, independent of m:

$$c_{k-1}^{(1)} = \sum_{i_k>i_{k-1}>\cdots>i_1\geqq 1} \frac{1}{\lambda_{i_1}^2}\cdot\frac{1}{\lambda_{i_2}^2}\cdots\frac{1}{\lambda_{i_k}^2};$$

from which it is apparent also that $c_{k,m}\leqq c_k^{(1)}$ $(m\geqq 1,\ k\geqq 0)$. Since $\sum(-1)^k c_k^{(1)}H^{(2k)}(s)$ converges absolutely and uniformly in $\overline{\Delta}$, the series $\sum(-1)^k c_{k,m}H^{(2k)}(s)$ also converges absolutely and uniformly in $\overline{\Delta}$ to a function $h_m(s)$ holomorphic in $\overline{\Delta}$. For each s in $\overline{\Delta}$ the convergence is uniform with respect to m also, so that $\lim_{m\to\infty}h_m(s)=H(s)$ by (10). Symbolically

$$(11)\qquad h_m(s) = [(1 - D^2/\lambda_{m+1}^2)(1 - D^2/\lambda_{m+2}^2)\cdots]H(s).$$

By (8) and the definition of the $h_m(s)$, for each $m\geqq 1$,

$$(12)\qquad \begin{aligned} &(1 - D^2/\lambda_2^2)(1 - D^2/\lambda_3^2)\cdots(1 - D^2/\lambda_m^2)\ \ h_m(s) \equiv 0,\\ &(1 - D^2/\lambda_1^2)(1 - D^2/\lambda_3^2)\cdots(1 - D^2/\lambda_m^2)\ \ h_m(s) \equiv 0,\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,\\ &(1 - D^2/\lambda_1^2)(1 - D^2/\lambda_2^2)\cdots(1 - D^2/\lambda_{m-1}^2)h_m(s) \equiv 0. \end{aligned}$$

The operator $\prod_{j=1(j\neq n)}^{j=m}(1-D^2/\lambda_j^2)$ is evidently equivalent to $\sum_{j=1}^{m-1}(-1)^j a_{j,m}^{(n)}D^{2j}$, the $a_{j,m}^{(n)}$ being the coefficients in the Taylor's expansion of $\prod_{j=1(j\neq n)}^{m}(1-z^2/\lambda_j^2)$. Since we have $(\sum_{j=1}^{m-1}(-1)^j a_{j,m}^{(n)}z^{2j})\cdot(\sum_{k=0}^{\infty}(-1)^k c_{k,m}z^{2k})\equiv\sum_{u=0}^{\infty}(-1)^u c_u^{(n)}z^{2u}$, one easily verifies that $\sum_{j=1}^{m-1}(-1)^j a_{j,m}^{(n)}D^{2j}(\sum_{k=0}^{\infty}(-1)^k c_{k,m}D^{2k}H(s))\equiv\sum_{u=0}^{\infty}(-1)^u c_u^{(n)}D^{2u}H(s)$ $\equiv 0$, these being equations (12).

This is a system of m differential equations of finite order with constant coefficients. In order to satisfy the first of these equations, $h_m(s)$ must be a linear combination of powers $\pm\lambda_2 s$, $\pm\lambda_3 s$, $\pm\lambda_4 s$, $\cdots$, $\pm\lambda_m s$ of e; to satisfy the second, a linear combination of powers $\pm\lambda_1 s$, $\pm\lambda_3 s$, $\cdots$, $\pm\lambda_m s$, and so on. It follows that $h_m(s)\equiv 0$ $(m\geqq 1)$. Since $H(s)=\lim_{m\to\infty}h_m(s)$, also $H(s)\equiv 0$, which completes the proof of the lemma; and with Lemma 1, completes the proof of Theorem 1.

3. **Proof of II.** In his lectures at The Rice Institute referred to above, Mandelbrojt proved the following lemma.

LEMMA 3. *Representation with respect to an asymptotic sequence* $\{A_n(\sigma)\}$ *implies representation for each positive integer n with logarithmic precision* $p(\sigma)-K_n$ *for certain positive constants* K_n, *where* $p(\sigma)=-\log A(\sigma)$, *and where* $A(\sigma)$ *is the lower envelope of* $\{A_n(\sigma)\}$: $A(\sigma)=\text{g.l.b.}_{n\geqq 1}A_n(\sigma)$.

By definition (3)–(5), $\text{l.u.b.}_{\sigma\geqq x}\left|F(s)-\sum_1^m d_k e^{-\lambda_k s}\right| \leqq A_m(x)$; and there exist constants $B_2, B_3, \cdots$ such that for $x > \sigma_0$ the inequalities $A_2(x) < B_2A_1(x)$, $A_3(x) < B_3A_2(x), \cdots, A_n(x) < B_nA_{n-1}(x)$ hold simultaneously. Hence $A_n(x) < B_nA_{n-1}(x) < B_nB_{n-1}A_{n-2}(x) < \cdots < B_nB_{n-1} \cdots B_2A_1(x)$. Choose R_n greater than 1 so as to be greater than $\max_{0\leqq j\leqq n-2}\prod_{i=0}^{j} B_{n-i}$ also. Then for each fixed n, $\text{g.l.b.}_{m\geqq n} R_nA_m(x) > \text{g.l.b.}_{m\geqq n}\, A_m(x)$, and $\text{g.l.b.}_{m<n}\, R_nA_m(x) > A_n(x)$; hence we have $\text{g.l.b.}_{m\geqq 1}\, R_nA_m(x) > \text{g.l.b.}_{m\geqq n}\, A_m(x)$. Thus

$$\begin{aligned}\underset{m\geqq n}{\text{g.l.b.}}\ \underset{\sigma\geqq x}{\text{l.u.b.}}\left|F(s)-\sum_1^m d_k e^{-\lambda_k s}\right| &\leqq \underset{m\geqq n}{\text{g.l.b.}}\ A_m(x)\\ &< \underset{m\geqq 1}{\text{g.l.b.}}\ R_nA_m(x) = R_nA(x) = \exp\left(\log A(x) + K_n\right)\\ &= \exp - \left[p(x) - K_n\right] \qquad \text{for } x > \sigma_0,\end{aligned}$$

where $K_n = \log R_n > 0$, from which the lemma follows.

It will be sufficient, then, to show that hypothesis (iv′) of Theorem II implies (iv) of Theorem I, with (2a) replacing (2). Mandelbrojt shows in [1] that (iv′) implies the existence of a function $h(\sigma)$, continuous and decreasing to $\overline{D}^{\cdot}$, and an increasing function $C(\sigma)$ such that

$$2\nu(h(\sigma)) - p(\sigma) < -C(\sigma),$$

$$\int^{\infty} C(\sigma)\exp\left(-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-h(u)}\right)d\sigma = \infty.$$

Put $p_n(\sigma) = p(\sigma) - K_n$, $h_n(\sigma) = h(\sigma)$, and $C_n(\sigma) = C(\sigma) - K_n$ for $n \geqq 1$. Since $\lim_{u\to\infty} g(u) > \overline{D}^{\cdot}$, the denominator in the integral appearing as a power of e is bounded above and below by positive numbers for u sufficiently large. Hence $\int^{\infty} K_n\{\exp\,[2^{-1}\int^{\sigma} du/(g(u)-h(u))]\}d\sigma < \infty$ for $n \geqq 1$, and

$$2\nu(h_n(\sigma)) - p_n(\sigma) < -C_n(\sigma),$$

$$\int^{\infty} C_n(\sigma)\exp\left(-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)-h(u)}\right)d\sigma = \infty \tag{2a}$$

holds. The hypotheses of Theorem I are then satisfied, and the conclusion, that $\sum d_k e^{-\lambda_k s}$ converges to $F(s)$ for σ sufficiently large, follows.

4. **Remarks.** Mandelbrojt has given a simpler and more direct proof of Lemma 2 for the case in which $A_n(\sigma) = M_n e^{-\lambda_n \sigma}$, under less restrictive conditions: those obtained in removing the restriction

$\lim g(\sigma)>\overline{D}^{\cdot}$ (still assuming $\lim g(\sigma)>0$) and in replacing $g(u)-\overline{D}^{\cdot}(p(u))$ in (2′) by $g(u)$. This proof depends on writing

$$\begin{aligned} |H(s)| = |F(s)-\Phi(s)| &= \left|F(s)-\sum_{1}^{n} d_k e^{-\lambda_k s}-\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}\right| \\ &\leqq M_n e^{-\lambda_n \sigma}+\left|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}\right|, \end{aligned}$$

and considering separately the two possibilities: we have either $\liminf_{n\to\infty}(\log M_n)/\lambda_n<\infty$, or $\lim_{n\to\infty}(\log M_n)/\lambda_n=\infty$. If $B_1(\sigma)=\text{g.l.b.}_{n\geq 1}\left[M_n e^{-\lambda_n\sigma}+\left|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}\right|\right]$, and $B_1^*(\sigma)=\text{g.l.b.}_{n\geq 1} M_n e^{-\lambda_n \sigma}$, it may be shown that, in either case, if $B_1^*(\sigma)$ satisfies the condition

$$(13) \qquad \int^{\infty} -\log B_1^*(\sigma)\exp\left[-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)}\right]d\sigma=\infty$$

then $B_1(\sigma)$ does also. For by Theorem II (Lemma 1) if (13) is satisfied the series $\sum_1^{\infty} d_k e^{-\lambda_k s}$ converges for σ sufficiently large. One then sees that if $s=a$ (a real) is a point of convergence of the series, then $\left|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}\right|=\left|\sum_{n+1}^{\infty} d_k e^{-\lambda_k(s-a)}e^{-\lambda_k a}\right|\leqq M e^{-\lambda_k(\sigma-a)}$ where $M=\sum_1^{\infty}|d_k|e^{-\lambda_k a}$. Then if $\liminf_{n\to\infty}(\log M_n)/\lambda_n<\infty$, $B_1(\sigma)\equiv 0$; while if $\lim_{n\to\infty}(\log M_n)/\lambda_n=\infty$, then $M_n>Me^{\lambda_n a}$ for n sufficiently large, and $\left|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}\right|<M_n e^{-\lambda_n\sigma}$, hence $B_1(\sigma)<2B_1^*(\sigma)$. It follows that $H(s)\equiv 0$, by a theorem of Mandelbrojt and MacLane ([**2**, Theorem I]; cf. also [**1**, Lemma 1, p. 360], obtainable from [**2**, Theorem 1], by the obvious transformation).

Similar proofs of Lemma 2 may be given provided the $\{A_n(\sigma)\}$ satisfy any of a number of conditions which may easily be formulated: in particular if $A^*(\sigma)$ satisfies (13), where $A^*(\sigma)=\liminf_{n\to\infty}A_n(\sigma)$, or, in general, if $B(\sigma)$ satisfies (13), where $B(\sigma)=\text{g.l.b.}_{n\geq 1}\left\{\left|\sum_{n+1}^{\infty} d_k e^{-\lambda_k s}\right|+A_n(\sigma)\right\}$. Professor Mandelbrojt has remarked to the author that it follows from (6) that

$$\sum |d_k| e^{-\lambda_k a}<A\sum \lambda_k e^{-\lambda_k(a-B-\sigma_0)}<K_\epsilon \sum e^{-\lambda_k(a-B-\sigma_0-\epsilon)}$$

for each positive number ϵ, K_ϵ being suitably chosen. Thus for $a>B+\sigma_0$, there exists a positive number $K=K(a)$ such that $\sum|d_k|e^{-\lambda_k a}<K$ (since $\sum e^{-\lambda_k s}$ converges for $\sigma>0$), and such that $\sum_{n+1}^{\infty}|d_k|e^{-\lambda_k\sigma}=\sum_{n+1}^{\infty}|d_k|e^{-\lambda_k(\sigma-a)}e^{-\lambda_k a}\leqq Ke^{-\lambda_n(\sigma-a)}$. Thus the conclusion of Lemma 2 can be shown to hold if $B^*(\sigma)$ satisfies (13), where $B^*(\sigma)=\text{g.l.b.}_{n\geq 1}(Ke^{-\lambda_n(\sigma-a)}+A_n(\sigma))$.

It may be remarked that a theorem of Mandelbrojt and MacLane ([**2**, Theorem II], a suitable transformation being made if $0<\lim g(\sigma)\not\leq\pi/2$) constitutes a converse of Lemma 2 for asymptotic sequences, $g(u)-\overline{D}^{*}(p(u))$ in (2′) being replaced by $g(u)$, as it may be under conditions on the $A_n(\sigma)$ discussed above. For by their theorem, given

$$\int^{\infty} p(\sigma) \exp\left[-\frac{1}{2}\int^{\sigma}\frac{du}{g(u)}\right]d\sigma < \infty,$$

there exists a function $F(s)$ holomorphic in Δ, not identically zero, such that $\left|F(s)-\sum_1^n 0e^{-\lambda_k s}\right| < e^{-p(\sigma)}$, hence $\left|F(s)-\sum_1^n 0e^{-\lambda_k s}\right| < A_n(\sigma)$ if $\{A_n(\sigma)\}$ is any asymptotic sequence with g.l.b.$_{n\geqq 1}$ $A_n(\sigma)=A(\sigma)$ $=e^{-p(\sigma)}$; so that $F(s)$ is represented asymptotically in Δ by the series $\sum d_k e^{-\lambda_k s}$ with $d_k=0$ $(k\geqq 1)$ with respect to the asymptotic sequence $\{A_n(\sigma)\}$, without being identically zero.

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R. D. Carmichael, *On Euler's ϕ function*, vol. 13, pp. 241–243; vol. 54, p. 1192.

Vol. 54, p. 1192, lines 2 and 9. For "Hedburg" read "Hedberg."

Vol. 54, p. 1192, line 10. For "$2^{28}+1$ and $2^{29}+1$" read "$2^{2^8}+1$ and $2^{2^9}+1$."

COMBINATORIAL HOMOTOPY. I

J. H. C. WHITEHEAD

1. **Introduction.** This is the first of a series of papers, whose aim is to clarify the theory of "nuclei" and "n-groups" and its relation to Reidemeister's[1] *Überlagerungen.* Here we give a new definition of "n-groups," or *n-types* as we now propose to call them. This is stated in terms of $(n-1)$-homotopy types, which were introduced by R. H. Fox.[2] In a later paper we shall show that this is equivalent to the definition in terms of elementary transformations, which was given in [1]. The series of n-types $(n=1, 2, \cdots)$ is a hierarchy of homotopy, and a fortiori of topological invariants. That is to say, if two complexes,[3] K, L, are of the same n-type, then they are of the same m-type for any $m<n$, where $n \leqq \infty$ and the ∞-type means the homotopy type. If dim K, dim $L \leqq n$ then K, L are of the same homotopy type if they are of the same $(n+1)$-type. Two complexes are of the same 2-type if, and only if, their fundamental groups are isomorphic. Moreover any (discrete) group is isomorphic to the fundamental group of a suitably constructed complex. Therefore the classification of complexes according to their 2-types is equivalent to the classification of groups by the relation of isomorphism. Thus the n-type $(n>2)$ is a natural generalization of a geometrical equivalent of an abstract group.[4]

Following up this idea we look for a purely algebraic equivalent of an n-type when $n>2$. An important requirement for such an algebraic system is "realizability," in two senses. In the first instance this means that there is a complex which is in the appropriate relation to a given one of these algebraic systems, just as there is a complex whose fundamental group is isomorphic to a given group. The second kind, whose importance is underlined by theorems in [5; 6] and in this paper, is the "realizability" of homomorphisms, chain mappings, etc., by maps of the corresponding complexes. Thus realizability means that the algebraic representation is not subject to conditions which can only be expressed geometrically.

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[1] See [1], [3] and [8, p. 177]. Numbers in brackets refer to the references cited at the end of the paper.

[2] See [9, p. 343] and [10, p. 49].

[3] I.e., CW-complexes, as defined in §5 below.

[4] I.e., the class of groups which are isomorphic to a given group.

What has been achieved so far is a purely algebraic description of the homotopy type of any 3-dimensional complex (see [3]) and of any finite, simply-connected, 4-dimensional complex. An account of the former will be given in Paper II of this series and of the latter in [5]. This and Theorem 6 below lead to an algebraic description of the 3-type of any complex and of the 4-type of any simply-connected, finite complex. The theorem on the realizability of chain-mappings, which applies to the 3-dimensional complexes, will be generalized to what we call J_m-complexes, which are defined in §3 below.

The algebraic apparatus used in [5] and in Paper II is inadequate for the classification of homotopy classes of maps of such complexes. Each homotopy class of maps induces a unique equivalence class of algebraic maps. Moreover each algebraic map, of the appropriate algebraic type, has a geometrical realization. But in general the homotopy class of the latter is not unique. For example in the case of a simply-connected, 3-dimensional complex our algebraic system simply consists of the ordinary chain groups and all maps of a 3-sphere in a 2-sphere lead to the same equivalence class of chain mappings. Thus our results are complementary to much of the recent work on the algebraic classification of mappings (e.g. [11]).

In this presentation we abandon simplicial complexes in favor of cell complexes. This first part consists of geometrical preliminaries, including some elementary propositions concerning what we call *closure finite complexes with weak topology*, abbreviated to CW-*complexes*, which are defined in §5 below. There are two main reasons why we do not confine ourselves to finite complexes. The first is that we want to include such simple spaces as open manifolds or, more generally, infinite but locally finite complexes. The second is that we have a great deal to do with covering complexes. We do not restrict ourselves to locally finite complexes because this restriction would be troublesome when considering "mapping cylinders" of infinite complexes. Also we want to allow for such operations as shrinking an infinite subcomplex (e.g., a tree containing all the 0-cells) into a point.

We recall from [6] that a space, P, *dominates* a space, X, if, and only if, there are maps, $\lambda : X \rightarrow P$, $\lambda' : P \rightarrow X$, such that $\lambda'\lambda \simeq 1$. We show that the theorems in [6] can be extended to non-compact spaces, which are dominated by CW-complexes. To do this we give a new and possibly[5] more restrictive definition of n-homotopy. According to Fox, maps $f, g : X \rightarrow Y$, of any space X in a space Y, are n-homotopic, written $f \simeq_n g$, if, and only if, $f\phi \simeq g\phi$, for every map,

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[5] See §7 below.

$\phi: K^n \to X$, of every finite simplicial complex, K^n, of at most n dimensions. We alter this by defining $f \simeq_n g$ if, and only if, $f\phi \simeq g\phi$ for every map, $\phi: K^n \to X$, of every CW-complex, K^n, of at most n dimensions. As in [6] we describe a map, $f: X \to Y$, as an n-homotopy equivalence if, and only if, it has an ***n*-homotopy inverse**, meaning a map, $g: Y \to X$, such that $gf \simeq_n 1$, $fg \simeq_n 1$. A homotopy equivalence is similarly defined in terms of ordinary homotopy. We shall use the symbol

$$X \equiv_n Y$$

to mean that X and Y are of the same n-homotopy type (i.e. there is an n-homotopy equivalence $f: X \to Y$) and $X \equiv Y$ will mean that X and Y are of the same homotopy type.

Let α be the class of all connected spaces, each of which is dominated by some CW-complex.[6] Let $\alpha_0 \subset \alpha$ be the subclass consisting of spaces which are locally simply-connected in the weak sense. That is to say, if $X \in \alpha_0$ there is a basic set of neighborhoods in X, such that any closed curve in one of these neighborhoods is contractible to a point in X. The universal covering space, $\tilde{X}$, of such a space has the usual properties. If X is dominated by a CW-complex of finite dimensionality we shall use ΔX to denote the minimum dimensionality of all CW-complexes which dominate X. If none of the CW-complexes, which dominate X, has finite dimensionality we write $\Delta X = \infty$. We restate the theorems in [6], which now refer to spaces X, $Y \subset \alpha$ and to the new definition of n-homotopy. Let $f_n: \pi_n(X) \to \pi_n(Y)$ be the homomorphism induced by a map $f: X \to Y$ ($n = 1, 2, \cdots$) and let $N = \max(\Delta X, \Delta Y) \leqq \infty$.

THEOREM 1. *The map $f: X \to Y$ is a homotopy equivalence if, and only if, $f_n: \pi_n(X) \to \pi_n(Y)$ is an isomorphism onto for every n such that $1 \leqq n < N+1$.*

THEOREM 2. *The map $f: X \to Y$ is an $(N-1)$-homotopy equivalence if, and only if, f_n is an isomorphism onto for every n such that $1 \leqq n < N$.*

Let X, $Y \subset \alpha_0$ and let $\tilde{X}$, $\tilde{Y}$ be the universal covering spaces of X, Y, with base points $x_0 \in X$ and $y_0 = fx_0 \in Y$. As explained in [6], the map $f: X \to Y$ induces homomorphisms $H_n(\tilde{X}) \to H_n(\tilde{Y})$, where H_n indicates the nth homology group, defined in terms of singular cycles. The argument given in [6] shows that $H_n(\tilde{X}) = 0$, $H_n(\tilde{Y}) = 0$ if $n > N$.

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[6] A space in α must satisfy the usual union and intersection axioms for closed and open sets but need not satisfy any separation axioms. For example, a single point need not be a closed subset. It follows from (M) in §5 below that any connected CW-complex is arcwise connected. Hence it is easily proved that every space in α is arcwise connected.

THEOREM 3. *The map $f: X \to Y$ is a homotopy equivalence if, and only if, each of the induced homomorphisms $f_1: \pi_1(X) \to \pi_1(Y)$ and $H_n(\tilde{X}) \to H_n(\tilde{Y})$ is an isomorphism onto.*

Let $N < \infty$.

THEOREM 4. *The map $f: X \to Y$ is an $(N-1)$-homotopy equivalence if*

(a) *each of the induced homomorphisms $f_1: \pi_1(X) \to \pi_1(Y)$, $H_n(\tilde{X}) \to H_n(\tilde{Y})$ is an isomorphism onto, for $n = 0, \cdots, N-2$, and also*

(b) *the induced homomorphism $H_{N-1}(\tilde{X}) \to H_{N-1}(\tilde{Y})$ is onto and*

(c) *$f_{N-1}: \pi_{N-1}(X) \to \pi_{N-1}(Y)$ is an isomorphism into.*

Conversely if $f: X \to Y$ is an $(N-1)$-homotopy equivalence,[7] *so is the lifted map, $\tilde{f}: \tilde{X} \to \tilde{Y}$, and $H_n(\tilde{X}) \to H_n(\tilde{Y})$ is an isomorphism onto for $n = 0, \cdots, N-1$.*

2. **n-types.** Let K, L be CW-complexes. A map $f: K \to L$ is said to be *cellular* if, and only if, $fK^n \subset L^n$ for each $n \geqq 0$, where K^n, L^n are the n sections of K, L. A homotopy $f_t: K \to L$ is said to be *cellular* if, and only if, the maps f_0, f_1 are cellular and $f_t K^n \subset L^{n+1}$ for each $n \geqq 0$. According to (L), in §5 below, any map $K \to L$ is homotopic to a cellular map and if $f_0 \simeq f_1: K \to L$, where f_0, f_1 are cellular, then there is a cellular homotopy $f_t: K \to L$. Let f_0, $f_1: L \to X$ be maps of L in any space X and let $f_0 \simeq_n f_1$. Then $f_0\phi \simeq f_1\phi$ for any map, $\phi: K^n \to L$, of any CW-complex, K^n, of at most n dimensions. In particular $f_0 | L^n \simeq f_1 | L^n$. But ϕ is homotopic to a cellular map, $\phi': K^n \to L$, and if $f_0 | L^n \simeq f_1 | L^n$ we have $f_0\phi \simeq f_0\phi' \simeq f_1\phi' \simeq f_1\phi$. Therefore $f_0 \simeq_n f_1$ if, and only if, $f_0 | L^n \simeq f_1 | L^n$. If dim $L \leqq n$ then $f_0 \simeq_n f_1$ obviously means $f_0 \simeq f_1$. If will be convenient to allow n to take the value ∞ on the understanding that $L^\infty = L$ and that $f_0 \simeq_\infty f_1$ and $K \equiv_\infty L$ mean $f_0 \simeq f_1$ and $K \equiv L$.

We shall say that CW-complexes, K, L, are of the same *n-type* $(1 \leqq n \leqq \infty)$ if, and only if, $K^n \equiv_{n-1} L^n$. If dim K, dim $L \leqq n$, then $X^{n+1} = X^n = X (X = K \text{ or } L)$ and $K^{n+1} \equiv_n L^{n+1}$ means $K \equiv L$.

THEOREM 5. *If K and L are of the same n-type $(2 \leqq n \leqq \infty)$ then they are of the same m-type for any $m < n$.*

Let $f: K^n \to L^n$ be an $(n-1)$-homotopy equivalence and let $g: L^n \to K^n$ be an $(n-1)$-homotopy inverse of f. We may assume that f and g are cellular maps and also that there are cellular homotopies, $\xi_t: K^{n-1} \to K^n$, $\eta_t: L^{n-1} \to L^n$, such that $\xi_0 = gf | K^{n-1}$, $\xi_1 = 1$, $\eta_0 = fg | L^{n-1}$,

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[7] By taking X to be a 2-sphere and Y to be a complex projective plane one sees that $f: X \to Y$ need not be an $(N-1)$-homotopy equivalence $(N=4)$ even though the induced homomorphisms f_1 and $H_n(\tilde{X}) \to H_n(\tilde{Y})$ are isomorphisms onto for $n = 1, \cdots, N-1$.

$\eta_1 = 1$, where 1 stands for both identical maps $K^{n-1} \to K^n$, $L^{n-1} \to L^n$. Let $f^n: K^n \to L^n$, $g^n: L^n \to K^n$ be the maps induced by f, g (i.e. $f^n p = fp$, $g^n q = gq$, where $p \in K^n$, $q \in L^n$). Since $\xi_t K^{n-1} \subset K^n$ it follows that $g^n f^n | K^{n-1} \simeq 1$ (in K^n). Therefore $g^n f^n \simeq_{n-1} 1$. Similarly $f^n g^n \simeq_{n-1} 1$, whence $K^n \equiv_{n-1} L^n$ and the theorem is proved.

It follows from Theorem 5 that the n-type is an invariant of the homotopy type (i.e. is the same for two complexes of the same homotopy type). It is a fortiori a topological invariant. Since a CW-complex is locally contractible, according to (M) in §5 below, each component is arcwise connected. Therefore two CW-complexes have the same 1-type if, and only if, they have the same cardinal number of components. It also follows from the results in §5 below, and the argument used in the finite case,[8] that connected CW-complexes, K, L, have the same 2-type if, and only if, $\pi_1(K) \approx \pi_1(L)$. We now assume, until §4, that any given complex is connected and that $n \geqq 2$ in statements concerning n-homotopy and n-types.

These ideas provide a generalization of a result due to Hopf,[9] on the relation between the fundamental group and the second homology group of a polyhedron. For let X be any space and let $G_n(X)$ be the residue group

$$G_n(X) = H_n(X) - S_n(X),$$

where $S_n(X) \subset H_n(X)$ is the sub-group whose elements are represented by spherical cycles. Let $f, g: X \to Y$ be maps of X in a space Y. It may be proved that,[10] if $f \simeq_{n-1} g$, then f and g induce the same homomorphism $G_n(X) \to G_n(Y)$. It follows that, if $X \equiv_{n-1} Y$, then $G_n(X) \approx G_n(Y)$. In case X, Y are CW-complexes this also follows from:

THEOREM 6. *If* $K_1^n \equiv_{n-1} K_2^n$ *then*

$$K_1^n \cup \Sigma_1^n \equiv K_2^n \cup \Sigma_2^n,$$

where Σ_i^n $(i = 1, 2)$ *is a set of (disjoint) n-cells, whose closures are n-spheres, attached to* K_i^n *at a single point,* Σ_i^n *being a finite set of cells if* K_1^n, K_2^n *are finite complexes.*

This is Theorem 13 in[11] [1], restated in terms of CW-complexes and the new definition of the n-type. We shall give another proof in a later paper.

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[8] See [16, pp. 217, 213], and [18, 3.1, 3.2 on p. 29].

[9] See [17] and [18]. See also [1, Theorems 12 and 13, pp. 266, 269].

[10] This is easily proved on the assumption that any homology class in $H_n(X)$ has a representative (singular) cycle, which is also a continuous cycle (i.e., the image in a map, $K^n \to X$, of a cycle carried by a finite, n-dimensional, simplicial complex K^n).

[11] [1], p. 269 for finite and p. 324 for infinite complexes.

Another corollary of Theorem 6, and of Theorem 19 in [1] is:

COROLLARY 1. *If* $K_1^n \equiv_{n-1} K_2^n$ *then*[12]

$$\pi_n(\tilde{K}_1^n) + M_1 \approx \pi_n(\tilde{K}_2^n) + M_2,$$

where + *indicates direct summation and* M_i *is a free* $\mathfrak{R}_i$*-module,* $\mathfrak{R}_i$ *being the group ring of* $\pi_1(K_i)$ *with integral coefficients.*

Another obvious corollary of Theorem 6, which is also easy to prove directly, is:

COROLLARY 2. *If* K_1^n *and* K_2^n *are of the same n-type, so are their universal covering complexes,* $\tilde{K}_1^n$ *and* $\tilde{K}_2^n$.

Hence, using $G_n(X)$ to mean the same as before, with $X = \tilde{K}_i^n$, we have:

COROLLARY 3. *If* $K_1^n \equiv_{n-1} K_2^n$ *then* $G_n(\tilde{K}_1^n) \approx G_n(\tilde{K}_2^n)$.

3. J_m**-complexes.** Let K be a (connected) CW-complex and let a 0-cell, e^0, be taken as base point for all the groups $\pi_n(K^n, K^{n-1})$, $\pi_r(K^r)$ $(n \geqq 2, r \geqq 1)$. Let $\rho_n = \pi_n(K^n, K^{n-1})$ and let $j_n: \pi_n(K^n) \to \rho_n$ be the homomorphism induced by the identical map $(K^n, e^0) \to (K^n, K^{n-1})$. We shall describe K as a J_m*-complex* if, and only if, j_n is an isomorphism into ρ_n for each $n = 2, \cdots, m$. Obviously K is a J_m-complex if, and only if, K^m is a J_m-complex. We shall prove that the property of being a J_m-complex is an invariant of the m-type. That is to say, if $K^m \equiv_{m-1} L^m$ and if K is a J_m-complex so is L.

Let $\beta_n: \rho_n \to \pi_{n-1}(K^{n-1})$ $(n \geqq 2)$ be the (homotopy) boundary homomorphism and let

$$d_n = j_{n-1}\beta_n : \rho_n \to \rho_{n-1},$$

where $\rho_1 = \pi_1(K^1)$ and $j_1 = 1$. Then $d_n j_n = 0$ and $d_n d_{n+1} = 0$ since $\beta_n j_n = 0$. Let $i_n: \pi_n(K^{n-1}) \to \pi_n(K^n)$ be the homomorphism induced by the identical map $K^{n-1} \to K^n$.

We now prove two theorems which are closely related to Theorem 1 in [25].

THEOREM 7. *The homomorphism* $j_n: \pi_n(K^n) \to \rho_n$ *is into* $d_n^{-1}(0)$ $(n \geqq 2)$.

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[12] In each of Corollaries 1 and 3 the symbol $\approx$ means that there is an operator isomorphism (onto), which is defined in terms of the isomorphism, $\pi_1(K_1^n) \to \pi_1(K_2^n)$, induced by an $(n-1)$-homotopy equivalence $K_1^n \to K_2^n$. Compare Corollary 3 with Theorem 28.1 on p. 414 of [13].

It is:

(a) *an isomorphism (into) if, and only if,* $i_n\pi_n(K^{n-1})=0$ *and is*

(b) *onto* $d_n^{-1}(0)$ *if* $i_{n-1}\pi_{n-1}(K^{n-2})=0$.

The fact that $j_n\pi_n(K^n)\subset d_n^{-1}(0)$ follows from the relation $d_nj_n=0$. The assertion (a) follows at once from the exactness of the homotopy sequence

$$\pi_r(K^{r-1}) \xrightarrow{i} \pi_r(K^r) \xrightarrow{j} \pi_r(K^r, K^{r-1}) \xrightarrow{\beta} \pi_{r-1}(K^{r-1}), \tag{3.1}$$

with $r=n$. Let $a\in d_n^{-1}(0)$ be given ($n\geqq 2$). Since $d_na=j_{n-1}\beta_na=0$ it follows from the exactness of (3.1) with $r=n-1$ and $\pi_1(K^0)=\pi_0(K^0)=0$ in case $n=2$, that $\beta_na\in i_{n-1}\pi_{n-1}(K^{n-2})$. Therefore $\beta_na=0$ if $i_{n-1}\pi_{n-1}(K^{n-2})=0$ and it follows from the exactness of (3.1), with $r=n$, that $a\in j_n\pi_n(K^n)$. This completes the proof.

Since $\pi_1(K^1)=0$ it follows from Theorem 7 that any complex is a J_2-complex. Also j_2 is onto $d_2^{-1}(0)=\beta_2^{-1}(0)$ by the exactness of (3.1) with $r=2$. In general we have:

COROLLARY. *A complex, K, is a* J_m*-complex if, and only if,* $i_n\pi_n(K^{n-1})=0$ *for* $n=2, \cdots, m$. *In this case* j_r *is onto* $d_r^{-1}(0)$ *for* $r=2, \cdots, m+1$.

Let $\pi_1(K)=1$. Then the groups ρ_n ($n>2$) and ρ_1, ρ_2 made Abelian may be taken as chain-groups.[13] The homology group, $H_n(K)$, may be defined as the residue group

$$H_n(K) = \overset{-1}{d_n}(0) - d_{n+1}\rho_{n+1} = \overset{-1}{d_n}(0) - j_n\beta_{n+1}\rho_{n+1}. \tag{3.2}$$

It follows from (L), in §5 below, that we may take

$$\pi_n(K) = \pi_n(K^{n+1}) = \pi_n(K^n) - \beta_{n+1}\rho_{n+1}. \tag{3.3}$$

Therefore j_n determines a homomorphism $j_n^*:\pi_n(K)\to H_n(K)$.

THEOREM 8. *If* $\pi_1(K)=1$ *and if K is a* J_m*-complex, then* $j_n^*:\pi_n(K)\to H_n(K)$ *is an isomorphism onto for* $n=1, \cdots, m$ *and* j_{m+1}^* *is onto.*

This is an immediate consequence of (3.2), (3.3) and Theorem 7 and its corollary.

We shall say that a space Y, *n-dominates* a space X if, and only if, there is a map, $f:X\to Y$, which has a left n-homotopy inverse, meaning a map $g: Y\to X$ such that $gf\simeq_n 1$.

THEOREM 9. *If a* CW*-complex, K, is* $(m-1)$*-dominated by a* J_m*-complex, L, then K is a* J_m*-complex.*

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[13] Cf. [6], [5]. For the definition of $H_n(K)$ in terms of the relative homology groups $H_n(K^n, K^{n-1})$, treated as chain groups, see [14].

Let $f:K\to L$ and $g:L\to K$ be cellular maps such that $gf\simeq_{m-1}1$. Since f, g are cellular they induce homomorphisms

$$\phi_n: i_n\pi_n(K^{n-1}) \to i_n\pi_n(L^{n-1}),$$
$$\psi_n: i_n\pi_n(L^{n-1}) \to i_n\pi_n(K^{n-1}).$$

Let $\xi_t: K^{m-1}\to K^m$ be a cellular homotopy such that $\xi_0=gf\,|\,K^{m-1}$, $\xi_1=1$. Let $2\leqq n\leqq m$. Then $\xi_t K^{n-1}\subset K^n$ and it follows that $\psi_n\phi_n=\alpha_n$, where α_n is an automorphism of $i_n\pi_n(K^{n-1})$. Therefore ψ_n is onto. But $i_n\pi_n(L^{n-1})=0$, by the corollary to Theorem 7. Therefore $i_n\pi_n(K^{n-1})=0$, whence K is a J_m-complex and Theorem 9 is proved.

If $K^m\equiv_{m-1}L^m$ then K^m and L^m $(m-1)$-dominate each other. Therefore we have the corollary:

COROLLARY. *The property of being a J_m-complex is an invariant of the m-type.*

Let K be a CW-complex such that $\pi_n(K)=0$ for $n=1,\cdots,m-1$. Then it follows from Theorem 2 that the map $K^m\to L^0$ is an $(m-1)$-homotopy equivalence, where L^0 is a single point. Since L^0 is (obviously) a J_m-complex it follows from Theorem 9 that K^m and hence K is a J_m-complex. Therefore Theorem 8 includes Hurewicz's result[14] that $\pi_m(K)\approx H_m(K)$ if $\pi_n(K)=0$ for $n=1,\cdots,m-1$. The following example of a simply connected J_4-complex, K, with $\pi_3(K)\neq 0$, shows that Theorem 8 is more general than Hurewicz's theorem, restricted to CW-complexes. Let $K=e^0\cup e^3\cup e^4$, where e^0 is a 0-cell, e^3 is a 3-cell whose closure is a 3-sphere, $S^3=e^0\cup e^3$, and e^4 is attached[15] to S^3 by a map, $f:\partial E^4\to S^3$, of degree $(2r+1)(r>0)$. Then $\pi_3(K)$ is cyclic of order $2r+1$. If $1\leqq n\leqq 3$ then $K^{n-1}=K^0$, whence $\pi_n(K^{n-1})=0$. Let $g:S^4\to\partial E^4$ be an essential map. Then $fg:S^4\to S^3$ is essential [4, Theorem 9, p. 268] and hence represents the nonzero element of $\pi_4(S^3)$. Therefore [2, Lemma 4, p. 418] $i_4\pi_4(K^3)=0$, whence K is a J_4-complex.

According to (M) and (N) in §5 below, a CW-complex, K, is locally contractible and its universal covering complex, $\tilde{K}$, is also a CW-complex.

THEOREM 10. *K is a J_m-complex if, and only if, $\tilde{K}$ is a J_m-complex.*

This is an obvious consequence of the corollary to Theorem 7.

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[14] [15, p. 522]. See also [19, p. 314]. Hurewicz's theorem applies to a larger class of spaces than CW-complexes.

[15] See §8 below. We use E^n to denote an n-element (i.e. a homeomorph of an n-simplex) and ∂E^n to denote its boundary. By a simplex we mean a simplex with boundary.

Let $\tilde{K}$ be the universal covering complex of a CW-complex K. Since $\pi_1(\tilde{K})=1$, $\pi_n(\tilde{K})\approx\pi_n(K)$ if $n>1$ and since L is a J_m-complex if $\pi_r(L)=0$ for $r=1, \cdots, m-1$ it follows from Theorem 10 that K is a J_m-complex if $\pi_n(K)=0$ for $n=2, \cdots, m-1$. In particular K is a J_m-complex if its universal covering space is an m-sphere or any Euclidean space.

In Paper II we shall consider "Reidemeister" systems of chain groups, $C(K)$, $C(L)$ and chain mappings $C(K)\rightarrow C(L)$, where K, L, are CW-complexes. We shall prove that, if L is a J_m-complex and if $\dim K\leqq m+1$, then any chain mapping, $C(K)\rightarrow C(L)$, can be realized by some (cellular) map, $K\rightarrow L$. Hence it follows from Theorem 3 that $K\equiv L$ if there is a chain equivalence $C(K)\rightarrow C(L)$. If, in addition [5, Lemma 4], K and L are finite, simply-connected complexes, then $K\equiv L$ if $H_n(K)\approx H_n(L)$ for all values of n.

4. **Cell complexes.**[16] By a *cell complex*, K, or simply a *complex*, we mean a Hausdorff space, which is the union of disjoint (open) cells, to be denoted by e, e^n, e_i^n, etc., subject to the following condition. The closure, $\bar{e}^n$, of each n-cell, $e^n\in K$, shall be the image of a fixed n-simplex, σ^n, in a map, $f:\sigma^n\rightarrow\bar{e}^n$, such that

(4.1) (a) *$f|\sigma^n-\partial\sigma^n$ is a homeomorphism onto e^n,*

(b) *$\partial e^n\subset K^{n-1}$, where $\partial e^n=f\partial\sigma^n=\bar{e}^n-e^n$ and K^{n-1} is the $(n-1)$-section*[17] *of K, consisting of all the cells whose dimensionalities do not exceed $n-1$.*

Such a map will be called a *characteristic map* for the cell e^n. If $f:\sigma^n\rightarrow\bar{e}^n$ is a characteristic map for e^n, so obviously is $fh:\sigma^n\rightarrow\bar{e}^n$, where $h:(\sigma^n, \partial\sigma^n)\rightarrow(\sigma^n, \partial\sigma^n)$ is any map such that $h|\sigma^n-\partial\sigma^n$ is a homeomorphism of $\sigma^n-\partial\sigma^n$ onto itself. No restriction other than $\partial e^n\subset K^{n-1}$ is placed on $f|\partial\sigma^n$. Therefore $\bar{e}^n$ need not coincide, as a point set, with a subcomplex of K. Since K, and hence $\bar{e}^n$, is a Hausdorff space and since σ^n is compact it follows that $\bar{e}^n$ has the identification topology determined[18] by f. A complex is defined as a topological space with a certain cell structure. Therefore we shall not need a separate letter to denote a complex and the space on which it lies.[19] Notice that, in the absence of further restrictions, any (Hausdorff) space may be regarded as a complex. For example, we may take it to be the complex $K=K^0$, which consists entirely of 0-cells, each point in K being a 0-cell.

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[16] The use of these complexes was suggested in [3, p. 1235]. They are now called cell complexes, rather than membrane complexes, in conformity with [14].

[17] K^n is defined for every value of n. If there are no m-cells in K for $m>n$ then $K^n=K$.

[18] I.e., $Y\subset\bar{e}^n$ is closed if, and only if, $f^{-1}Y$ is closed. In other words the closed sets in $\bar{e}^n$ are precisely the sets fX for every closed set, $X\subset\sigma^n$, which is saturated with respect to f, meaning that $f^{-1}fX=X$ (cf. [23, pp. 61, 95] and [24, p. 52]).

[19] N.B. $e\in K$ will mean that e is a cell of the complex K and $e\subset K$, $\bar{e}\subset K$, etc., will mean that the sets of points e, $\bar{e}$, etc., are subsets of the space K.

A subcomplex, $L\subset K$, is the union of a subset of the cells of K, which are the cells of L, such that, if $e\subset L$ then $\bar{e}\subset L$. Clearly L is a subcomplex if it is the union of a subset of the cells in K, which is a closed set of points in K. However the above example shows that a subcomplex need not be a closed set of points. Clearly K^n is a subcomplex, for each $n\geqq 0$, and we admit the empty set as the subcomplex K^{-1}. Also the union and intersection of any set of subcomplexes, finite or infinite, are obviously subcomplexes. If $X\subset K$ is an arbitrary set of points we shall use $K(X)$ to stand for the intersection of all the subcomplexes of K, which contain X. Obviously $K(p)=K(e)=K(\bar{e})$, where p is any point in K and $e\in K$ is the cell which contains p. A finite subcomplex, L (i.e. one which contains but a finite number of cells) is a closed, and indeed a compact subset of K. For it is the union of the finite aggregate of compact sets, $\bar{e}$, for each cell $e\in L$.

The topological product, $K_1\times K_2$, of complexes K_1, K_2 is a complex, whose cells are the products, $e^{n_1+n_2}=e_1^{n_1}\times e_2^{n_2}$, of all pairs of cells $e_1^{n_1}\in K_1$, $e_2^{n_2}\in K_2$. For let $f_i:\sigma^{n_i}\rightarrow\bar{e}^{n_i}$ $(i=1, 2)$ be a characteristic map for e^{n_i}, let $g:\sigma^{n_1}\times\sigma^{n_2}\rightarrow\bar{e}^{n_1+n_2}$ be given by $g(p_1, p_2)=(f_1p_1, f_2p_2)$ and let $h:\sigma^{n_1+n_2}\rightarrow\sigma^{n_1}\times\sigma^{n_2}$ be a homeomorphism (onto). Then $gh:\sigma^{n_1+n_2}\rightarrow\bar{e}^{n_1+n_2}$ obviously satisfies the conditions (4.1). Therefore $K_1\times K_2$ is a complex, with this cell structure. In particular $K\times I$ is a complex, which consists of the cells $e\times 0$, $e\times 1$, $e\times(0, 1)$, for each cell $e\in K$, where $(0, 1)$ is the open interval $0<t<1$.

Let K be a locally connected complex, let $\tilde{K}$ be a (locally connected) covering space of K and let $p:\tilde{K}\rightarrow K$ be the *covering map*. That is to say there is a basis, $\{U\}$, for the open sets in K such that, if $U\in\{U\}$ then p maps each component of $p^{-1}U$ homeomorphically onto U (cf. [20, p. 40]). Let $\tilde{x}\in\tilde{K}$ be a given point and let $e^n\in K$ be the cell which contains $x=p\tilde{x}$. Then a characteristic map, $f:\sigma^n\rightarrow\bar{e}^n$, can be "lifted" into a unique map,[10] $\tilde{f}:\sigma^n\rightarrow\tilde{K}$, such that $f=p\tilde{f}$ and $\tilde{f}(f^{-1}x)=\tilde{x}$. Let $\tilde{e}^n=\tilde{f}(\sigma^n-\partial\sigma^n)$ and let $p_0=p\mid\tilde{e}^n$. Then $f\mid\sigma^n-\partial\sigma^n=p_0(\tilde{f}\mid\sigma^n-\partial\sigma^n)$ and since $f\mid\sigma^n-\partial\sigma^n$ is a (1-1) map onto e^n it follows that p_0 is (1-1) and is onto e_n. Since p, and hence p_0, is an open mapping it follows that p_0 is a homeomorphism. Since

$$\tilde{f}\mid\sigma^n-\partial\sigma^n=p_0^{-1}(f\mid\sigma^n-\partial\sigma^n)$$

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[10] See [21, Theorem 2, p. 40] or [22]. We shall sometimes use the same symbol, f or g, to denote two maps, $f:A\rightarrow B$, $g:A\rightarrow C\subset B$, such that $fa=ga$ for each point $a\in A$, even though $B\neq C$.

it follows that $\tilde{f} | \sigma^n - \partial\sigma^n$ is a homeomorphism, which, according to the definition of $\tilde{e}^n$, is onto $\tilde{e}^n$. Also $\tilde{f}\partial\sigma^n \subset \tilde{K}^{n-1} = p^{-1}K^{n-1}$. Therefore $\tilde{f}$ satisfies the conditions (4.1). It follows that $\tilde{K}$ is a complex, each of whose cells is mapped by p homeomorphically onto a cell of K.

Let Q be a subcomplex of $\tilde{K}$ and let e be a given cell in Q. Then $p\bar{e}$ is closed, since $\bar{e}$ is compact, and $p\bar{e} \subset pQ$. Therefore $\overline{pe} = p\bar{e} \subset pQ$. Therefore pQ is a subcomplex of K, which consists of the cells pe for each cell $e \in Q$.

5. **CW-complexes.** We shall describe a complex, K, as *closure finite* if, and only if, $K(e)$ is a finite subcomplex, for every cell $e \in K$. Since $K(p) = K(e)$ if $p \in e$ this is equivalent to the condition that $K(p)$ is finite for each point $p \in K$. If $L \subset K$ is a subcomplex and $e \in L$ then obviously $L(e) = K(e)$. Therefore any subcomplex of a closure finite complex is closure finite.

We shall say that K has the *weak topology* (cf. [1, pp. 316, 317]) if, and only if, a subset $X \subset K$ is closed (open) provided $X \cap \bar{e}$ is closed (relatively open) for each cell $e \in K$. If K is closure finite this is equivalent to the condition that X is closed provided $X \cap L$ is closed for every finite subcomplex $L \subset K$. For $X \cap L$ is the union of the finite number of sets $X \cap \bar{e}$ $(e \in L)$. Therefore $X \cap L$ is closed if each set $X \cap \bar{e}$ is closed. Conversely, if $X \cap L$ is closed for each finite subcomplex, L, and if $K(\bar{e})$ is finite, then $X \cap \bar{e}$ is closed, since $X \cap \bar{e} = X \cap K(\bar{e}) \cap \bar{e}$.

By a CW-*complex* we mean one which is closure finite and has the weak topology. Any finite complex, K, is obviously closure finite and it has the weak topology since $X \subset K$ is the union of the finite number of sets $X \cap \bar{e}$ $(e \in K)$. Therefore any finite complex is a CW-complex. Also a complex, K, is a CW-complex if it is *locally finite*, meaning that each point $p \in K$ is an inner point of some finite subcomplex of K. For let K be locally finite. Then $K(p)$ is finite, for each point $p \in K$. Therefore K is closure finite. Let $X \subset K$ be such that $X \cap L$ is closed for each finite subcomplex $L \subset K$. Let L be a finite subcomplex of which a given point $p \in K - X$ is an inner point. Since $X \cap L$ is closed, p is an inner point of $L - X = L - (X \cap L)$. Therefore X is closed and K has the weak topology. It may be verified that the number of cells, and hence the number of finite subcomplexes of a connected, locally finite complex, K, is countable. Hence, and from (G) below, it may be proved that K is a separable metric space.

If the cells in a CW-complex, K, have a maximum dimensionality we call this the dimensionality, dim K, of K. If there is no such maximum we write dim $K = \infty$.

Examples of complexes which are not CW-complexes are:

(1) $\partial\sigma^n$ ($n>1$) regarded as a "0-dimensional" complex, K^0, whose cells are the points of $\partial\sigma^n$. This is closure finite but does not have the weak topology.

(2) σ^n ($n>1$), regarded as a complex $K^n=K^0\cup e^n$, where $e^n=\sigma^n-\partial\sigma^n$ and $K^0=\partial\sigma^n$, as in (1). This has the weak topology, since $\bar{e}^n=K^n$, but is not closure finite.

(3) a simplicial complex, which has a metric topology but which is not locally finite (e.g. a complex covering the coordinate axes in Hilbert space). The weak topology in such a complex cannot be metricized (cf. [1, pp. 316, 317]).

Let K be a CW-complex. We establish some properties of K.

(A) *A map, $f:X\rightarrow Y$, of a closed (open) subset, $X\subset K$, in any space, Y, is continuous provided $f|X\cap\bar{e}$ is continuous for each cell $e\in K$.*

Let $f_e=f|X\cap\bar{e}$ be continuous, for each cell $e\in K$. Let Y_0 be any closed (open) subset of Y. Obviously $\bar{e}\cap f^{-1}Y_0=f_e^{-1}Y_0$ and it follows from the continuity of f_e that $\bar{e}\cap f^{-1}Y_0$ is a relatively closed (open) subset of $X\cap\bar{e}$. But $X\cap\bar{e}$ is a closed (relatively open) subset of $\bar{e}$, whence $\bar{e}\cap f^{-1}Y_0$ is closed (relatively open) in $\bar{e}$. Therefore $f^{-1}Y_0$ is closed (open) in K, and a fortiori in X. Therefore f is continuous.

(B) *A subcomplex, $L\subset K$, is a closed subspace of K and the topology induced by K is the weak topology in L.*

Let $Y\subset L$ be such that $Y\cap L_0$ is closed, and hence compact, for each finite subcomplex $L_0\subset L$. Since $Y\cap L_0$ is compact it is a closed subset of K. Let K_0 be any finite subcomplex of K. Then $L_0=L\cap K_0$ is a finite subcomplex of L and

$$Y\cap K_0=Y\cap L\cap K_0=Y\cap L_0.$$

Therefore $Y\cap K_0$ is closed, whence Y is closed in K, and a fortiori in L. Therefore L has the weak topology. Also, taking $Y=L$, it follows that L is closed, which establishes (B).

(C) *If K is connected so is K^n for each $n>0$.*

Let $n>0$ and let K^n be the union of disjoint, nonvacuous closed sets K_1^n, K_2^n. Since the closure of a cell $e\in K$ is connected it follows that $\bar{e}\subset K_i^n$ if $e\cap K_i^n\neq 0$ ($i=1, 2$). Therefore K_i^n is a subcomplex of K. Clearly ∂e^{n+1} is connected ($e^{n+1}\in K$), whence it lies either in K_1^n or in K_2^n. Therefore K^{n+1} is the union of disjoint subcomplexes, K_1^{n+1}, K_2^{n+1}, where $K_i^n\subset K_i^{n+1}$ and $e^{n+1}\in K_i^{n+1}$ if $\partial e^{n+1}\subset K_i^n$. A similar (inductive) argument shows that K^m is the union of disjoint subcomplexes, K_1^m, K_2^m, such that $K_i^m\subset K_i^{m+1}$ ($m=n, n+1, \cdots$). Let K_i be the union of the K_i^m for $m=n, n+1, \cdots$. Then $K_i\cap K^m=K_i^m$ and

$$K_1\cap K_2=\bigcup_m (K_1\cap K_2)\cap K^m=\bigcup_m K_1^m\cap K_2^m=0.$$

Also $K_i \neq 0$, since $K_i^n \subset K_i$, and K_i is a closed subset of K, according to (B). Therefore K is not connected, which establishes (C).

(D) *If $X \subset K$ is compact, then $K(X)$ is a finite complex.*

If X meets but a finite number of cells, $e_1, \cdots, e_k \subset K$, it is contained in the finite union of the (finite) subcomplexes $K(e_1), \cdots, K(e_k)$. Assume that there is an infinite set of cells, $\{e_i\}$, each of which meets X and let $p_i \in X \cap e_i$. Then a finite subcomplex, $L \subset K$, contains but a finite set of the cells in $\{e_i\}$ and $e_i \cap L = 0$ unless $e_i \in L$. Therefore L contains but a finite number of points in the set $P = \{p_i\}$, whence P is closed. Similarly any subset of P is closed, whence P is discrete. But this is absurd, since P is compact, being a closed subset of X. Therefore (D) is established.

(E) *If a complex L, and also L^n for each $n \geqq 0$, all have the weak topology, then L is a CW-complex.*

Certainly L^0 is closure finite. Assume that L^{n-1} is closure finite, and hence a CW-complex, for some $n > 0$. Let e^n be a given n-cell in L^n. Since ∂e^n is compact it follows from (D) that $L(\partial e^n)$ is finite. But obviously $L(e^n) = L(\partial e^n) \cup e^n$ and it follows from induction on n that L is closure finite, which establishes (E).

Let $f: K \rightarrow L$ be a map of K onto a closure finite complex L, which has the indentification topology[18] determined by f. Further let the subcomplex $L(f\bar{e})$ be finite for each cell $e \in K$.

(F) *Subject to these conditions L is a CW-complex.*

Let $Y \subset L$ be such that $Y \cap L_0$ is closed for each finite subcomplex $L_0 \subset L$. Let $L_0 = L(f\bar{e})$ for a given cell $e \in K$. Then $\bar{e} \subset f^{-1}L_0$ and

$$f^{-1}Y \cap \bar{e} = f^{-1}Y \cap \bar{e} \cap \bar{e} \subset (f^{-1}Y \cap f^{-1}L_0) \cap \bar{e} \subset f^{-1}(Y \cap L_0) \cap \bar{e},$$

since $f^{-1}A \cap f^{-1}B \subset f^{-1}(A \cap B)$ for any sets $A, B \subset L$. But $f^{-1}(Y \cap L_0) \subset f^{-1}Y$. Therefore

$$f^{-1}Y \cap \bar{e} = f^{-1}(Y \cap L_0) \cap \bar{e}.$$

Since $Y \cap L_0$ is closed it follows that $f^{-1}Y \cap \bar{e}$ is closed. Therefore $f^{-1}Y$ is closed, since K has the weak topology. Since L has the identification topology determined by f it follows that Y is closed. Therefore L has the weak topology. Since L is closure finite by hypothesis this proves (F).

(G) *K is a normal space.*

Let $X_1, X_2 \subset K$ be disjoint, closed subsets and let $X_i^r = X_i \cap K^r$ $(i = 1, 2; r \geqq 0)$. Clearly K^0 is a discrete set, and hence normal. Let $n > 0$ and assume that there are disjoint, relatively open subsets, $U_1^{n-1}, U_2^{n-1} \subset K^{n-1}$, such that $X_i^{n-1} \subset U_i^{n-1}$. Then $X_i \cap \bar{U}_j^{n-1} = 0$ $(i, j = 1, 2; i \neq j)$. If $K^n = K^{n-1}$ we define $U_i^n = U_i^{n-1}$. Otherwise let

$f:\sigma^n \to \bar{e}^n$ be a characteristic map for a given n-cell $e^n \in K$ and let

$$V_i = \overset{-1}{f} U_i^{n-1} \subset \partial \sigma^n, \qquad Y_i = \overset{-1}{f} X_i \subset \sigma^n.$$

Since $X_1 \cap X_2 = 0$, $X_i \cap \bar{U}_j^{n-1} = 0$ we have $Y_1 \cap Y_2 = 0$, $Y_i \cap \bar{V}_j = 0$. Let p_0 be the centroid of σ^n and let r, p be polar coordinates for σ^n ($r \in I$, $p \in \partial\sigma^n$) such that (r, p) is the point which divides the rectilinear segment p_0p in the ratio $r:1-r$. Let $V_i' \subset \sigma^n$ be the (open) subset, which consists of all points (r, p) with $p \in V_i$ and $1-\epsilon < r \leqq 1$, where $0<\epsilon<1$. Since $Y_i \cap \bar{V}_j = 0$ it follows that, if ϵ is sufficiently small, then $Y_i \cap \bar{V}_j' = 0$, which we assume to be the case. Since $fY_i \subset X_i$, $f\partial\sigma^n \subset K^{n-1}$ and $X_i \cap K^{n-1} \subset U_i^{n-1}$ it follows that $Y_i \cap \partial\sigma^n \subset V_i$. Let V_i'' be an η-neighborhood of Y_i, defined in terms of a metric for σ^n, where η is so small that $V_1'' \cap V_2'' = 0$, $V_i'' \cap \bar{V}_j' = 0$ and $V_i'' \cap (\partial\sigma^n - V_i) = 0$. Then $V_i'' \cap \partial\sigma^n \subset V_i$. Let

$$W_i = V_i' \cup V_i''.$$

Then $Y_i \subset W_i$ and $W_1 \cap W_2 = 0$. Obviously $V_i' \cap \partial\sigma^n = V_i$, whence

$$W_i \cap \partial\sigma^n = V_i = \overset{-1}{f} U_i^{n-1}. \tag{5.1}$$

Since $f:\sigma^n - \partial\sigma^n$ is a (1-1) map onto e^n and $f\partial\sigma^n \cap e^n = 0$ it follows that W_i is saturated[18] with respect to f. Therefore fW_i is a relatively open subset of $\bar{e}^n$. From (5.1) we have

$$fW_i \cap K^{n-1} = U_i^{n-1} \cap \partial e^n \tag{5.2}$$

and it follows that $fW_1 \cap fW_2 = 0$.

Let us write $W_i = W_i(e^n)$ and let

$$U_i^n = U_i^{n-1} \cup \bigcup_{e^n \in K} fW_i(e^n).$$

Then it follows from (5.2) that $U_i^n \cap K^{n-1} = U_i^{n-1}$ and that

$$U_i^n \cap \partial e^n = U_i^{n-1} \cap \partial e^n = fW_i(e^n) \cap K^{n-1}.$$

Also $fW_i(e^n) \subset K^{n-1} \cup e^n$ and $U_i^n \cap e^n = fW_i(e^n) \cap e^n$. Therefore

$$\begin{aligned} U_i^n \cap \bar{e}^n &= (U_i^n \cap \partial e^n) \cup (U_i^n \cap e^n) \\ &= fW_i(e^n) \cap (K^{n-1} \cup e^n) \\ &= fW_i(e^n). \end{aligned}$$

Therefore U_i^n is a relatively open subset of K^n. Obviously $X_i^n \subset U_i^n$ and $U_1^n \cap U_2^n = 0$. Therefore such sets, U_i^n, may be defined inductively for every value of n. Let them be so defined and let

$$U_i = \bigcup_n U_i^n.$$

Since $U_i^{n+1} \cap K^n = U_i^n$ it follows by induction on $m > n$ that

$$U_i^m \cap K^n = U_i^m \cap K^{m-1} \cap K^n = U_i^{m-1} \cap K^n = U_i^n$$

and hence that $U_i \cap K^n = U_i^n$. Therefore it follows, first that U_i is an open subset of K and second that $U_1 \cap U_2 = 0$. Obviously $X_i \subset U_i$, which completes the proof of (G).

(H) *If L is a locally finite*[21] *complex then $K \times L$ is a CW-complex.*

If $e \in K$, $e' \in L$ are cells in K and L respectively, then the cell $e \times e' \in K \times L$ is contained in the finite subcomplex $K(e) \times L(e') \subset K \times L$. Therefore $K \times L$ is closure finite.

Let the cells in K be indexed and with each m-cell, $e_i^m \in K$, $(m = 0, 1, \cdots)$ let us associate an m-element, E_i^m, as follows. The points in E_i^m shall be the pairs (x, e_i^m), for every point $x \in \sigma^m$, and E_i^m shall have the topology which makes the map $x \to (x, e_i^m)$ a homeomorphism. No two of these elements have a point in common and we unite them into a topological space,

$$P = \bigcup_{m,i} E_i^m,$$

in which each E_i^m, with its own topology, is both open and closed. Let $f_i^m : \sigma^m \to \bar{e}_i^m$ be a characteristic map for e_i^m and let $\phi : P \to K$ be the map which is given by $\phi(x, e_i^m) = f_i^m x$, for each point $(x, e_i^m) \in P$. Since $\bar{e}_i^m$ has the identification topology determined by f_i^m it follows that the weak topology in K is the identification topology determined by ϕ.

Let a space,

$$Q = \bigcup_{n,j} E_j^n,$$

and a map, $\psi : Q \to L$, be similarly associated with L. Then $K \times L = \theta(P \times Q)$, where $\theta : P \times Q \to K \times L$ is given by $\theta(p, q) = (\phi p, \psi q)$ $(p \in P, q \in Q)$. Also $P \times Q$ is the union of the $(m+n)$-elements $E_i^m \times E_j^n$, and $\theta(E_i^m \times E_j^n) = \bar{e}_{ij}^{m+n}$, where $e_{ij}^{m+n} = e_i^m \times e_j^n$. Therefore the weak topology in $K \times L$ is obviously the same as the identification topology determined by θ.

Let $V \subset L$ be an open subset and $y \in V$ an arbitrary point in V. Since y is an inner point of a finite subcomplex, $L_0 \subset L$, it is contained in a subset, $V_0 \subset V \cap L_0$, which is open in L. Since L is normal there is a neighborhood, W, of y such that $\overline{W} \subset V_0$. Since $\overline{W} \subset V_0 \subset L_0$ and since V_0 is open in L, it follows that $\overline{W} \cap \bar{e} = 0$ for any cell $e \in L - L_0$. Therefore there are only a finite number of cells in L, whose closures meet $\overline{W}$. Therefore $\psi^{-1}\overline{W}$ is contained in the union of a finite subset of the components $E_i^n \subset Q$. Therefore $\psi^{-1}\overline{W}$ is compact and (H) follows from Lemma 4 in [7].

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[21] I do not know if this restriction on L is necessary.

(I) *A homotopy, $f_t: X \rightarrow Y$, of a closed (open) subset, $X \subset K$, in an arbitrary space, Y, is continuous provided $f_t | X \cap \bar{e}$ is continuous for each cell $e \in K$.*

This follows from (H), with $L = I$, and (A), applied to the subset $X \times I \subset K \times I$ and the map $f: X \times I \rightarrow Y$, which is given by $f(x, t) = f_t x$.

(J) (*Homotopy extension.*) *Let $f_0: K \rightarrow X$ be a given map of K in an arbitrary space X. Let $g_t: L \rightarrow X$ be a homotopy of $g_0 = f_0 | L$, where L is a subcomplex of K. Then there is a homotopy, $f_t: K \rightarrow X$, such that $f_t | L = g_t$.*

Let $K_r = L \cup K^r$ $(r \geqq -1;\ K_{-1} = L)$ and assume that g_t has been extended to a homotopy, $f_t^{n-1}: K_{n-1} \rightarrow X$, such that $f_0^{n-1} = f_0 | K_{n-1}$, $f_t^{n-1} | L = g_t$ $(n \geqq 0)$. The homotopy f_t^{n-1} can be extended throughout $K_{n-1} \cup e^n$, for each n-cell[22] $e^n \in K_n - L$, and hence, by (I), to a (continuous) homotopy $f_t^n: K_n \rightarrow X$. Starting with $f_t^{-1} = g_t$ it follows by induction on n that there is a sequence of homotopies, $f_t^n: K_n \rightarrow X$ $(n = 0, 1, \cdots)$, such that $f_0^n = f_0 | K_n$, $f_t^n | K_{n-1} = f_t^{n-1}$. It follows from (I) that a homotopy, $f_t: K \rightarrow X$, which satisfies the requirements of (J), is given by $f_t | K_n = f_t^n$.

Let $X_0 \subset X_1 \subset \cdots$ be a sequence of subspaces of a given space, X, such that any map, $(\sigma^n, \partial\sigma^n) \rightarrow (X, X_{n-1})$, is homotopic, rel. $\partial\sigma^n$, to a map[23] $\sigma^n \rightarrow X_n$ $(n = 0, 1, \cdots)$. Let $L \subset K$ be a given subcomplex, which may be empty, and let $f_0: K \rightarrow X$ be a map such that $f_0 L^n \subset X_n$, for each $n = 0, 1, \cdots$.

(K) *There is a homotopy, $f_t: K \rightarrow X$, rel. L, such that $f_1 K^n \subset X_n$ for each $n = 0, 1, \cdots$.*

Since each point in X is joined by an arc to some point in X_0 there is a homotopy, $f_t^0: K^0 \rightarrow X$, rel. L^0, such that $f_0^0 = f_0 | K^0$ and $f_1^0 K^0 \subset X_0$. Let $n > 0$ and assume that there is a homotopy $f_t^{n-1}: K^{n-1} \rightarrow X$, rel. L^{n-1}, such that $f_0^{n-1} = f_0 | K^{n-1}$, $f_1^{n-1} K^{n-1} \subset X_{n-1}$. It follows from (J) that f_t^{n-1} can be extended, first throughout L^n by writing $f_t^{n-1} | L^n = f_0 | L^n$, and then to a homotopy, $\xi_t: K^n \rightarrow X$, rel. L^n $(\xi_0 = f_0 | K^n)$. Since $\xi_1 K^{n-1} \subset X_{n-1}$ it follows from a standard argument (see [6, §8]), and the condition on $X_0, X_1, \cdots$, that there is a homotopy, $\eta_t: K^n \rightarrow X$, rel. $(K^{n-1} \cup L^n)$, such that $\eta_0 = \xi_1$, $\eta_1 K^n \subset X_n$. If $\dim K < \infty$ we define $f_t^n: K^n \to X$ as the resultant of ξ_t followed by η_t. Then f_t^n may be defined inductively for every $n \geqq 0$ and we take $f_t = f_t^m$, where $m = \dim K$. But if $\dim K = \infty$ this method fails and we shall define f_t^n as an extension of f_t^{n-1}, not as the resultant of ξ_t followed by η_t.

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[22] See [5, Lemma 10 in §16].

[23] If $n = 0$ this simply means that each point in X is joined by an arc to some point in X_0.

If $K^n = K^{n-1}$ we define $f_t^n = f_t^{n-1}$. Otherwise let $g: \sigma^n \to \bar{e}^n$ be a characteristic map for a given n-cell $e^n \in K$. Let r, p be polar coordinates for σ^n, defined as in (G), and let $\rho_t: \bar{e}^n \to X$ be defined by

$$\begin{aligned} \rho_t g(r, p) &= \xi_{2t/(1+r)} g(r, p) && (\text{if } 0 \leqq 2t \leqq 1 + r) \\ &= \eta_{(2t-1-r)/(1-r)} g(r, p) && (\text{if } 1 + r < 2t \leqq 2). \end{aligned} \tag{5.3}$$

Since $\eta_0 = \xi_1$ and $g^{-1} \mid e^n$ is a homeomorphism onto $\sigma^n - \partial\sigma^n$ it follows that $\rho_t \mid e^n$ is single-valued and continuous. Since $\rho_t x = \xi_t x$ for any point $x = g(1, p) \in \partial e^n$ it follows that ρ_t is single-valued. Also ρ_t is continuous at $\{g(r, p), t\}$ if $r < 1$ and, obviously, if $t < 1$. I say that it is continuous at $\{g(1, p), 1\} = (gp, 1)$. For $gp \in K^{n-1}$ and $\eta_t \mid K^{n-1} = \eta_0 \mid K^{n-1} = \xi_1 \mid K^{n-1}$. Therefore, given a neighborhood, $U \subset X$, of $\xi_1 gp = \eta_t gp$, it follows from the compactness of I that there is a neighborhood, $V \subset \bar{e}^n$, of gp such that $\eta_t x \in U$ for every $t \in I$, provided $x \in V$. There is also a neighborhood, $V' \subset \bar{e}^n$, of gp, and a $\delta > 0$ such that $\xi_t x \in U$ if $x \in V'$, $1 - 2\delta < t \leqq 1$. Since $(2-2\delta)/(1+r) > 1 - 2\delta$ it follows that $\rho_t x \in U$ if $x \in V \cap V'$, $1 - \delta < t \leqq 1$. Therefore ρ_t is continuous. Also

$$\begin{aligned} \rho_t g(1, p) &= \xi_t g(1, p) = f_t^{n-1} g(1, p), \\ \rho_0 g(r, p) &= \xi_0 g(r, p) = f_0 g(r, p), \\ \rho_1 g(r, p) &= \eta_1 g(r, p) \in X_n. \end{aligned}$$

Therefore a homotopy, $f_t^n: K^n \to X$, rel. L^n, such that

$$f_t^n \mid K^{n-1} = f_t^{n-1}, \qquad f_0^n = f_0 \mid K^n, \; f_1 K^n \subset X_n,$$

is defined by $f_t^n \mid K^{n-1} = f_t^{n-1}$, $f_t^n \mid \bar{e}^n = \rho_t$, for each n-cell $e^n \in K^n$. It follows from induction on n that such a homotopy is defined for each $n \geqq 0$ and a homotopy, $f_t: K \to X$, which satisfies the requirements of (K), is defined by $f_t \mid K^n = f_t^n$.

Let $f_0: K \to P$ be a map of K into a CW-complex, P, such that $f_0 \mid L$ is cellular, where $L \subset K$ is a subcomplex. Also let $g_t: K \to P$ be a homotopy such that the maps g_0, g_1 and the homotopy $g_t \mid L$ are cellular.

(L) *There is a homotopy, $f_t \mid K \to P$, rel. L, of f_0 into a cellular map f_1. There is a cellular homotopy, $g_t': K \to P$, such that $g_0' = g_0$, $g_1' = g_1$, $g_t' \mid L = g_t \mid L$.*

Since any continuous image of σ^n in P is compact it is contained in a finite subcomplex $Q \subset P$, according to (D). Any map $(\sigma^n, \partial\sigma^n) \to (Q, Q^{n-1})$ is homotopic,[24] rel. $\partial\sigma^n$, in Q to a map $\sigma^n \to Q^n$. Therefore the first part follows from (K). The second part follows from the first part with K, L, f_0 replaced by $K \times I$, $(K \times 0) \cup (L \times I) \cup (K \times 1)$, $g: K \times I \to P$, where $g(p, t) = g_t p$.

(M) *K is locally contractible.*

Let $a_0 \in K$ be a given point, let $U \subset K$ be a given neighborhood of a_0 and let $e^r \in K$ be the cell which contains a_0. Let $E^r \subset U \cap e^r$ be an r-element, which contains a_0 in its interior, $V^r = E^r - \partial E^r$, and let $f_t^r: V^r \to V^r$ be a homotopy such that $f_0^r = 1$, $f_1^r V^r = a_0$. Using induction on n we shall define sequences of relatively open subsets, $V^n \subset K^n$ $(n = r, r+1, \cdots)$, such that $V^{n+1} \cap K^n = V^n$, $\overline{V}^n \subset U$, and of homotopies $f_t^n: V^n \to V^n$, such that $f_t^{n+1} \mid V^n = f_t^n$, $f_0^n = 1$, $f_1^n V^n = a_0$. Assuming that this has been done, let

$$V = \bigcup_n V^n$$

and let $f_t: V \to V$ be defined by $f_t \mid V^n = f_t^n$. Then it follows from the definition of the weak topology and from (I) that V is open in K and f_t continuous. Obviously $V \subset U$, $f_0 = 1$, $f_1 V = a_0$ and (M) follows.

Assume that V^{n-1} and f_t^{n-1} satisfy the above conditions for some $n > r$. Let $g: \sigma^n \to \bar{e}^n$ be a characteristic map for a given n-cell, $e^n \in K$, and let polar coordinates, r, p, for σ^n be defined as in (G). If $g\partial\sigma^n \cap V^{n-1} = 0$, let $W \subset \sigma^n$ be the empty set. Otherwise let $W \subset \sigma^n$ be the (open) subset, which consists of all points, (r, p), such that

$$1 - \epsilon < r \leqq 1, \qquad p \in g^{-1}V^{n-1},$$

where $0 < \epsilon < 1$. Since $\overline{V}^{n-1} \subset U$, whence $g^{-1}\overline{V}^{n-1} \subset g^{-1}U$, it follows that $\overline{W} \subset g^{-1}U$ if ϵ is sufficiently small, which we assume to be the case. Let $\xi_t: W \to W$ be the "radial projection," which is defined by

$$\xi_t(r, p) = (r + t - rt, p),$$

and let $\theta_t gW \to V^{n-1} \cup gW$ be given by

$$\begin{aligned} \theta_t g(r, p) &= g\xi_{2t/(1-r)}(r, p) & &(\text{if } 0 \leqq 2t < 1 - r) \\ &= f^{n-1}_{(2t-1+r)/(1+r)} g(1, p) & &(\text{if } 1 - r \leqq 2t \leqq 2). \end{aligned}$$

Since $\xi_t(1, p) = (1, p)$ and $g\xi_1(r, p) = g(1, p) = f_0^{n-1} g(1, p)$ it follows from an argument similar to the one which comes after (5.3) that θ_t is single-valued and continuous. Also

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[24] [5, §16, Theorem 6]. It follows from Theorem 6 in [5] that the condition $f_t x \in P(f_0 x)$ $(x \in K)$ may be imposed on the homotopy f_t in (L).

$$
\begin{aligned}
&\theta_t g(1, p) = f_t^{n-1} g(1, p),\\
(5.4)\qquad &\theta_0 g(r, p) = g\xi_0(r, p) = g(r, p),\\
&\theta_1 g(r, p) = f_1^{n-1} g(1, p) = a_0.
\end{aligned}
$$

Let V^n be the union of V^{n-1} and the sets gW, which are thus defined for all the n-cells in K. Arguments used in (G) show that V^n is a relatively open subset of K^n and that $V^n \cap K^{n-1} = V^{n-1}$. Also it follows from the definition of W that $\partial\sigma^n \cap \overline{W} \subset g^{-1}\overline{V}^{n-1}$, whence

$$K^{n-1} \cap g\overline{W} \subset \overline{V}^{n-1}.$$

Hence it follows from the definition of the weak topology that $\overline{V}^n$ is the union of $\overline{V}^{n-1}$ and the sets $g\overline{W}$, which are closed since $\overline{W}$ is compact. Since $\overline{V}^{n-1} \subset U$, $\overline{W} \subset g^{-1}U$ it follows that $\overline{V}^n \subset U$. Finally define $f_t^n: V^n \rightarrow V^n$ by $f_t^n \mid V^{n-1} = f_t^{n-1}$, $f_t^n \mid gW = \theta_t$. It follows from (5.4) and from (I) that f_t^n is single-valued and continuous and that $f_0^n = 1$, $f_1^n V^n = a_0$. Therefore (M) follows by induction on n.

(N) *Any covering complex, $\tilde{K}$, of K is a* CW-*complex.*

Since $\tilde{K}$ is locally connected, by the definition of a covering space, each of its components is both open and closed and is a covering complex of a component of K. A locally connected complex is obviously a CW-complex if, and only if, each of its components is a CW-complex. Therefore (N) will follow when we have proved it in case K and $\tilde{K}$ are connected. We assume that this is so and also, to begin with, that $\tilde{K}$ is a regular covering complex of K. That is to say the group, G, of covering transformations[25] in $\tilde{K}$ operates transitively on the set $p^{-1}q$, for any point $q \in K$, where $p: \tilde{K} \rightarrow K$ is the covering map. We shall describe an open set, $U \subset K$, as an *elementary neighborhood* if, and only if, each component of $p^{-1}U$ is mapped by p topologically onto U. We shall describe an elementary neighborhood in K as a *basic neighborhood* if, and only if, its closure is contained in an elementary neighborhood. We shall describe a subset of $\tilde{K}$ as a *basic neighborhood* if and only if it is a component of $p^{-1}U$, where U is a basic neighborhood in K. If $\tilde{U} \subset \tilde{K}$ is a basic neighborhood the component of $p^{-1}(p\tilde{U})$ are the sets $T\tilde{U}$ for every $T \in G$. It follows from the definition of $\tilde{K}$ and the normality of K that the basic neighborhoods constitute a basis for the open sets, both in K and in $\tilde{K}$.

Let $U \subset K$ be a basic neighborhood and let V be an elementary neighborhood such that $\overline{U} \subset V$. Then the components of $p^{-1}V$ are disjoint open sets in $\tilde{K}$, each of which contains exactly one component of $p^{-1}\overline{U}$. Let $Q \subset p^{-1}\overline{U}$ be a set of points, of which at most one lies in each component of $p^{-1}\bar{U}$. Then Q is a closed discrete set. For if Q has a limit point, $\bar{q}$, then $p\bar{q}\in\bar{U}\subset V$, whence $\bar{q}$ lies in one of the components, $\tilde{V}$, of $p^{-1}V$. But this is absurd, since $\tilde{V}$ contains at most one point of Q. Therefore Q is closed and discrete.

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[25] I.e., the group of homeomorphisms, $T: \tilde{K} \rightarrow \tilde{K}$, such that $pT = p$.

Let $\tilde{U}\subset\tilde{K}$ be a basic neighborhood, let U^* be its closure[26] and let $C\subset\tilde{K}$ be compact. I say that only a finite number of the sets TC meet U^*, where $T\in G$. For if TC meets U^* then C meets $T^{-1}U^*$. Let $q_T\in C\cap T^{-1}U^*$. Since $T'U^*\cap T''U^*=0$ if $T'\neq T''$ it follows from the preceding paragraph that the aggregate of points q_T, for every T such that $U^*\cap TC\neq 0$, is a discrete, closed subset of C. Since C is compact the set $\{q_T\}$ is finite, which proves our assertion.

We now prove that $\tilde{K}$ has the weak topology. Let $\tilde{X}\subset\tilde{K}$ be a subset such that $\tilde{X}\cap e^*$ is closed, for every cell $\tilde{e}\in\tilde{K}$. In order to prove that $\tilde{X}$ is closed it is enough to prove that $\tilde{X}\cap U^*$ is closed, where U^* is the closure of an arbitrary basic neighborhood $\tilde{U}\subset\tilde{K}$. For this implies that $\tilde{U}-\tilde{X}=\tilde{U}-(\tilde{X}\cap U^*)$ is open, whence it follows that $\tilde{K}-\tilde{X}$ is open. Therefore, to simplify the notation, we assume that $\tilde{X}\subset U^*$, where $\tilde{U}$ is a basic neighborhood in $\tilde{K}$. Let $X=p\tilde{X}$ and let e be a given cell in K. Then[27]

$$X\cap\bar{e}=p(\tilde{X}\cap p^{-1}\bar{e}).$$

Let $\tilde{e}\in\tilde{K}$ be a cell which covers e. Then $p^{-1}\bar{e}$ consists of the sets Te^* for every $T\in G$, and Te^* is the closure of the cell $T\tilde{e}\in\tilde{K}$. Since e^* is compact it follows from the preceding paragraph that only a finite number of the sets Te^*, say $T_1e^*,\cdots,T_ke^*$, meet U^*. Let $P_i=\tilde{X}\cap T_ie^*$ $(i=1,\cdots,k)$. Then

$$X\cap\bar{e}=p(\tilde{X}\cap p^{-1}\bar{e})=p(P_1\cup\cdots\cup P_k).$$

But P_i is closed, by the hypothesis concerning $\tilde{X}$, and hence compact, since T_ie^* is compact. Therefore, $P_1\cup\cdots\cup P_k$ and hence $X\cap\bar{e}$ are compact. Since the cell $e\in K$ is arbitrary it follows that X is closed. Therefore $p^{-1}X$ is closed. Since $U^*\cap TU^*=0$ if $T\neq 1$ it follows that

$$\tilde{X}=U^*\cap\bigcup_T T\tilde{X}=U^*\cap p^{-1}X.$$

Therefore $\tilde{X}$ is closed and it follows that $\tilde{K}$ has the weak topology.

Since K^0 is discrete it follows that $\tilde{K}^0=p^{-1}K^0$ is a discrete set of points. That is to say, $\tilde{K}^0$ has the weak topology. If $n>0$ then K^n is connected, according to (C), and $\tilde{K}^n$ is obviously a covering complex of K^n. It follows from (L) that the injection homomorphism, $\pi_1(K^n)$

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[26] We shall denote the closure of a set $\tilde{P}\subset\tilde{K}$ by P^*.

[27] If $f:P\to Q$ is any map and $A\subset P$, $B\subset Q$, then $f(A\cap f^{-1}B)=(fA)\cap B$.

$\to\pi_1(K)$, is onto, whence $\tilde{K}^n$ is connected. Obviously $T\tilde{K}^n=\tilde{K}^n$ for any $T\in G$ and it follows that $\tilde{K}^n$ is a regular covering complex of K^n. Therefore $\tilde{K}^n$ has the weak topology, according to what we have just proved. It follows from (E) that $\tilde{K}$ is a CW-complex.

Now let $\tilde{K}$ be a (connected) covering complex of K, which is not regular. Then a universal covering complex, $\hat{K}$ of $\tilde{K}$ is a universal covering complex of K. Therefore $\hat{K}$ is a CW-complex. Let $p:\hat{K}\to\tilde{K}$ be the covering map. Since p is an open map it follows that $\tilde{K}$ has the identification topology determined by p. It follows from the final paragraph in §4 that $\tilde{K}$ is closure finite and that the remaining condition of (F) is satisfied. Therefore it follows from (F) that $\tilde{K}$ is a CW-complex, which completes the proof of (N).

6. **Proof of Theorems 1–4.** Let X, Y be spaces in the class α and let $f:X\to Y$ be a map such that the induced homomorphism, $f_*:\pi_n(X)\to\pi_n(Y)$, is an isomorphism onto if $1\leqq n<N+1$, where $N=\max(\Delta X, \Delta Y)$. Let P and Q be CW-complexes, of dimensionalities ΔX and ΔY, which dominate X and Y respectively. Let $\lambda:X\to P$, $\lambda':P\to X$ and $\mu:Y\to Q$, $\mu':Q\to Y$ be maps such that $\lambda'\lambda\simeq 1$, $\mu'\mu\simeq 1$. Let Z be the mapping cylinder[28] of f and R the mapping cyclinder of the map $\mu f\lambda':P\to Q$, assuming that no two of the spaces X, Y, $X\times I$, etc., have a point in common.

Assume that there is a homotopy, $h_t:(Z, X)\to(Z, X)$, such that $h_0=1$, $h_1Z\subset X$. Let $h:Z\to X$ be given by $hz=h_1z$ $(z\in Z)$ and let $i:X\to Z$ be the identical map. Then $ih=h_1:Z\to Z$, whence $ih\simeq 1$. Also $hi\simeq 1$, since $k_t:X\to X$, given by $k_tx=h_tx$ $(x\in X)$, is a deformation of $k_0=1$ into $k_1=hi$. Therefore i is a homotopy equivalence. As in [6] it follows that $f:X\to Y$ is a homotopy equivalence.

We now prove the existence of the homotopy $h_t:(Z, X)\to(Z, X)$. The argument in [6], which refers to the homotopy sequence

$$\pi_n(X)\to\pi_n(Z)\to\pi_n(Z, X)\to\pi_{n-1}(X)\to\pi_{n-1}(Z), \tag{6.1}$$

shows that $\pi_n(Z, X)=0$ if $1\leqq n<N+1$. Since $\dim Q\leqq N$ it follows from (K), in §5 above, that there is a homotopy, $\rho_t:Q\to Z$, such that $\rho_0=j\mu'$, $\rho_1Q\subset X$, where j is the identical map $j:Y\to Z$. Therefore $j\simeq j\mu'\mu\simeq\rho_1\mu$. Since $\rho_1\mu Y\subset\rho_1Q\subset X$, the argument used in [6] shows that $\pi_{N+1}(Z, X)=0$, in case $N<\infty$.

Let $K=(P\times I)\cup Q$ and let R be formed from the CW-complex K by identifying[28] $(p, 0)$ with p and $(p, 1)$ with $\mu f\lambda'p\in Q$ for each point $p\in P$. Let $\phi:K\to R$ be given by $\phi(p, 0)=p$, $\phi(p, 1)=\mu f\lambda'p$, $\phi(p, t)=(p, t)$ if $0<t<1$, $\phi|Q=1$. The arguments in [6, §2] show that there are maps

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[28] See §7 below.

$$\nu:(Z, X) \rightarrow (R, P), \qquad \nu':(R, P) \rightarrow (Z, X),$$

with $\nu' | Q = j\mu'$, and a homotopy, $\zeta_t:(Z, X) \rightarrow (Z, X)$, such that $\zeta_0 = \nu'\nu$, $\zeta_1 = 1$. Let $L = (P\times 0)\cup(P\times 1)\cup Q \subset K$, let $\rho_t: Q \rightarrow Z$ mean the same as before and let $\rho'_t: L \rightarrow Z$ be given by

$$\rho'_t(p, 0) = \nu'\phi(p, 0),$$
$$\rho'_t(p, 1) = \rho_t\phi(p, 1),$$
$$\rho'_t q = \rho_t\phi q \qquad (p \in P,\ q \in Q).$$

Then $\rho'_0 = \nu'\phi | L$, since $\rho_0 = j\mu' = \nu' | Q$, and $\rho'_1 L \subset X$. Let $\psi_0 = \nu'\phi: K \rightarrow Z$. Then it follows from (J) that ρ'_t has an extension $\psi_t: K \rightarrow Z$. Since $\psi_1 L = \rho'_1 L \subset X$ and since $\dim K \leqq N+1$ and $\pi_n(Z, X) = 0$ if $1 \leqq n < N+2$ it follows from (K) that there is a homotopy, $\psi'_t: K \rightarrow Z$, rel. L, such that $\psi'_0 = \psi_1$, $\psi'_1 K \subset X$. Let $\theta_t: K \rightarrow Z$ be the resultant of ψ_t followed by ψ'_t. Then it is easily verified that $\chi_t = \theta_t\phi^{-1}: R \rightarrow Z$ is single-valued, and hence continuous [7, §5]. Moreover $\chi_0 = \psi_0\phi^{-1} = \nu'\phi\phi^{-1} = \nu'$ and $\chi_1 R = \psi'_1\phi^{-1}R \subset X$. Therefore the required homotopy, $h_t:(Z, X) \rightarrow (Z, X)$, may be defined as the resultant of ζ_{1-t} followed by $\chi_t\nu$. This proves Theorem 1.

The proofs of Theorems 2, 3, 4 are the same as those of the corresponding theorems in [6].

7. **Note on n-homotopy.** Let us return to Fox's original definition of n-homotopy, in terms of finite polyhedra. It is not certain, for reasons indicated in [6], that an n-homotopy equivalence, $f: X \rightarrow Y$, induces an isomorphism of $\pi_r(X)$ onto $\pi_r(Y)$ $(r \leqq n)$, where X, Y are arbitrary, arcwise connected spaces. This question reduces to the following. Let $h_r: \pi_r(X) \rightarrow \pi_r(X)$ be the endomorphism induced by a map, $h: X \rightarrow X$, such that $h \simeq_n 1$. Is h_r an automorphism if $1 \leqq r \leqq n$? Consider the case $r = 1$. It is easily proved that, given any finite set of elements, $a_1, \cdots, a_k \subset \pi_1(X)$, there is an element, $\eta \in \pi_1(X)$, such that $h_1 a_i = \eta a_i \eta^{-1}$ $(i = 1, \cdots, k)$. It follows that h_1 is an isomorphism into. Also it is an inner automorphism if $\pi_1(X)$ has a finite set of generators. Is h_1 an automorphism if $\pi_1(X)$ is arbitrary?

Meanwhile we show that these questions can be answered with our definition. Let X be an arbitrary arcwise connected space and let a point $x_0 \in X$ be chosen as base point for all the groups $\pi_r(X)$. We construct a CW-complex, $K = K(X, x_0)$, which is somewhat analogous to the singular complex of homology theory. Let K^0 consist of a single 0-cell e^0. With each map, $\theta:(\sigma^r, \partial\sigma^r) \rightarrow (X, x_0)$ $(r = 1, 2, \cdots)$, we associate a cell, $e^r_\theta \in K$, such that $\partial e^r_\theta = e^0$. The points of e^r_θ are the pairs (p, θ), for each point $p \in \sigma^r - \partial\sigma^r$, and $g: \sigma^r \rightarrow \bar{e}^r_\theta$, given by $gp = (p, \theta)(p \in \sigma^r - \partial\sigma^r)$, $g\partial\sigma^r = e^0$, is a characteristic map for e^r_θ. The complex thus defined is obviously closure finite and we give it the weak topology.

Let $f, f': X \rightarrow Y$ be maps of X in any arcwise connected space Y, and let $f_r, f'_r: \pi_r(X) \rightarrow \pi_r(Y)$ be the homomorphisms induced by f, f', when the base-point $y_0 \in Y$ is joined to $fx_0, f'x_0$ by paths (see [12] and [1, p. 279]) $\lambda, \lambda': (I, 0) \rightarrow (Y, y_0)$. If $\eta \in \pi_1(Y)$, $b \in \pi_r(Y, y_0) (r \geqq 1)$ we shall use $\eta \cdot b$ to denote the image of b in the automorphism of $\pi_r(Y, y_0)$, which is determined by η. Thus $\eta \cdot b = \eta b \eta^{-1}$ if $r = 1$. Using our definition of n-homotopy we have:

THEOREM 11. *If* $f \simeq_n f'$ *there is a fixed element,* $\eta \in \pi_1(Y)$, *such that* $f'_r a = \eta \cdot f_r a$ *for every* $a \in \pi_r(X)$ *and every* $r = 1, \cdots, n$.

Let $K = K(X, x_0)$ mean the same as before and let $\phi: K \rightarrow X$ be given by $\phi e^0 = x_0$, $\phi(p, \theta) = \theta p$, for each point $(p, \theta) \in K - e^0$. It follows from (A) in §5 that ϕ is continuous. The homomorphism, $\phi_r: \pi_r(K) \rightarrow \pi_r(X)$, induced by ϕ is obviously onto. Since $f \simeq_n f'$ there is a homotopy, $\xi_t: K^n \rightarrow Y$, such that $\xi_0 = f\phi | K^n$, $\xi_1 = f'\phi | K^n$. Let $\mu: I \rightarrow Y$ be given by $\mu(t) = \xi_t e^0$. Then $\mu(0) = \lambda(1)$, $\mu(1) = \lambda'(1)$. Let $\eta \in \pi_1(Y)$ be the element represented by the circuit, which consists of λ', followed by μ^{-1}, given by $\mu^{-1}(t) = \mu(1-t)$, followed by λ^{-1}. Then it is easily verified that $f'_r \phi_r a = \eta \cdot f_r \phi_r a$, where $a \in \pi_r(K)$ and $1 \leqq r \leqq n$. Since ϕ_r is onto this proves the theorem.

8. **A process of identification.** We shall prove a theorem, which is analogous to Theorem 2 in [7]. For this we shall need a lemma concerning a process of identification, which is often used (e.g., in constructing mapping cylinders, attaching cells k to complexes, etc.) but which, as far as I know, has never been described in full generality and detail.

Let X and Y_0 be given topological spaces and let $f_0: X_0 \rightarrow Y_0$ be a map of a closed subset, $X_0 \subset X$, into Y_0. We allow X and Y_0 to have common points, but only if $X \cap Y_0 \subset X_0$, $X \cap Y_0$ being a closed subset, both of X_0 and of Y_0, and $f_0 | X_0 \cap Y_0 = 1$. Let Y be the space which consists of the points in $X - X_0$ and in Y_0, with the identification topology determined by the map[29] $\phi: X \cup Y_0 \rightarrow Y$, where $\phi | X - X_0 = 1$, $\phi | Y_0 = 1$, $\phi | X_0 = f_0$. Notice that a set $B_0 \subset Y_0$ is closed in Y if, and only if, $Y_0 \cap \phi^{-1}B_0$ $(=B_0)$ and $X \cap \phi^{-1}B_0$ $(=f_0^{-1}B_0)$ are both closed in $X \cup Y_0$. Since X_0 is closed it follows from the continuity of f_0 that $f_0^{-1}B_0$ is closed in X if B_0 is closed in Y_0. Therefore Y_0 keeps its own topology and is a closed subset of Y. Similarly $X-X_0$ keeps its own topology. We shall say that Y is formed by *attaching* X, *or* $X-X_0$ *to* Y_0 *by means of the map* f_0. We shall also say that Y is formed by *identifying* each point $x \in X_0$ with $f_0x \in Y_0$.

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[29] $X \cup Y_0$ denotes the space, which consists of the points in X and in Y_0 and in which a subset, A, is closed if, and only if, $A \cap X$ and $A \cap Y_0$ are closed subsets of the spaces X and Y_0. Since $X \cap Y_0$ is closed in both X and Y_0 it follows that X and Y_0 keep their own topologies in $X \cup Y_0$.

In some cases we may be given the spaces X, Y and a map, $f:(X, X_0) \rightarrow (Y, Y_0)$, where $X_0 \subset X$ and $Y_0 \subset Y$ are closed subsets, subject to the conditions:

(8.1) (a) $f | X-X_0$ *is a homeomorphism onto* $Y-Y_0$;

(b) *a subset* $B \subset Y$ *is closed* (*open*) *if, and only if,* $B \cap Y_0$ *is closed* (*relatively open*) *and if* $f^{-1}B$ *is closed* (*open*).

In this case we may always regard Y as formed by attaching a homeomorph of X to Y_0. For let $h_0: X_0 \rightarrow X_0'$ be a homeomorphism of X_0 onto a new space, X_0', which does not meet Y. Let X' be the space consisting of the points in X_0' and in $Y-Y_0$ with the topology which makes $h:(X, X_0) \rightarrow (X', X_0')$ a homeomorphism, where $h | X_0 = h_0$, $h | X-X_0 = f | X-X_0$. The conditions (8.1) are satisfied by X', Y and the map fh^{-1}: $(X', X_0') \rightarrow (Y, Y_0)$. Since $X'-X_0' = Y-Y_0$ and $fh^{-1} | X'-X_0' = 1$ it follows that Y is the result of attaching X' to Y_0 by means of the map $fh^{-1} | X_0'$.

A particularly important application of this construction is that of the mapping cylinder of a map, $g: A \rightarrow B$, where A, B are disjoint spaces. After replacing $A \times I$ by a homeomorph, if necessary, we assume that it has no point in common with A or B. Then the mapping cylinder is defined as above, with $Y_0 = A \cup B$, $X = A \times I$, $X_0 = (A \times 0) \cup (A \times 1)$ and $f_0(a, 0) = a$, $f_0(a, 1) = ga$.

Let X, Y and $f:(X, X_0) \rightarrow (Y, Y_0)$ satisfy (8.1) and let $X-X_0 = Y-Y_0$, $f | X-X_0 = 1$. Let X be a normal, Hausdorff space and let Y_0 satisfy the following "weak regularity" condition: distinct points $y_1, y_2 \subset Y_0$ have neighborhoods $V_1, V_2 \subset Y_0$ such that $\overline{V}_1 \cap \overline{V}_2 = 0$.

LEMMA 1. *Under these conditions* Y *is a Hausdorff space.*

Let y_1, y_2 be distinct points in Y. First assume that at least one of them, say y_1, is in $Y-Y_0 = X-X_0$. Since Y_0 and X are Hausdorff spaces, the sets y_1, y_2 and $f^{-1}y_2$ are closed. Since X is normal there is therefore a neighborhood, $U \subset X$, of y_1, such that

$$\overline{U} \subset X - (X_0 \cup f^{-1}y_2).$$

Then $f\overline{U} = \overline{U} \subset Y-Y_0$ and $y_1 \in U$, $y_2 \in Y-\overline{U}$. Also U is open and $\overline{U}$ closed in Y. Therefore U, $Y-\overline{U}$ are disjoint, open subsets of Y, which contain y_1, y_2 respectively.

Let $y_1, y_2 \subset Y_0$ and let $V_1^0, V_2^0 \subset Y_0$ be neighborhoods of y_1, y_2 such that $\bar{V}_1^0 \cap \bar{V}_2^0 = 0$. Let $U_i^0 = f^{-1}V_i^0 \subset X_0$ $(i = 1, 2)$. Then $\bar{U}_i^0 \subset f^{-1}\bar{V}_i^0$, whence $\bar{U}_1^0 \cap \bar{U}_2^0 = 0$. Since X is normal there are disjoint open sets, U_1', $U_2' \subset X$, such that $\bar{U}_i^0 \subset U_i'$. Let

$$U_i = U_i' - (X_0 - U_i^0).$$

Then U_i is open, since $X_0 - U_i^0$ is closed, and $U_1 \cap U_2 = 0$, $U_i \cap X_0 = U_i^0$. Let

$$V_i = (U_i - U_i^0) \cup V_i^0 \subset Y.$$

Since $U_i - U_i^0 \subset Y - Y_0$ it follows that $Y_0 \cap V_i = V_i^0$, which is open in Y_0. Also

$$f^{-1}V_i = f^{-1}(U_i - U_i^0) \cup f^{-1}V_i^0 = (U_i - U_i^0) \cup U_i^0 = U_i.$$

Therefore V_i is open in Y. Also $y_i \in V_i$ and, obviously, $V_1 \cap V_2 = 0$. This completes the proof.

Now let X and Y_0 be disjoint CW-complexes and let X_0 be a subcomplex of X. Since X, Y_0 are normal, Hausdorff spaces it follows from Lemma 1 that Y is a Hausdorff space. Let $g: \sigma^n \to \bar{e}^n$ be a characteristic map for a given cell $e^n \in X - X_0$. Then $fg \mid \sigma^n - \partial\sigma^n$ is a homeomorphism onto fe^n, where $f: X \to Y$ means the same as before. It follows that Y is a complex, consisting of the cells in Y_0 and the cells fe $(e \in X - X_0)$, provided[30] $f(X_0 \cap \partial e^n) \subset Y_0^{n-1}$ for each n-cell $e^n \in X - X_0$ $(n = 0, 1, \cdots)$. This being so, I say that Y is closure finite. For any cell $e \in Y_0$ is contained in the finite subcomplex $Y_0(e)$. Let e be a given cell in $X - X_0$, let $K = X(e)$ and let $K_0 = K \cap X_0$. Then K_0, and hence fK_0, is compact, and it follows from (D) in §5 that $L_0 = Y_0(fK_0)$ is finite. The subset $L_0 \cup fK$ is compact and is the union of the cells in L_0 and in $f(K - K_0)$. It is therefore a subcomplex. Moreover it is finite and contains fe. Therefore Y is closure finite. Also Y has the identification topology, which is determined by the map $\phi: X \cup Y_0 \to Y$, where $\phi \mid X = f$, $\phi \mid Y_0 = 1$. Obviously $X \cup Y_0$ is a CW-complex, consisting of the cells in X and in Y_0, and the cells in Y are the cells ϕe for each cell $e \in (X - X_0) \cup Y_0$. Since Y is closure finite it follows from this and from (D) in §5 that $Y(\phi\bar{e})$ is finite for any $e \in X \cup Y_0$. Therefore Y is a CW-complex, according to (F) in §5. Under these conditions[31] we say that $Y - Y_0$ is *isomorphic* to $X - X_0$ and that $f \mid X - X_0$ is an *isomorphism* of $X - X_0$ onto $Y - Y_0$.

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[30] E.g., if $f_0: X_0 \to Y_0$ is cellular or if $X^{n-1} \subset X_0$, $fX_0 \subset Y_0^{n-1}$.

[31] In this definition we allow X_0 and Y_0 to be empty.

We mention two applications of the preceding paragraph. The first is that the mapping cylinder of a cellular map, $f:K\to L$, where K, L are CW-complexes, is itself a CW-complex. Secondly, let X be the union of a set of n-elements, $\{E_i^n\}$, which are disjoint from each other and from Y_0, each E_i^n being a subcomplex, $E_i^n=e_i^0\cup e_i^{n-1}\cup e_i^n$. Let X_0 be the union of the boundaries, $\partial E_i^n=e_i^0\cup e_i^{n-1}$, and let f_0 be any map of the form $f_0:X_0\to Y_0^{n-1}$. Then we say that Y is formed by *attaching the set of cells* $\{e_i^n\}$ *to* Y_0 *by means of the maps* $f_0|\partial E_i^n$.

Let X, Y and $f:(X, X_0)\to(Y, Y_0)$ satisfy (8.1), where X, Y are CW-complexes, X_0, Y_0 are subcomplexes and the map f is cellular. Let $f_0:X_0\to Y_0$ be the map given by $f_0x=fx$ $(x\in X_0)$. Then we have:

THEOREM 12. *If* $f_0:X_0\to Y_0$ *is a homotopy equivalence, so is* $f:X\to Y$.

The proof is the same[32] as that of Theorem 2 in [7].

If Y_0 is a single point we describe the above process of identification as shrinking X_0 into the point Y_0. It follows from Theorems 1 and 12 that a simply connected, aspherical subcomplex, $X_0\subset X$, may be shrunk into a point without altering the homotopy type of X.

Let $X_0=X^n$ and let $Y_0=Y_0^n$. Then $Y^n=Y_0^n$ and we have:

COROLLARY 1. *Given a* CW-*complex,* $Y_0^n\equiv X^n$, *there is a* CW-*complex,* Y, *such that* $Y^n=Y_0^n$, $X\equiv Y$ *and* $Y-Y^n$ *is isomorphic to* $X-X^n$.

Let X and Y_0 be connected, let $Y_0=Y_0^n$ and let $Y_0^n\equiv_{n-1}X^n$ $(n\geqq 2)$. Then

$$X^n\cup\Sigma_1^n\equiv Y_0^n\cup\Sigma_2^n,$$

where Σ_1^n, Σ_2^n mean the same as in Theorem 6. Let us "fill up" the closure, S^n, of each cell $e^n\in\Sigma_1^n$ with an $(n+1)$-cell, e^{n+1}, in such a way that $\bar{e}^{n+1}$ is an $(n+1)$-element bounded by S^n. The result is a CW-complex,[33] $K=X\cup\Sigma_1^n\cup\{e^{n+1}\}$. There is obviously a retracting deformation of K into X in which each $(n+1)$-element, $\bar{e}^{n+1}$, is retracted into the point $X\cap\bar{e}^{n+1}$. Therefore $K\equiv X$. Also $K^n=X^n\cup\Sigma_1^n$, $K^{n-1}=X^{n-1}$. Let $L^n=Y_0^n\cup\Sigma_2^n$. Then $L^{n-1}=Y_0^{n-1}$. Therefore, replacing X, X^n, Y_0^n in Corollary 1 by K, K^n, L^n, we have:

COROLLARY 2. *If* $Y_0^n\equiv_{n-1}X^n$ *there is a* CW-*complex,* Y, *such that* $X\equiv Y$ *and* $Y^{n-1}=Y_0^{n-1}$.

For example, if $\pi_r(X)=0$ for $r=1, \cdots, n-1$, we may take Y

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[32] Notice that CW-complexes and subcomplexes have all the homotopy extension properties required by [10].

[33] We assume that the cells in Σ_1^n and the new $(n+1)$-cells, $\{e^{n+1}\}$, are disjoint from each other and from X.

to be a single point, as observed in §3. Then $X \equiv Y$, where $Y^{n-1} = Y_0^n$ and Y^n consists of the n-spheres, which are the closures of the n-cells in Σ_2^n.

9. **Countable complexes.** We describe a complex as *countable* if, and only if, the number of its cells is countable. By a locally finite (finite) polyhedron we mean a space which is covered by a locally finite (finite) simplicial complex. The purpose of this section is to prove:

THEOREM 13. *Any countable (finite), p-dimensional* CW-*complex* ($p \leqq \infty$) *is of the same homotopy type as a locally finite (finite), p-dimensional polyhedron.*

We shall need three lemmas for the proof of this. A CW-complex, K', will be described as a *subdivision* of a CW-complex K if, and only if, both complexes cover the same space and if each cell of K' is contained in a cell of K. If K' is a simplicial complex we shall describe it as a simplicial subdivision of K. Let K' be a simplicial subdivision of K and let a set of disjoint n-cells, $\{e_i^n\}$, be attached to K by maps, $g_i:\partial E_i^n \to K^{n-1}$ ($K \cap E_i^n = 0$), each of which is simplicial with respect to K' and some triangulation of ∂E_i^n. Then we have:

LEMMA 2. *The complex $K \cup \{e_i^n\}$ has a simplicial subdivision, of which K' is a subcomplex.*

This will be proved in §10 below.

Let K and $\{e_i^n\}$ mean the same as in Lemma 2, except that K need not have a simplicial subdivision. Let $f:K \to L$ be a homotopy equivalence of K into a CW-complex L. Let

$$h_i \simeq fg_i:\partial E_i^n \to L,$$

where $h_i\partial E_i^n \subset L^{n-1}$ but $\simeq$ indicates homotopy in L. Let the cells $\{e_i^n\}$ be attached to L by the maps h_i, thus forming a complex $L \cup \{e_i^n\}$ ($L \cap E_i^n = 0$).

LEMMA 3. *The map $f:K \to L$ can be extended to a homotopy equivalence $h:K \cup \{e_i^n\} \to L \cup \{e_i^n\}$.*

If $h_i = fg_i$ this follows from Theorem 12, taking $h | e_i^n = 1$. In general it follows from Theorem 12 and the proof of Lemma 5 in [5].

Let a connected, CW-complex, K, be the union of an "expanding" set of subcomplexes, $K_0 \subset K_1 \subset \cdots$, such that if $n > 0$ then K_n is connected and the injection homomorphism

$$i_n:\pi_n(K_{n+1}) \to \pi_n(K)$$

is an isomorphism onto (e.g., $K_n = K^n$). Let L and $L_0 \subset L_1 \subset \cdots \subset L$ satisfy the same conditions and let $\{K_n\}$, $\{L_n\}$ be related by a sequence of homotopy equivalences,

$$f^n : (K_n, K_{n-1}) \to (L_n, L_{n-1}),$$

such that $f^n | K_{n-1} = f^{n-1}$.

LEMMA 4. *The map, $f: K \to L$, which is given by $f | K_n = f^n$, is a homotopy equivalence.*

Let i_n mean the same in L as in K and let

$$f_n : \pi_n(K) \to \pi_n(L), \qquad f_n^{n+1} : \pi_n(K_{n+1}) \to \pi_n(L_{n+1})$$

be the homomorphisms induced by f, f^{n+1}. Then f_n^{n+1} is an isomorphism onto since f^{n+1} is a homotopy equivalence. Also

$$i_n : \pi_n(K_{n+1}) \to \pi_n(K), \qquad i_n : \pi_n(L_{n+1}) \to \pi_n(L)$$

are isomorphisms onto and obviously $f_n i_n = i_n f_n^{n+1}$. Therefore f_n is an isomorphism onto, for each $n = 1, 2, \cdots$, and the lemma follows from Theorem 1.

We are now ready to prove Theorem 13. This will obviously follow when we have proved it for a countable CW-complex, K, which is connected if $p = \dim K > 0$. We first assume K to be finite and use induction on the number of cells in K, starting with the trivial case $K = K^0$. Let $p > 0$ and let $K_0 = K - e^p$, where e^p is any p-cell in K. Assume that there is a finite complex, L_0, with a simplicial subdivision L_0', which is related to K_0 by a homotopy equivalence $f_0 : K_0 \to L_0$. Let e^p be attached to K_0 by a map $g : \partial E^p \to K_0$, where $e^p = E^p - \partial E^p$. Assuming that $L_0 \cap E^p = 0$ let e^p be attached to L_0 by a map

$$h_0 \simeq f_0 g : \partial E^p \to L_0,$$

which is simplicial with respect to L_0' and some triangulation of ∂E^p. The result is a complex, L, which has a simplicial subdivision, according to Lemma 2. The theorem, for K finite, now follows from the induction and Lemma 3.

Now let K be infinite. In order to repeat the above construction an infinite number of times we shall, so to speak, "spread L_0 out" so that L may be locally finite. Assume that there is a sequence of locally finite complexes, $L_0 \subset L_1 \subset \cdots$, such that L_n is connected if $n > 0$, and a sequence of subcomplexes, $P_n \subset L_n$, such that:

(9.1) (a) $L_{n-1} - P_{n-1} = U_{n-1}$ is open in L_n and $L_{n-1} \subset U_n$ ($L_{-1} = P_{-1} = 0$),

(b) P_n is a deformation retract of L_n,

(c) if $n>m>1$ the injection homomorphism $\pi_{m-1}(L_m)\rightarrow\pi_{m-1}(L_n)$ is an isomorphism onto,

(d) there is a sequence of homotopy equivalences, $f^n:(K^n, K^{n-1})\rightarrow(L_n, L_{n-1})$, such that $f^n | K^{n-1}=f^{n-1}$,

(e) L_n has a simplicial subdivision, L_n', of which L_{n-1}' is a subcomplex.

(f) $\dim L_n=\dim K^n\leqq n$.

Then we define the topological space

$$L = \bigcup_{n=0}^{p} L_n \qquad (p = \dim K \leqq \infty),$$

in which a subset $X\subset L$ is closed (open) if, and only if, $X\cap L_n$ is a closed (relatively open) subset of L_n, for each $n\geqq 0$. Clearly L_n keeps its own topology in L and since each L_n is a locally finite complex it follows that L is a CW-complex, which consists of the cells in $L_0, L_1, \cdots$. I say that L is locally finite. For $U_{n-1}\subset L_{n-1}\subset U_n$ and U_{n-1} is open in L_n and hence in U_n. Therefore, given $m\geqq 0$, it follows by induction on n that U_m is open in U_n and hence in L_n for every $n>m$. Therefore U_m is open in L. Since $L_{n-1}\subset U_n$ it follows that any point $q\in L$ is in U_n for some n. Let $q\in U_n$ and let $Q\subset L_n$ be a finite subcomplex, of which q is an inner point, according to the topology of L_n. Since U_n is open in L it follows that q is an inner point of Q, according to the topology of L. Therefore L is locally finite.

Since L is locally finite it follows from (D), in §5 above, that any compact subset is contained in a finite subcomplex of L. It is therefore contained in L_n, for some $n\geqq 0$, and it follows without difficulty from (9.1c) that the injection homomorphism $\pi_{m-1}(L_m)\rightarrow\pi_{m-1}(L)$ $(m>1)$ is an isomorphism onto. Therefore it follows from (9.1d) and Lemma 4 that $K\equiv L$. Clearly

$$L' = \bigcup_n L_n'$$

is a (locally finite) simplicial subdivision of L. Finally $\dim L=p=\dim K$ and the theorem follows.

Now assume that L_n, P_n have been constructed for $n=1, \cdots, r\geqq 0$ so as to satisfy (9.1) with $n\leqq r$. If $K^{r+1}=K^r$ we define $L_{r+1}=L_r$, $P_{r+1}=P_r$. Let $K^{r+1}\neq K^r$ and let X_a be the complex covering the half line $\langle a, \infty)$, which has 0-cells at the points $a+i$ $(i=0, 1, \cdots)$. Assuming that the topological product $P_r\times X_0$ has no point in common with L_r, we identify each point $x\in P_r$ with $(x, 0)\in P_r\times 0$ so as to form a locally finite complex

$$M_r = L_r \cup (P_r \times X_0).$$

This has the simplicial subdivision $L_r' \cup (P_r \times X_0)'$, where $(P_r \times X_0)'$ is formed by starring the prisms $\sigma \times \langle i, i+1 \rangle$ $(i=0, 1, \cdots)$, for each simplex, σ, in the subcomplex of L_r' which covers P_r. Clearly L_r is a deformation retract of M_r, whence $g^r: K^r \rightarrow M_r$, given by $g^r x = f^r x$ $(x \in K^r)$, is a homotopy equivalence. It obviously follows from (9.1b) that $P_r \times X_0$ is a deformation retract of M_r and hence that $P_r \times k$ is a deformation retract of M_r, for each $k=1, 2, \cdots$. Therefore any map $\partial E^{r+1} \rightarrow M_r$ is homotopic to a map in $P_r \times k$. We now repeat the construction used in the finite case, attaching an $(r+1)$-cell, e_k^{r+1}, to M_r by a suitable map $\partial E_k^{r+1} \rightarrow P_r \times k$ $(k>0)$, where $e_1^{r+1}, e_2^{r+1}, \cdots$ are the $(r+1)$-cells in K. The result is a locally finite complex, L_{r+1}, which satisfies (9.1d, e, f) with $n \leqq r+1$. We define

$$P_{r+1} = (P_r \times X_1) \cup \left\{e_k^{r+1}\right\}.$$

Clearly $P_r \times X_1$ is a deformation retract of $P_r \times X_0$, and hence of M_r. Therefore P_{r+1} is a deformation retract of L_{r+1}. Therefore (9.1b), and obviously (9.1a), is satisfied with $n \leqq r+1$. Since L_r is a deformation retract of M_r and L_{r+1} is formed by attaching $(r+1)$-cells to M_r it follows that (9.1c), and hence all the conditions (9.1), is satisfied with $n \leqq r+1$. Starting with $L_0 = P_0 = K^0$ the theorem now follows by induction on r.

10. **Proof of Lemma 2.** An n-element, E^n, in an n-sphere, S^n, will be called a *hemisphere* if, and only if, the closure of $S^n - E^n$ is also an n-element. Let $f: \sigma^n \rightarrow \sigma^k$ be a barycentric map of an n-simplex, σ^n, onto a k-simplex σ^k $(k \leqq n)$. Assuming that σ^k does not meet σ^n let Γ be the mapping cylinder of f. We shall prove:

Lemma 5. *Γ is an $(n+1)$-element and σ^n is a hemisphere in $\partial\Gamma$.*

Before proving this we deduce a corollary and show that it implies Lemma 2. Let σ_1^n be a face of an $(n+1)$-simplex, σ^{n+1}. Let $f_1: \sigma_1^n \rightarrow \sigma^k$ be a barycentric map of σ_1^n onto σ^k and, assuming that σ^k does not meet σ_1^{n+1}, let E^{n+1} be the space formed by identifying each point $p \in \sigma_1^n$ with $f_1 p$. Obviously E^{n+1} is homeomorphic to $\sigma_0^{n+1} \cup \Gamma$, where Γ means the same as in Lemma 5 and σ_0^{n+1} is an $(n+1)$-simplex, of which $\sigma^n = \sigma_0^{n+1} \cap \Gamma$ is a face. Since σ^n is a hemisphere both of $\partial\sigma_0^{n+1}$ and of $\partial\Gamma$ we obviously have the corollary:

Corollary. *E^{n+1} is an $(n+1)$-element.*

Let an n-cell, e^n, be attached to a complex, K, by means of a map, $\partial E^n \rightarrow K$, which is simplicial with respect to a simplicial subdivision, K', of K and some triangulation, S, of ∂E^n. Let p_0 be an interior point of E^n. When e^n is attached to K each simplex $\sigma^{r+1}=p_0\sigma^r$, where σ^r is a simplex of S, is transformed into an $(r+1)$-element E^{r+1}, according to the above corollary. We triangulate $K'\cup e^n$ by starring all these $(r+1)$-elements, in order of increasing dimensionality. The result is a simplicial subdivision of $K\cup e^n$, which contains K' as a subcomplex. Lemma 2 follows from this construction, carried out in each of the cells e_i^n.

We now prove Lemma 5, using a method suggested by M. H. A. Newman. Let σ^n be a rectilinear simplex in Euclidean space R^n. Let $b_0, \cdots, b_k$ be the vertices of σ^k and let σ_i be the face of σ^n, whose vertices are those which are mapped by f on b_i. Let a_i be the centroid of σ_i, let σ_0^k be the k-simplex $a_0 \cdots a_k$ and let $g:\sigma^n\to\sigma_0^k$ be the barycentric map, which is determined by $g\sigma_i=a_i$. Then $ga_i=a_i$, whence $ggx=gx$ for each $x\in\sigma^n$, or $g^2=g$. Also $c\in\sigma_0^k$, where c is the centroid of σ^n, since a_i is the centroid of σ_i, and it follows from the definition of a barycentric map that $gc=c$.

We treat R^n as a vector space, taking c to be the origin or null vector, which we also denote by 0. We shall use small Roman letters to denote vectors in R^n and small Greek letters, as in $\lambda x+\mu y$, to denote (real) scalar multipliers. Let $\theta_\lambda:\sigma^n\to\sigma^n$ be the map which is given by

$$\theta_\lambda x = (1-\lambda)x + \lambda gx/2 \qquad (0 \leqq \lambda \leqq 1;\ x \in \sigma^n). \tag{10.1}$$

I say that if $\lambda<1$ the map θ_λ is nonsingular. For θ_λ is a linear map, such that $\theta_\lambda 0=0$, and if it is singular, then $\theta_\lambda x=0$ for some nonzero vector $x\in\sigma^n$. Since $g^2=g$ it follows from the linearity of g that

$$g\theta_\lambda x = (1-\lambda)gx + \lambda gx/2 = (1-\lambda/2)gx. \tag{10.2}$$

Therefore $\theta_\lambda x=0$ implies, first, $gx=0$, and then $(1-\lambda)x=0$, or $x=0$ if $\lambda<1$, which proves the assertion.

I say that, if $0\leqq\lambda<\mu\leqq 1$, then $\theta_\mu\sigma^n$ is in the interior of $\theta_\lambda\sigma^n$. For $\theta_\lambda x$ is the point which divides the segment joining x to $gx/2$ in the ratio $\lambda:(1-\lambda)$. When we have shown that the simplex $\theta_1\sigma_0^k$ is in the interior of $\theta_\lambda\sigma^n(\lambda<1)$ it will follow that the vertices of $\theta_\mu\sigma^n(\mu>\lambda)$ lie on rectilinear segments, which join the vertices of $\theta_\lambda\sigma^n$ to interior points of $\theta_\lambda\sigma^n$. Hence, and since $\mu>\lambda$, they are interior points of $\theta_\lambda\sigma^n$. Since θ_ρ is a linear map the assertion will follow. Thus we have to prove that $\theta_1\sigma_0^k$ is inside $\theta_\lambda\sigma^n$, or that any point $ga/2$ $(a\in\sigma^n)$ is of the form $\theta_\lambda x$, for some interior point $x\in\sigma^n$. Let $\rho=1/(2-\lambda)$. Then

$$\begin{aligned}\theta_\lambda\rho ga &= (1-\lambda)\rho ga + \lambda\rho g^2a/2\\ &= \rho(1-\lambda/2)ga = ga/2.\end{aligned}$$

But $\rho < 1$ since $\lambda < 1$. Therefore ρga is an interior point of σ^n and the assertion is proved.

The set of values $\lambda \in I$, such that a given point, $x \in \sigma^n$, does not lie in $\theta_\lambda \sigma^n$, is obviously open. Therefore there is a maximum, $\phi(x)$, of the values of λ such that $x \in \theta_\lambda \sigma^n$. It follows without difficulty from the preceeding paragraph that ϕ is a continuous, real-valued function of x, such that

$$(10.3) \qquad \begin{aligned} &\text{(a) } \phi(x) > 0 && \text{if } x \in \sigma^n - \partial\sigma^n, \\ &\text{(b) } \phi(x) = 0 && \text{if } x \in \partial\sigma^n, \\ &\text{(c) } x \in \theta_\lambda\sigma^n && \text{if, and only if, } \lambda \leqq \phi(x). \end{aligned}$$

We now imbed σ^n in $\sigma^n \times I$, in such a way that $x = (x, 0)$ if $x \in \sigma^n$, and take σ^k to be the simplex $\theta_1 \sigma_0^k \times 1$ and $f: \sigma^n \to \sigma^k$ to be the map given by

$$fx = (\theta_1 x, 1) = (gx/2, 1).$$

If x is fixed and λ varies from 0 to 1 we may regard the locus of the point $(\theta_\lambda x, \lambda)$ as the linear segment in $\sigma^n \times I$, which joins $x \in \sigma^n$ to $fx \in \sigma^k$. If $\theta_\lambda x = \theta_\lambda y$ it follows from (10.2) that $gx = gy$ and from (10.1) that $x = y$ if $\lambda < 1$. Therefore no two of these linear segments have an inner point in common and we may take Γ to be the union of the sets $\theta_\lambda \sigma^n \times \lambda$ for each $\lambda \in I$. It follows from (10.3c) that Γ is the set of all points (x, λ) such that $0 \leqq \lambda \leqq \phi(x)$.

Let $\psi: \sigma^n \to I$ be the real-valued function which is defined by $\psi(\rho v) = 1 - \rho$ for each vector $v \in \partial\sigma^n$ and each $\rho \in I$. Then the subset of $\sigma^n \times I$, which consists of all points (x, λ) such that $0 \leqq \lambda \leqq \psi(x)$, is the join, σ^{n+1}, of σ^n and the point $(0, 1)$. A homeomorphism (onto), $h: \Gamma \to \sigma^{n+1}$, is defined by

$$h(x, \lambda) = \{x, \rho(x)\lambda\},$$

where $0 \leqq \lambda \leqq \phi(x)$ and $\rho(x) = \psi(x)/\phi(x)$ if $x \in \sigma^n - \partial\sigma^n$, $\rho(x) = 0$ if $x \in \partial\sigma^n$. Clearly $h\sigma^n = \sigma^n$ and σ^n is a hemisphere of $\partial\sigma^{n+1}$. Therefore Lemma 5 is established.

Let K, L be simplicial complexes and let $f: K \to L$ be a simplicial map. Let C be the "combinatorial" mapping cylinder of f, defined as in [1], and Γ the "topological" mapping cylinder defined as in §8 above. Let σ be any simplex in K and let $C_\sigma \subset C$, $\Gamma_\sigma \subset \Gamma$ be the mapping cylinders of the map, $\sigma \to f\sigma$, which is determined by f. It follows from Lemma 5 that Γ is a polyhedral complex, whose (closed) cells are the simplexes in K and in L and the elements Γ_σ, for each simplex, σ, in K. Since Γ_σ is an element it may be starred from an internal point. Since C_σ is also a star an easy induction shows that C is combinatorially isomorphic to the triangulation of Γ', which is obtained by starring the elements Γ_σ in order of increasing dimensionality. Hence the conjecture in the footnote on p. 1235 of [3] is easily verified.

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RECENT PROGRESS IN THE GOLDBACH PROBLEM

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1. **Introduction.** The problem under consideration had its origin in a letter written by Goldbach to Euler in 1742 [4].[1] In the letter, Goldbach made two conjectures concerning the representation of integers as a sum of primes. They are equivalent to

(A) *every even integer greater than* 2 *is a sum of two primes,*

and

(B) *every integer greater than* 5 *is a sum of three primes.*

The two conjectures are, of course, equivalent. If $2n-2=p_1+p_2$, then $2n=p_1+p_2+2$ and $2n+1=p_1+p_2+3$. Conversely, if $2n=p_1+p_2+p_3$ one of the primes must be 2 and $2n-2=p_1+p_2$.

An impressive collection of numerical evidence indicating the truth of the conjectures has accumulated in the years since Goldbach's letter was written, but is it not known to this day whether the conjectures are true or false. What progress has been made towards the solution of the problem has been through two principal methods of attack.

The first of these is the sieve method (see §§2 and 3) due originally to Brun [1], and improved by Rademacher [16], Esterman [5], Ricci [17], [18], and Buchstab [2], [3]. The best result by this method, due to Buchstab in 1940, is

every sufficiently large even integer is a sum of two integers, each having at most four prime factors.

The sieve method has also been used in combination with results on the density of sequences of integers. (See §4.) Contributions have been made by Schnirelmann [20], Landau [12], [13], Heilbronn, Landau, Scherk [7], and Ricci [19]. Schnirelmann proved that

every integer >1 *is a sum of a finite number of primes.*

The best result, due to Ricci in 1937, a minor improvement on the Landau, Heilbronn, Scherk result of 1936, is

every sufficiently large integer is a sum of at most 67 *primes.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

This result did not stand for long. It was superseded almost immediately by the Vinogradov result described in the next paragraph.

The second principal method is the analytic method developed by Hardy and Littlewood [6] in their series of papers on Partitio Numerorum. In 1937, Vinogradov [23] established an asymptotic formula for the number of representations of an odd integer as a sum of three primes. From this it follows that

every sufficiently large odd integer is a sum of three primes.

The work of Page [15] and Siegel [21] is an important part of the Vinogradov method. Vinogradov's result is close to Conjecture B and is now known in the literature as the Goldbach-Vinogradov Theorem. The same method also leads to a proof of the result

almost all even integers are sums of two primes.

The precise meaning of "almost all" is explained in §5.

Linnik [14] and Tchudakoff [22] have given different proofs of Vinogradov's result, but the Goldbach-Vinogradov Theorem still stands as the nearest approach of modern mathematics to the Goldbach conjecture of 1742.

2. The sieve method. Brun's contribution. The method of Brun is an adaptation of the ancient Sieve of Eratosthenes, in which the mechanical process of striking out the multiples of primes is replaced by an algebraic recursion formula. Brun's significant contribution is the way in which the formula is suitably modified. This important point will be explained later at the appropriate time.

Let d be a given positive integer, and let $p_1, p_2, \cdots, p_k$ be the primes in order, which do not divide d and do not exceed y, where y is any positive number. Let $a_1; b_1; \cdots; a_k, b_k$ be a set of integers with $0 \leqq a_i < p_i$, $0 \leqq b_i < p_i$, $a_i \neq b_i$, $i = 1, 2, \cdots, k$. Let a be any positive integer and x any positive number. Then, following Rademacher, Buchstab, and Landau, let $F(x; d, y) = F(x; d, y; a_i, b_i; p_i)$ denote the number of integers n which satisfy the conditions

$$\begin{aligned} &n \leqq x, \qquad n \equiv a \pmod{d}, \\ &(n - a_i)(n - b_i) \not\equiv 0 \pmod{p_i}, \qquad i = 1, \cdots, k. \end{aligned} \tag{2.1}$$

The arguments a, a_i, b_i, p_i, need not be written in the function since the results will hold for every a and every set a_i, b_i of the type described. $F(x; d, 1)$ is nothing but the number of integers $n \leqq x$, $n \equiv a \pmod{d}$ and will be abbreviated $F(x; d)$.

The connection between $F(x; d, y)$ and the Goldbach problem is indicated by the following considerations. Let $d=2$, $a=1$, $y=x^{1/u}$, where x is even integer and u is an integer not less than 2. Let $a_i=0$, $b_i\equiv x \pmod{p_i}$ if $p_i\nmid x$; $b_i\not\equiv x \pmod{p_i}$ if $p_i | x$. Then the function $F(x; 2, x^{1/u})$ is the number of odd integers $n\leqq x$ such that neither n nor $x-n$ is divisible by any prime not exceeding $x^{1/u}$. Hence all the prime factors of n and $x-n$ are greater than $x^{1/u}$ and there cannot be more than $u-1$ of them. If $u=2$ each of n and $x-n$ is either a prime or equal to 1. Thus, if it could be shown that $F(x; 2, x^{1/2})\geqq 2$, it would follow that there is at least one representation $x=n+(x-n)$ in which each of n and $x-n$ is a prime. Buchstab's result, quoted in the introduction, comes from establishing a similar inequality for $F(x; 2, x^{1/5})$.

The fundamental recursion formula is not difficult to prove. Clearly, the number of integers satisfying (2.1) is equal to the number of integers satisfying

$$\begin{aligned} &n \leqq x, \qquad n \equiv a \pmod{d}, \\ &(n-a_i)(n-b_i) \not\equiv 0 \pmod{p_i}, \qquad i = 1, 2, \cdots, k-1, \end{aligned} \tag{2.2}$$

diminished by the number of integers satisfying

$$n \leqq x, \qquad n \equiv a \pmod{d}, \tag{2.3}$$

$$(n-a_k)(n-b_k) \equiv 0 \pmod{p_k}, \tag{2.4}$$

$$(n-a_i)(n-b_i) \not\equiv 0 \pmod{p_i} \qquad i = 1, 2, \cdots, k-1.$$

Since $a_k\not\equiv b_k$, $p_k\nmid d$, the conditions (2.3), (2.4) are equivalent to

$$n \leqq x, \quad n \equiv a' \pmod{dp_k}; \qquad n \leqq x, \quad n \equiv a'' \pmod{dp_k}.$$

It follows that

$$\begin{aligned} F(x; d, p_k; a) = F(x; d, p_{k-1}; a) \\ - F(x; dp_k, p_{k-1}; a') - F(x; dp_k, p_{k-1}; a''), \end{aligned} \tag{2.5}$$

where the arguments a, a', a'' are temporarily indicated in the function. Since the results are to hold for every a, a', a'' it is usual to write (2.5) symbolically in the form

$$F(x; d, p_k) = F(x; d, p_{k-1}) - 2F(x; dp_k, p_{k-1}), \tag{2.6}$$

where, of course, if $k=1$, $F(x; d, p_1)=F(x; d)-2F(x; dp_1)$. Equation (2.6) is the fundamental recursion formula.

From (2.6) it is but a short step to

$$F(x; d, p_k) = F(x; d) - 2\sum_{r=1}^{k} F(x; dp_r, p_{r-1})$$

and

$$
\begin{aligned}
F(x;\, d,\, p_k) &= F(x;\, d) - 2\sum_{r=1}^{k} F(x;\, dp_r) \\
&\quad + 4\sum_{r=1}^{k}\sum_{s=1}^{r-1} F(x;\, dp_r p_s) - \cdots \\
&\quad + (-2)^k F(x;\, dp_k p_{k-1} \cdots p_2 p_1).
\end{aligned} \tag{2.7}
$$

Since $F(x;\, d)$ is the number of integers $n \leqq x$, $n \equiv a \pmod{d}$, $F(x;\, d) = x/d + \theta$, where $|\theta| \leqq 1$. Hence, from (2.7),

$$
\begin{aligned}
F(x;\, d,\, p_k) &= (x/d)\{1 - 2\sum (1/p_r) \\
&\quad + 4\sum\sum (1/p_r p_s) - \cdots \\
&\quad + (-2)^k (1/p_k \cdots p_1)\} + R,
\end{aligned}
$$

or

$$
F(x;\, d,\, p_k) = (x/d)\prod_{r=1}^{k} (1 - 2/p_r) + R, \tag{2.8}
$$

where $|R|$ does not exceed the number of terms in the brackets, that is, $|R| \leqq 2^{k+1} - 1$.

Unfortunately, formula (2.8) is of no help as it stands. The difficulty is that, if $p_1, p_2, \cdots, p_k$ are the primes up to $x^{1/u}$ then $\prod(1 - 2/p_r) \sim c_1 u^2/\log^2 x$, and $k = \pi(x^{1/u}) \sim ux^{1/u}/\log x$, so that R is of higher order than the principal term in (2.8). This is where Brun's contribution comes in. He replaces the equation (2.8) by an inequality in which the number of terms is sufficiently reduced to make R of lower order than the principal term.

It would take too long to reproduce the details here. The interested reader may find them in Brun's original memoir or in Rademacher's paper. The final result is that, if $k_1, k_2, \cdots, k_t$ are integers satisfying $1 \leqq k_t \leqq \cdots \leqq k_1 \leqq k$, then

$$
F(x;\, d,\, p_k) = (x/d)E - R \tag{2.9}
$$

where $E = 1 - 2\sum(1/p_{r_1}) + 4\sum\sum(1/p_{r_1} p_{r_2}) - \cdots$, the summation indices satisfy

$$
\begin{aligned}
&1 \leqq r_1 \leqq k, \\
&1 \leqq r_{2j} \leqq k_j, \qquad\qquad j = 1, 2, \cdots, t, \\
&1 \leqq r_{2j+1} \leqq k_j, \\
&1 \leqq r_{2t+1} < r_{2t} < \cdots < r_2 < r_1,
\end{aligned}
$$

and R does not exceed the number of terms in the product

$$(1 - 2\sum (1/p_{r_1}))(1 - 2\sum (1/p_{r_2})) \cdots ,$$

that is, $R \leqq (2k+1)(2k_1+1)^2 \cdots (2k_t+1)^2$.

By suitably choosing the integers $k_1, k_2, \cdots, k_t$ it can be shown, for example, that

$$E > \frac{3}{10} \prod_{r=1}^{k} (1 - 2/p_r), \qquad R = O(p_k^{79/10}). \tag{2.10}$$

This is Rademacher's result, and illustrates the point that reducing the number of terms only changes the principal term to a numerical constant times its former value, but reduces the remainder term by a considerable amount.

It follows from (2.9) and (2.10) that, if $p_1, p_2, \cdots, p_k$ are the primes up to $x^{1/8}$, then $F(x; 2, x^{1/8}) > c_2 x/\log^2 x - c_3 x^{79/80}$. Hence every sufficiently large even integer x is a sum $x = n + (x - n)$ of two odd integers, each with at most seven prime factors. The reduction from seven to four prime factors is accomplished by the method of Buchstab, which is discussed in the next section.

It should be noted in passing that the Brun method will yield upper bounds for $F(x; d, p_k)$ as well. It is only necessary to stop the series for E with r_{2t} instead of r_{2t+1}. Such upper bounds are needed in Buchstab's work (§3) and in the density-sieve method (§4).

3. The sieve method. Buchstab's contribution. Buchstab's results are based on a modification of the fundamental recursion formula (2.6), and uses a different estimate for the number of integers satisfying (2.3), (2.4). (See also James [8].)

The number of integers n such that

$$\begin{gathered} n \leqq x, \qquad n \equiv a \pmod{d}, \\ n \equiv a_k \pmod{p_k}, \qquad (n - a_i)(n - b_i) \not\equiv 0 \pmod{p_i} \end{gathered} \tag{3.1}$$

is equal to the number of integers m such that

$$\begin{gathered} a_k + mp_k \leqq x, \qquad a_k + mp_k \equiv a \pmod{d}, \\ (a_k + mp_k - a_i)(a_k + mp_k - b_i) \not\equiv 0 \pmod{p_i}, \end{gathered}$$

or

$$\begin{gathered} m \leqq (x - a_k)/p_k, \qquad m \equiv a' \pmod{d}, \\ (m - a_i')(m - b_i') \not\equiv 0 \pmod{p_i}. \end{gathered} \tag{3.2}$$

The number of integers satisfying (3.2) and hence (3.1) is $F((x - a_k)/p_k; d, p_{k-1})$. Similarly the number of integers satisfying

(3.1) with a_k replaced by b_k is $F((x-b_k)/p_k;\, d,\, p_{k-1})$, and therefore

$$F(x;\, d,\, p_k) = F(x;\, d,\, p_{k-1}) - F((x - a_k)/p_k;\, d,\, p_{k-1}) - F((x - b_k)/p_k;\, d,\, p_{k-1}). \tag{3.3}$$

Since $0 \leqq a_k < p_k$, $0 \leqq b_k < p_k$, the difference between the last two terms on the right of (3.3) and $2F(x/p_k;\, d,\, p_{k-1})$ is at most 2. Therefore

$$F(x;\, d,\, p_k) = F(x;\, d,\, p_{k-1}) - 2F(x/p_k;\, d,\, p_{k-1}) + \theta_k, \tag{3.4}$$

where $0 \leqq \theta_k \leqq 2$. This is the recursion formula in the Buchstab method.

Let u and v be integers such that $2 \leqq u < v$, and let $p_t, p_{t+1}, \cdots, p_{k-1}, p_k$ be the primes in order for which

$$p_t \leqq x^{1/v} < p_{t+1} < \cdots < p_k \leqq x^{1/u} < p_{k+1}.$$

By repeated application of (3.4) it follows that

$$F(x;\, d,\, x^{1/u}) = F(x;\, d,\, x^{1/v}) - 2\sum_{i=t}^{k} F(x/p_i;\, d,\, p_{i-1}) + \sum_{i=t}^{k} \theta_i. \tag{3.5}$$

If a lower bound for $F(x;\, d,\, x^{1/v})$ and upper bounds for each term in the sum are known, formula (3.5) provides a lower bound for $F(x;\, d,\, x^{1/u})$.

Formula (3.5) is one of the important parts of Buchstab's contribution. The larger v is the easier it is to get a good lower bound for $F(x;\, d,\, x^{1/v})$ and reasonably good upper bounds for the terms in the sum, using the Brun methods. This leads to a better lower bound for $F(x;\, d,\, x^{1/u})$ than can be obtained with the sieve method by direct calculation.

Another important part of the method is the way in which (3.5) is used. It can be shown by the Brun method that there exists a pair of non-negative step functions $g_0(u)$, $G_0(u)$ defined for $u \geqq 2$ such that

$$g_0(u)cx/\log^2 x \leqq F(x;\, d,\, x^{1/u}) \leqq G_0(u)cx/\log^2 x \tag{3.6}$$

for all sufficiently large x, where c is an absolute constant. In Buchstab's first paper [2], for example, he starts with

$$\begin{aligned} g_0(u) &= \begin{cases} 0, & 2 \leqq u < 10, \\ 98, & u = 10; \end{cases} \\ G_0(u) &= 101.6, \qquad 2 \leqq u \leqq 10. \end{aligned} \tag{3.7}$$

It then follows from (3.6) that $g_0(u)$ and $G_0(u)$ may be replaced by

$g_1(u)$ and $G_1(u)$, respectively, where

$$g_1(u) < g_0(v) - 2\int_{u-1}^{v-1} G_0(z)(z+1)z^{-2}dz,$$

$$G_1(u) > G_0(v) - 2\int_{u-1}^{v-1} g_0(z)(z+1)z^{-2}dz,$$

for $2 \leqq u < v$.

The process may be repeated as long as the new bounds are better than the old. Starting with the rough estimate (3.7), Buchstab [2] obtains the following:

$$(3.8)\qquad \begin{aligned} g_8(u) &= \begin{cases} 0, & 2 \leqq u < 6, \\ 0.3, & 6 \leqq u < 8, \\ 53.51, & 8 \leqq u < 9, \\ 75.58, & 9 \leqq u < 10, \\ 98.0, & u = 10; \end{cases} \\ G_8(u) &= \begin{cases} 67.58, & 2 \leqq u \leqq 7, \\ 72.86, & 7 < u \leqq 8, \\ 85.1, & 8 < u \leqq 9, \\ 101.6, & 9 < u \leqq 10. \end{cases} \end{aligned}$$

In a second note, Buchstab [3] starts with the set (3.8) with the addition of

$$g_8(u) = \begin{cases} 98.0, & 10 \leqq u < 15, \\ 244.9997, & u = 15; \end{cases}$$

$$G_8(u) = \begin{cases} 144.1328, & 10 < u \leqq 12, \\ 196.0022, & 12 < u \leqq 14; \end{cases}$$

and works down to $g_s(5) = 0.96438$. From this follows the result quoted in the introduction, the best so far obtained by this method, that every sufficiently large even integer x is a sum $x = n + (x - n)$, in which each of n and $x - n$ has at most four prime factors.

It is not entirely improbable that the newer computation methods may be used to work further down, perhaps even to $g_s(2) > 0$. If this should be the case, Goldbach's first conjecture and other similar ones would be proved, at least for sufficiently large integers.

4. **The density-sieve method.** While it is true that the results of

the density sieve method were soon superseded, the method itself is still of interest.

It is well known that, if $\pi(x)$ denotes, as usual, the number of primes not greater than x, then $\pi(x)/x \to 0$ as $x \to \infty$. On the other hand, Schnirelmann [20] proved that, if $M(x)$ denotes the number of integers $n \leqq x$ which are a sum of two primes, then $M(x)/x \geqq 1/h$, $x \geqq 4$, where h is a constant greater than 1. In other words, the set of integers $x \geqq 4$, which are sums of two primes, has positive density. It follows from standard theorems on the density of sequences of integers that every integer $x > 1$ is a sum of a finite number of primes. This is the Schnirelmann theorem. (See Landau [13].)

The Schnirelmann result depends on the sieve method in the following way. Let $A(m)$ denote the number of representations of an integer m as a sum of two primes, and let $M(x)$ be the function defined above. Then, by the Cauchy-Schwarz inequality,

$$\left(\sum_{m=4}^{x} A(m)\right)^2 \leqq M(x) \sum_{m=4}^{x} A^2(m)$$

and

$$M(x) \geqq \left(\sum_{m=4}^{x} A(m)\right)^2 \Big/ \left(\sum_{m=4}^{x} A^2(m)\right). \tag{4.1}$$

What is needed then is a lower bound for the numerator, and an upper bound for the denominator, of the term on the right of (4.1). A suitable lower bound is easily found by noting that

$$\sum A(m) \geqq \pi^2(x/2) > c_1 x^2/\log^2 x. \tag{4.2}$$

The sieve method enters into the problem of finding an upper bound. Clearly, the number of solutions of $m = p + p'$, where p and p' are primes, each greater than $m^{1/2}$, does not exceed $F(m;\, 1,\, m^{1/2})$. The number of solutions where $p \leqq m^{1/2}$ or $p' \leqq m^{1/2}$ does not exceed $2m^{1/2}$. Hence, an upper bound for $F(m;\, 1,\, m^{1/2})$ provides an upper bound for $A(m)$ and finally for $\sum A^2(m)$.

The details are given by Landau [13] and will be omitted here. The result is

$$F(m;\, 1,\, m^{1/2}) < c_2(m/\log^2 m) \prod_{p \mid m} (1 + 1/p), \tag{4.3}$$

where c_2 is a constant. It then follows that $\sum A^2(m) < c_3 x^3/\log^4 x$, and then from (4.1) and (4.2) that $M(x)/x \geqq 1/h$, $h = c_3/c_1^2$.

In the Heilbronn, Landau, Scherk paper [7], $M(x)$ is the number of integers $m \leqq x$ for which $30m = p + p'$, and the inequality (4.3) is replaced by

$$\limsup_{m\to\infty, m\equiv 0 \pmod{30}} \frac{F(m;\, 30,\, m^{1/2})}{(m/\log^2 m)\prod_{p\mid m}(1+1/(p-2))} \leqq c_4$$

where c_4 is a constant. There are similar changes for $\sum A(m)$ and $\sum A^2(m)$. By a careful consideration of the numerical constants involved, it is shown that

$$\liminf_{x\to\infty} M(x)/x \geqq \frac{\liminf x^{-4}\log^4 x(\sum A(m))^2}{\limsup x^{-3}\log^4 x(\sum A^2(m))} \geqq \frac{1}{34}\,.$$

Finally, from a general result of Khintchine [10] it follows that every sufficiently large integer u may be written in the form

$$u = \sum_{i=1}^{34} m_i + r, \qquad 0 \leqq r \leqq 33.$$

Hence, for each j such that $2 \leqq j \leqq 31$,

$$v = 30u + j = \sum_{i=1}^{34} (p_i + p_i') + s$$

where $2 \leqq s \leqq 1021$.

It is easily verified that every s in the given range is a sum of at most three primes, and therefore every sufficiently large integer v is a sum of at most 71 primes. Ricci [19] reduced the number 71 to 67, but the improvement is one of detail and no new principle is involved.

5. **The analytic method.** The method pioneered by Hardy and Littlewood [6] is based on the idea of a generating function and on the Cauchy residue theorem. Thus, if $\{a_j\}$ is an infinite sequence of positive integers with $a_j < a_{j+1}$, the function $f(z) = \sum_{j=1}^{\infty} z^{a_j}$, $|z| < 1$, is the generating function of the sequence. If $A_s(N)$ denotes the number of representations of an integer N as a sum of s integers of the sequence, then

$$f^s(z) = \sum_{j=1}^{\infty} A_s(j) z^j,$$

and

$$A_s(N) = (1/2\pi i)\int_C f^s(z) z^{-N-1} dz, \tag{5.1}$$

where C is the circle $|z| = r < 1$.

The question of convergence of the series, but not the difficulty of the proof, is avoided in the Vinogradov [23] version by considering not $f(z)$ but $F(x) = \sum_{a_j \leqq N} e(a_j x)$ where $e(a_j x)$ denotes $e^{2\pi i a_j x}$, and x is real. Then (5.1) is replaced by

$$A_s(N) = \int_0^1 F^s(x) e(-Nx) dx.$$

If the sequence a_j is the sequence of primes, the number of representations of an integer N as a sum of three primes is

$$A(N) = \int_0^1 F^3(x) e(-Nx) dx, \tag{5.2}$$

where

$$F(x) = \sum_{p \leqq N} e(px). \tag{5.3}$$

Formula (5.2) is Vinogradov's starting point, and since his work is not easily accessible, some details may be of interest.

As is well known, the first step is to divide the interval (0, 1) into sub-intervals whose centers are the points $x = a/q$, $(a, q) = 1$, $0 \leqq a \leqq q$, $1 \leqq q \leqq Q$, corresponding to a Farey series of some order Q. The sub-intervals corresponding to the smaller values of q form the intervals of Class I (major arcs), and the remaining sub-intervals, the intervals of Class II (minor arcs). If n is written for log N and if h is a number greater than 3, the classes for the Goldbach problem are defined by

$$\text{I:} \quad x = a/q + y, \qquad 1 \leqq q \leqq [n^{3h}], \quad |y| \leqq n^{2h}/N,$$

$$\text{II:} \quad x = a/q + y, \qquad [n^{3h}] < q \leqq N/n^{3h}, \quad |y| \leqq n^{2h}/qN.$$

If x belongs to an interval of Class I, the sum $F(x)$ is written in the form

$$\begin{aligned} F(a/q + y) &= \sum_{r=0}^{q-1} \sum_{p \leqq N, p \equiv r (\text{mod } q)} e(p(a/q + y)) \\ &= \sum_{r=0, (r,q)=1}^{q-1} e(ra/q) \sum_{p \leqq N, p \equiv r (\text{mod } q)} e(py) + O\Big(\sum_{p \mid q} 1 \Big). \end{aligned}$$

The inner sum is approximated by $(1/\phi(q))J(y)$, where

$$J(y) = \int_M^N (e(yt)/\log t) dt, \qquad M = N^{1/2}.$$

In Vinogradov's paper the lower limit in $J(y)$ is 2, but it is better to use M (see James and Weyl [9]). From the results of Page [15] and Siegel [21], the error made in approximating to the inner sum is $O(N/\phi(q)n^{15k+1})$.

Since

$$\sum_{r=0,\,(r,q)=1}^{q-1} e(ra/q) = \mu(q),$$

it follows in turn that

$$F(a/q + y) = (\mu(q)/\phi(q))J(y) + \text{an error term},$$
$$F^3(a/q + y) = (\mu(q)/\phi^3(q))J^3(y) + \text{an error term}.$$

The part of the integral in (5.2) over an interval of Class I is

$$\int_{-n^{3k}/N}^{n^{3k}/N} F^3(a/q + y)e(-N(a/q + y))dy$$
$$= (\mu(q)/\phi^3(q))e(-Na/q)\int_{-n^{3k}/N}^{n^{3k}/N} J^3(y)e(-Ny)dy$$
$$+ \text{ an error term}.$$

The limits in the integral may be extended from $-\infty$ to ∞ without making the error term any worse. Hence

$$\sum\int_{\mathrm{I}} F^3(x)e(-Nx)dx$$
$$(5.4)\qquad = \sum_{q=1}^{[n^{3k}]}\cdot\sum_{a=0,\,(a,q)=1}^{q-1} (\mu(q)/\phi^3(q))e(-Na/q)\int_{-\infty}^{\infty} J^3(y)e(-Ny)dy$$
$$+ \text{ an error term}.$$

If the series for q is extended from 1 to ∞ it becomes the so-called singular series, $S(N)$, and, if $R(N)$ is written for the integral in (5.4), then

$$(5.5)\qquad \sum\int_{\mathrm{I}} F^3(x)e(-Nx)dx = S(N)R(N) + \text{an error term}.$$

The singular series is a purely arithmetical function of an integer N (it is zero when N is even), whereas $R(N)$ takes only the size of N into account, being defined for all real values of N. In fact $R(N)\sim N^2/2n^3$ as $N\rightarrow\infty$.

One of Vinogradov's fundamental contributions is an estimate for $F(x)$ when x belongs to an interval of Class II. Let D denote the product $p_1p_2 \cdots p_k$ of all primes not exceeding $N^{1/2}$. Then

$$F(x) = \sum_{p \leqq N} e(px) = \sum_{m=2,(m,D)=1}^{N} e(mx) + O(N^{1/2}).$$

If $\mu(d)$ is the Möbius function and

$$F_d(x) = \sum_{m=2,\, d|m}^{N} e(mx),$$

then (the sieve again)

$$F(x) = \sum_{d \leqq N,\, d|D} \mu(d) F_d(x) + O(N^{1/2}). \tag{5.6}$$

It follows from (5.6) and Vinogradov's important results on exponential sums that, if x belongs to an interval of Class II, then

$$F(x) = O(N/n^{h-2}), \tag{5.7}$$

where h is an arbitrary number greater than 3. Hence

$$\begin{aligned} \sum \int_{\mathrm{II}} F^3(x) e(-Nx)dx &= O(N/n^{h-2}) \int_0^1 |F(x)|^2 dx \\ &= O(N/n^{h-2})\pi(N) = O(N^2/n^{h-1}). \end{aligned} \tag{5.8}$$

This is of lower order than the principal term in (5.5), and hence, by (5.2), (5.5), and (5.8), $A(N) = S(N)R(N) + O(N^2/n^{h-1})$.

There is an asymptotic expansion (James and Weyl [9])

$$R(N) \sim (N^2/n^3)(1/2 + b_1/n + b_2/n^2 + \cdots),$$

where the error in stopping with the term b_{j-1}/n^{j-1} is less than C_j/n^j, where C_j does not depend on N. Hence

$$A(N) \sim S(N)(N^2/n^3)(1/2 + b_1/n + b_2/n^2 + \cdots). \tag{5.9}$$

The singular series has a product representation (Hardy and Littlewood [6], Landau [11])

$$S(N) = \prod_p (1 + 1/(p-1)^3) \prod_{p|N} (1 - 1/(p^2 - 3p + 3)).$$

For even N the first factor of the second product is zero, but for odd N it can be shown that $S(N) > C$, where C is a positive constant. By (5.9), $A(N) > 0$ if N is sufficiently large, so that every sufficiently large odd integer is a sum of three primes.

For reasons which are not too clear, the more recent papers mark a return to the original Hardy-Littlewood method (Hardy and Littlewood [6], Landau [11]). Hardy and Littlewood start with the function

$$f(z) = \sum_{p \geqq 2} \log p\, z^p \tag{5.10}$$

and so does Tchudakoff [22]. Linnik [14], on the other hand, begins with

$$F(z) = \sum_{m=1}^{\infty} \Lambda(m) z^m, \tag{5.11}$$

where $\Lambda(m) = \log p$ if $m = p^t$, $t \geqq 1$, and $\Lambda(m) = 0$, otherwise.

Each of the functions $f(z)$ and $F(z)$ is easier to treat by classical methods than is $\sum z^p$, but there is no essential difference. Starting from (5.10) the first step is to follow the plan of the Vinogradov method, that is, to write $z = e^{-1/N} e(a/q + y)$, $e^{-1/N}$ being a convergence factor, and

$$f(z) = \sum_{r=0,(r,q)=1}^{q-1} \quad \sum_{p \geqq 2, p \equiv r \pmod q} \log p\, e^{-p/N} e(p(a/q+y))$$
$$+ \text{ an error term.}$$

It is well known that (Landau [11])

$$\sum_{\chi} \bar{\chi}(r)\chi(p) = \begin{cases} \phi(q) & \text{if } \quad p \equiv r \pmod q, \\ 0 & \text{if } \quad p \not\equiv r \pmod q, \end{cases}$$

where the summation is over all characters mod q. Hence $f(z)$ is equal to

$$(1/\phi(q)) \sum_{\chi} \sum_{r} e(ra/q)\bar{\chi}(r) \sum_{p} \chi(p) \log p\, e^{-p/N} e(py)$$

plus an error term. Also, the inner sum may be replaced by

$$\sum_{m=2}^{\infty} \Lambda(m)\chi(m) e^{-m/N} e(my)$$

with an error $O(N^{1/2}/n^{3/2})$, $n = \log N$. Thus

$$f(z) = (1/\phi(q)) \sum_{\chi} \sum_{r} e(ra/q)\bar{\chi}(r) \sum_{m} \Lambda(m)\chi(m) e^{-m/N} e(my)$$
$$+ O(q^{1/2} N^{1/2}/n^{3/2}).$$

Linnik, starting with (5.11) instead of (5.10), arrives at the same expression for $F(z)$, but with a slightly different error term.

By the classical formula of Mellin,

$$\sum_m \Lambda(m)\chi(m)e^{-m/N}e(my)$$
$$= (1/2\pi i)\int_{2-\infty i}^{2+\infty i} w^{-s}\Gamma(s)(L'(s, \chi)/L(s, \chi))ds,$$

where $w=1/N-2\pi iy$, $|y| \leqq n^{3h}/N$ (major arc). It is here that the original Hardy-Littlewood paper ran into trouble. Clearly, the zeros of $L(s, \chi)$ play an important part in the evaluation of the integral, and at that time information concerning the density of these zeros was not precise enough to be of help. The remainder of Hardy and Littlewood's paper depends on an unproved hypothesis concerning the zeros. No further progress was made until the appearance of the papers of Page and Siegel.

Linnik and Tchudakoff base their work on the approximate functional equations for the L-functions, analogous to the famous approximate functional equation for the zeta function, and on improved estimates of the number of zeros of the L-functions in the critical strip. Their work involves some of the most delicate and intricate analysis written on the subject. Since Tchudakoff's work is given in detail (and in English) it is not necessary to reproduce it here. The final result is that, if $Q(N)=\int_C f^3(z)z^{-N-1}dz=\sum_{p+p'+p''=N}\log p \log p' \log p''$, where C is the circle $|z|=e^{-1/N}$, then

$$Q(N) = S(N)((N + 1)(N + 2)/2) + \text{an error term.}$$

From this it follows that $A(N)\sim S(N)(N^2/2n^3)$, but not the more complete result (5.9).

No doubt something is gained by appealing to classical results in the theory of the distribution of primes, but Vinogradov's appears to be the more direct approach.

The analytic method also has something to contribute towards the solution of Goldbach's first conjecture. Let $B(x)$ denote the number of even integers $m \leqq x$ which are *not* a sum of two primes. Then, by the Vinogradov, Linnik, or Tchudakoff methods, it can be shown that

$$\lim_{x\to\infty} B(x)/x = 0.$$

This is the precise meaning of the statement that almost all even integers are sums of two primes. As Landau [11] puts it in the introduction to his chapter on the Goldbach conjecture, "The Goldbach conjecture is false for at most 0% of all even integers. This at most 0% does not exclude, of course, the possibility that there are infinitely many exceptions."

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UNIVERSITY OF BRITISH COLUMBIA

THE ANNUAL MEETING OF THE SOCIETY

The fifty-fifth Annual Meeting of the American Mathematical Society was held at the Ohio State University, Columbus, Ohio, Tuesday to Thursday, December 28–30, 1948, in conjunction with the Annual Meeting of the Association for Symbolic Logic, the Annual Meeting of the Mathematical Association of America, and the Christmas Conference of the National Council of Teachers of Mathematics.

Over 500 people registered for the meeting, among whom were the following 441 members of the Society.

A. W. Adkisson, J. E. Adney, R. P. Agnew, E. J. Akutowicz, L. U. Albers, A. A. Albert, C. B. Allendoerfer, Warren Ambrose, B. A. Amirà, E. W. Anderson, R. D. Anderson, R. V. Andree, Nachman Aronszajn, Max Astrachan, M. C. Ayer, Frank Ayres, W. L. Ayres, Reinhold Baer, F. R. Bamforth, G. M. Bareis, W. E. Barnes, I. A. Barnett, A. F. Bausch, H. M. Beatty, J. C. Bell, J. H. Bell, J. L. Bell, A. A. Bennett, Theodore Bennett, W. D. Berg, Stefan Bergman, Felix Bernstein, William Betz, R. H. Bing, Z. W. Birnbaum, A. H. Black, C. J. Blackall, H. D. Block, Henry Blumberg, L. M. Blumenthal, R. P. Boas, H. W. Bode, W. M. Boothby, J. G. Bowker, M. G. Boyce, G. F. Bradfield, H. J. Bradley, A. T. Brauer, Richard Brauer, H. E. Bray, H. W. Brinkmann, J. R. Britton, Foster Brooks, R. S. Burington, Herbert Busemann, L. E. Bush, Jewell H. Bushey, W. H. Bussey, S. S. Cairns, R. H. Cameron, C. C. Camp, A. B. Carson, K. C. Cartwright, C. R. Cassity, J. W. Cell, Lamberto Cesari, K. Chandrasekharan, Harold Chatland, Y. W. Chen, Alonzo Church, Edmund Churchill, R. V. Churchill, W. G. Clark, J. A. Clarkson, M. D. Clement, L. W. Cohen, J. A. Cooley, N. A. Court, R. R. Coveyou, W. H. H. Cowles, V. F. Cowling, E. H. Crisler, J. S. Cronin, J. C. Currie, H. B. Curry, D. A. Darling, L. A. V. DeCleene, C. H. Denbow, R. F. Deniston, A. H. Diamond, R. P. Dilworth, C. L. Dolph, M. D. Donsker, J. L. Doob, H. L. Dorwart, C. H. Dowker, W. C. Doyle, Arnold Dresden, Melvin Dresher, R. J. Duffin, W. L. Duren, Aryeh Dvoretzky, E. L. Eagle, E. D. Eaves, Samuel Eilenberg, H. M. Elliott, J. H. Engel, H. P. Evans, G. M. Ewing, A. B. Farnell, Herbert Federer, William Feller, F. A. Ficken, N. J. Fine, C. D. Firestone, M. P. Fobes, L. R. Ford, G. E. Forsythe, Tomlinson Fort, R. H. Fox, J. S. Frame, Orrin Frink, W. H. J. Fuchs, R. E. Fullerton, B. E. Gatewood, H. M. Gehman, H. H. Germond, F. J. Gerst, David Gilbarg, B. P. Gill, Leonard Gillman, Wallace Givens, A. M. Gleason, E. L. Godfrey, Casper Goffman, Michael Goldberg, Michael Golomb, A. W. Goodman, R. E. Goodman, V. G. Gorciu, Saul Gorn, M. J. Gottlieb, S. H. Gould, L. M. Graves, J. B. Greeley, J. W. Green, L. J. Green, L. Z. Greene, V. G. Grove, P. E. Guenther, William Gustin, S. W. Hahn, Elizabeth Hahnemann, C. B. Hailperin, Franklin Haimo, Marshall Hall, P. R. Halmos, Frank Harary, O. G. Harrold, W. L. Hart, E. E. Haskins, J. O. Hassler, G. E. Hay, C. T. Hazard, A. E. Heins, R. G. Helsel, Fritz Herzog, Edwin Hewitt, E. H. C. Hildebrandt, T. H. Hildebrandt, Einar Hille, J. J. L. Hinrichsen, A. J. Hoffman, D. L. Holl, Carl Holtom, E. M. Hove, J. A. Hratz, R. C. Huffer, Ralph Hull, P. M. Hummel, W. R. Hutcherson, L. C. Hutchinson, M. A. Hyman, R. M. Iwanowski, A. W. Jacobson, R. L. Jeffery, Herbert Jehle, E. D. Jenkins, Walter Jennings, A. W. Jones, B. W. Jones, M. E. Jones, P. S. Jones, Bjarni Jónsson, Mark Kac, Shizuo Kakutani, L. H. Kanter, Irving Kaplansky,

H. T. Karnes, Chosaburo Kato, M. E. Kellar, M. W. Keller, J. L. Kelley, A. J. Kempner, J. R. F. Kent, D. E. Kibbey, W. M. Kincaid, J. R. Kinney, S. C. Kleene, J. R. Kline, P. A. Knedler, L. C. Knight, L. A. Knowler, D. M. Krabill, M. S. Kramer, H. W. Kuhn, O. E. Lancaster, R. E. Langer, G. A. Larew, E. H. Larguier, H. D. Larsen, C. G. Latimer, V. V. Latshaw, J. S. Leech, Joseph Lehner, F. A. Lewis, B. J. Lockhart, Charles Loewner, E. R. Lorch, L. L. Lowenstein, Eugene Lukacs, N. H. McCoy, S. W. McCuskey, W. C. McDaniel, J. C. C. McKinsey, E. J. McShane, C. C. MacDuffee, G. R. MacLane, Saunders MacLane, Ingo Maddaus, Szolem Mandelbrojt, H. B. Mann, C. G. Maple, Morris Marden, R. H. Marquis, George Marsaglia, R. M. Martin, W. T. Martin, M. E. Martinson, W. S. Massey, F. I. Mautner, J. R. Mayor, P. E. Meadows, A. E. Meder, C. W. Mendel, G. M. Merriman, E. L. Mickelson, E. J. Mickle, C. C. Miesse, H. J. Miles, D. D. Miller, Frederic H. Miller, L. H. Miller, W. E. Milne, H. J. Miser, E. E. Moise, C. N. Moore, F. R. Morris, D. C. Morrow, Marston Morse, G. D. Mostow, W. B. Moye, J. R. Musselman, D. M. Nead, Zeev Nehari, W. J. Nemerever, C. V. Newsom, O. M. Nikodým, E. P. Northrop, I. L. Novak, F. S. Nowlan, C. O. Oakley, R. E. O'Connor, F. C. Ogg, Rufus Oldenburger, L. F. Ollmann, E. J. Olson, Paul Olum, Morris Ostrofsky, E. R. Ott, F. W. Owens, H. B. Owens, Gordon Pall, W. V. Parker, H. C. Parrish, Philip Peak, S. E. Pence, P. M. Pepper, Mary Pettus, Everett Pitcher, Harry Polachek, J. C. Polley, J. W. Ponds, G. B. Price, F. M. Pulliam, Tibor Rado, E. D. Rainville, J. F. Randolph, S. E. Rasor, M. O. Reade, L. M. Reagan, O. W. Rechard, Mina Rees, P. V. Reichelderfer, W. P. Reid, Eric Reissner, Daniel Resch, M. M. Resnikoff, C. N. Reynolds, D. E. Richmond, R. F. Rinehart, L. A. Ringenberg, R. A. Roberts, L. V. Robinson, V. N. Robinson, W. J. Robinson, L. D. Rodabaugh, T. G. Room, P. C. Rosenbloom, Arthur Rosenthal, M. F. Rosskopf, S. A. Rowland, L. R. Rubashkin, C. H. Rust, W. A. Rutledge, Raphael Salem, Charles Saltzer, R. G. Sanger, L. A. Santaló, S. W. Saunders, A. C. Schaeffer, Robert Schatten, S. A. Schelkunoff, M. M. Schiffer, E. R. Schneckenberger, K. C. Schraut, H. M. Schwartz, G. E. Schweigert, C. E. Sealander, C. L. Seeback, I. E. Segal, Wladimir Seidel, M. E. Shanks, H. C. Shaub, Seymour Sherman, S. S. Shü, C. N. Shuster, Edward Silverman, L. L. Silverman, F. C. Smith, R. E. Smith, J. L. Snell, W. S. Snyder, Andrew Sobczyk, T. H. Southard, D. C. Spencer, C. E. Springer, George Springer, G. W. Starcher, E. P. Starke, N. E. Steenrod, H. E. Stelson, Rothwell Stephens, Guy Stevenson, B. M. Stewart, R. W. Stokes, D. M. Stone, M. H. Stone, R. B. Stone, E. B. Stouffer, A. C. Sugar, E. G. Swafford, Otto Szasz, Alfred Tarski, J. S. Taylor, M. E. Taylor, William Charles Taylor, William Clare Taylor, H. P. Thielman, L. O. Thompson, R. M. Thrall, G. L. Tiller, H. E. Tinnappel, Leonard Tornheim, M. M. Torrey, H. M. Trent, Deonisie Trifan, W. J. Trjitzinski, C. A. Truesdell, A. W. Tucker, A. R. Turquette, J. L. Ullman, Gilbert Ulmer, E. P. Vance, Henry Van Engen, H. E. Vansant, H. E. Vaughan, R. W. Wagner, G. L. Walker, R. J. Walker, S. E. Walkley, J. L. Walsh, J. B. Walton, W. R. Wasow, André Weil, Alexander Weinstein, B. A. Welch, F. P. Welch, E. T. Welmers, I. W. Welmers, F. J. Weyl, Hermann Weyl, George Whaples, E. A. Whitman, P. M. Whitman, D. R. Whitney, Hassler Whitney, G. T. Whyburn, R. B. Wildermuth, F. B. Wiley, R. B. Wiley, S. S. Wilks, W. L. Williams, C. O. Williamson, R. L. Wilson, J. E. Yarnelle, J. W. T. Youngs, Arthur Zeichner, Daniel Zelinsky, M. A. Zorn, Antoni Zygmund.

The twenty-second Josiah Willard Gibbs Lecture, entitled *Ramifications, old and new, of the eigenvalue problem*, was delivered by Professor Hermann Weyl of the Institute for Advanced Study on Tuesday, December 28. Professor Einar Hille, President of the American Mathematical Society, was the presiding officer.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited three speakers. Professor Mark Kac of Cornell University lectured on *Probability methods in some problems of analysis and theory of numbers* on Tuesday, December 28. Professor Antoni Zygmund presided. On Wednesday, December 29, Professor J. L. Walsh presided over two lectures on area theory. Professor A. S. Besicovitch of Cambridge University and the University of Pennsylvania spoke on *Parametric surfaces*; Professor Lamberto Cesari of the University of Bologna, the Institute for Advanced Study, and the Ohio State University lectured on the topic *Area and representation of surfaces.*

The Annual Business Meeting and Election of Officers was held on Wednesday, December 29. Details of proceedings are reported in the sequel. The Bôcher Memorial Prize was awarded jointly to Professors A. C. Schaeffer of Purdue University and D. C. Spencer of Stanford University for their papers on *Coefficients of schlicht functions* published in Duke Math. J. vol. 10 (1943) pp. 611–635; vol. 12 (1945) pp. 107–125; Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 114–116. Professor Schaeffer gave a brief summary of the papers.

After the lecture by Professor Schaeffer, Professor Einar Hille of Yale University, President of the American Mathematical Society, delivered the Presidential Address entitled *Lie theory of semi-groups of linear transformations*. Professor J. L. Walsh, President Elect, was the presiding officer.

The organized social events provided for the Society consisted of:

A reception and tea on Tuesday afternoon in the Faculty Club.

A recital on Wednesday evening by a duo-piano team and a vocalist from the School of Music.

A motion picture on Wednesday afternoon and a tea for the ladies on Thursday afternoon given by the Math Circle, a group consisting of ladies who are connected with the Department of Mathematics as staff members or wives of staff members.

A dinner on Thursday night in the Mach-Canfield dining hall. Professor R. E. Langer acted as toastmaster, and there were speeches by Vice-President Harlan Hatcher (The Ohio State University), Professor Saunders MacLane and Professor Marston Morse. Professor R. P. Dilworth read a resolution of thanks to The Ohio State University for their hospitality, highly commending the Local Committee on Arrangements, Professor R. G. Helsel, Chairman, for the obvious attention to detail which characterized the meeting.

At the meeting of the Board of Trustees at 6:00 P.M. on December 28, 1948, there was no quorum present and the Board adjourned to January 15, 1949.

The Council met at 9:30 P.M. on December 28.

The Secretary announced the election of the following fifty-two persons to ordinary membership in the Society:

Mr. Andrew Norwood Aheart, West Virginia State College;
Mr. Bernard Altshuler, New York University;
Dr. Frederick Dewey Bennett, Ballistics Research Laboratory, Aberdeen Proving Ground, Md.;
Professor Rubens Betelman, Curitiba, Parana, Brazil;
Professor Thomas Alton Bickerstaff, University of Mississippi;
Mr. Donald Watson Blackett, Princeton University;
Miss Mildred Ellen Blackman, St. Ambrose College, Davenport, Iowa;
Mr. Isaac Edward Block, Harvard University;
Mr. Eleazer Bromberg, Reeves Instrument Corporation, New York City;
Mr. Howard Henry Burt, General Fireproofing Company, Youngstown, Ohio;
Mr. Kenneth Vincent Casey, Brown University;
Brother Damian Connolly, LaSalle College, Philadelphia, Pa.;
Mr. Stanley Fifer, Reeves Instrument Corporation, New York City;
Mr. Donald Mandel Friedlen, Illinois Institute of Technology;
Mr. Richard Francis Gabriel, St. Francis College, Brooklyn, N. Y.;
Mr. Paul Guy Galentine, Jr., Willow Run Village, Michigan;
Mr. Irving M. Garfunkel, Aero Research Center, Willow Run Airport, Ypsilanti, Michigan;
Dr. George Alvin Garrett, Carbide and Carbon Chemicals Corporation, Oak Ridge, Tenn.;
Mrs. Kathe Solis-Cohen Jacoby, Philadelphia, Pa.;
Mr. Leonard Guy Jones, University of Oregon;
Mr. Kenneth Edward Kain, Belleville, Ill.;
Miss Jane F. Kiefer, Columbia University;
Professor Pedro Laborde-Montaner, New York University;
Professor Werner Walter Leutert, University of Maryland;
Mr. Julius Lieblein, Statistical Engineering Laboratory, National Bureau of Standards, Washington, D. C.;
Mr. Harold Adrian Linstone, Aerophysics Laboratory, North American Aviation, Inc., Downey, Calif.;
Professor Henry Lisman, Yeshiva University;
Professor Everett William McClane, DePaul University;
Mr. John A. McKee, Wilkes College, Wilkes Barre, Pa.;
Mr. Robert Colegrove Meacham, Brown University;
Mr. Paul Alfred Moser, Coalinga Junior College, Coalinga, Calif.;
Professor Sam Naiditch, Chemistry Dept., Duquesne University;
Professor Paul Nastucoff, University of Notre Dame;
Mr. William Joseph Nemerever, University of Michigan;
Professor Loyd Clinton Oleson, Doane College, Crete, Neb.;
Mr. Alexander Orden, Massachusetts Institute of Technology;
Professor Mary Pettus, Lander College, Greenwood, S. C.;
Mr. Constantinos George Plithides, Columbia University;

Mr. Murray L. Polk, Naval Shipyard, Brooklyn, N. Y.;
Dr. Joseph Ehrman Pryor, Harding College, Searcy, Ark.;
Mr. Edward Rayher, Bergen Junior College, Teaneck, N. J.;
Mr. Lionel Israel Rebhun, Brooklyn, N. Y.;
Mr. Paulo Ribenboim, Rio de Janeiro, Brazil;
Mr. Alfredo Milton Vila-Flor Santos, Salvador, Bahia, Brazil;
Professor Gobind Ram Seth, Iowa State College of Agriculture and Mechanic Arts;
Professor Donald Arthur Steele, Fordham University;
Miss Herta Taussig, Hollins College, Hollins College, Va.;
Mr. John C. Thompson, State Teachers College, Dickinson, N. D.;
Professor Billie Braden Townsend, Louisiana State University;
Dr. Horace Maynard Trent, Naval Research Laboratory, Washington, D. C.;
Mr. Daniel Hobson Wagner, Brown University;
Mr. Stanley Simon Walters, University of California at Los Angeles.

It was reported that the following one hundred twenty-seven persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Mr. Louis Jaffe, Mrs. Ayrlene McGahey Jones;
Brooklyn College: Mr. Melvin Hausner;
Brown University: Professor Erastus Henry Lee and Mr. Elliot Samuel Wolk;
California Institute of Technology: Messrs. Robert James Diamond and Thomas George Macfarlane;
University of California: Messrs. James Michael Gardner Fell, Steven Lyle Jamison, Ralph Mortimer Lakness, Frederick Burtis Thompson, Robert Lawson Vaught, Donald Dines Wall, and Philip Starr Wolfe;
University of California at Los Angeles: Messrs. Eugene Howard Jacobs and Douglas Houston Moore;
University of Chicago: Messrs. Robert Gardner Bartle, Nathan Joseph Divinsky, Henry Abel Dye, Richard Vincent Kadison, Henry William Oliver, Charles Morton Price, Isadore Manual Singer, Grayson Letcher Tucker, Jr., Louis Max Weiner, and Edgar Samuel Williams;
City College, New York City: Messrs. David Finkelstein, Donald J. Newman, Harold Seymour Shapiro, and Robert Angelo Spinelli;
Columbia University: Messrs. Calvin Creston Elgot, Cornelius Wilde Langley, Irwin Mann, John Rausen, Hsien Chang Tsang, Hermann Valentin Waldinger, Jerome Harris Weiner, and David Wellinger;
Cornell University: Messrs. Wallace Edward Barnes, Ernest Slavko Elyash, George Seifert, and Murray R. Spiegel;
Duke University: Messrs. James Richard Garrett and Joseph Andrew Silva;
Illinois Institute of Technology: Mr. Howard Lahman Arnould, Miss Maria Davidson, Messrs. Pasquale Porcelli and Albert Soglin;
University of Illinois: Messrs. Paul F. Conrad, William Alvin Howard, George Roger Livesay, Shu-Teh Chen Moy, David Arthur Page, and William G. Rosen, Miss Jean Elaine Sammet, Messrs. Judson Samderson, Jr., James Harold Turnock, Jr., and John Eldon Whitesitt;
Indiana University: Mr. James Burton Serrin, Jr.;
Institute for Advanced Study: Professor Lamberto Cesari, Messrs. John George Kemeny, Leopold Alexander Pars, and Kollagunta Gopalaiyer Ramanathan, Professors Kurt Werner Reidemeister, and Herbert K. J. Seifert;

Iowa State College of Agriculture and Mechanic Arts: Mr. Robert N. Goss;
State University of Iowa: Mr. William Henry Marlow;
Johns Hopkins University: Messrs. Peter Henry Berning, Charles Henry Murphy, Jr., and Malcolm William Oliphant, Miss Mary McCulloch Templeton, Mr. Kenneth G. Wolfson;
University of Kansas: Miss Margaret Marie Pihlblad;
Lehigh University: Messrs. Henry Albert Seebald and Michael Tikson;
University of Maryland: Messrs. Edward Crownshield Higgison and William Russell Thickstun, Jr.;
Massachusetts Institute of Technology: Messrs. Frederick Sheppard Holt, Ernest Ray Keown, and Barrett O'Neill;
Michigan State College: Mr. Charles Hall Kraft;
University of Michigan: Messrs. Charles Francis Briggs, Morton Landers Curtis, Howard Raiffa, Ralph Leland Shively and William Kay Smith;
University of Minnesota: Mr. Robert H. Scherer;
University of Missouri: Mr. William Lee Stamey;
Northwestern University: Mr. Donald Y. Barrer;
Ohio State University: Messrs. Ray Edward Kidder, Norman Levine, Robert Vernon Mendenhall, and William Mackie Myers, Jr.;
Oklahoma Agricultural and Mechanical College: Mr. Franklin Arno Graybill;
College of St. Thomas: Mr. Ephriam Joseph Vitoff;
Stanford University: Mr. Halsey Lawrence Royden, Jr.;
Swarthmore College: Mr. Robert Zane Norman;
Syracuse University: Mr. Robert S. Finn;
Texas Technological College: Professor Emmett A. Hazelwood;
University of Texas: Messrs. Billy Joe Ball, John Herbert Barrett, C. Edmund Burgess, and George Copp;
University of Toronto: Messrs. Thomas Edward Hull and John Albert Rottenberg;
University of Virginia: Mr. Henry Francis DeFrancesco;
University of Washington: Messrs. Yeh Mo and Ernest Carl Schlesinger;
Wellesley College: Miss Zung-nyi Loh;
Wesleyan University: Messrs. James Louis Dolby and Robert Thomas Mathews;
Williams College: Professor Chester Henry Gordon;
University of Wisconsin: Mr. Henry Arthur Friedman, Miss Rachel Sherman Hodges, Messrs. Wallace Eugene Johnson, Benjamin Evans Mitchell, Harold Theodore Slaby, Kennan Tayler Smith, and Daniel Sokolowsky, Miss Margaret Alice Waugy, Dr. Chia-shuen Yih, Messrs. Frederick Harris Young and Allen Douglas Ziebur;
Yale University: Messrs. Robert Roland Christian, Charles Whittlesey Curtis, Eugene (Victor) Schonkman and Alfred Burton Willcox.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: French Mathematical Society: Professor Ferran Sunyer Balaguer, University of Barcelona; Professor Ratip Berker, Université Technique d'Istanbul; Professor Wenceslas Jardetzky, University of Graz; London Mathematical Society: Professor Hugh Patrick Mulholland, American University, Beirut, Lebanon; Professor Thomas Gerald Room, University of Sydney;

Swiss Mathematical Society: Dr. Hugo Ribeiro, University of California; Professor Ernest Trost, Technicum Winterthur, Zurich; Unione Matematica Italiana: Dr. Iacopo Barsotti, Princeton University.

It was reported that the Council had voted by mail to accept the recommendation of the Committee on the Role of the Society in Mathematical Publication that the Society publish three volumes of the Transactions in 1949, each volume to consist of approximately 500 pages and the subscription price to be $6.00 list per volume (members of the Society to receive the usual twenty-five per cent discount). The Board of Trustees also voted to approve this recommendation.

The Council had also voted by mail to accept the invitation of the University of Pennsylvania to meet in Philadelphia on April 29–30, 1949, this meeting to be in place of that previously scheduled by the Council for New York City.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 3811, including 333 nominees of institutional members and 53 life members. There are also 105 institutional members. The total attendance at all meetings in 1948 was 2424; the number of papers read was 610; there were 16 hour addresses, 6 addresses and 11 papers at the Applied Mathematics Symposium, 1 Gibbs Lecture, 4 Colloquium Lectures, and 1 Retiring Presidential Address; the number of members attending at least one meeting was 1432.

The following appointments by President Einar Hille of representatives of the Society were reported: Professor J. N. Michie at inauguration of James P. Cornette as President of West Texas State College on October 2, 1948; Professor E. B. Allen at inauguration of Jess Harrison Davis as President of Thomas S. Clarkson Memorial College of Technology on October 8, 1948; Professor J. R. Kline at installation of Dwight David Eisenhower as President of Columbia University on October 12, 1948; Professor C. N. Moore at Seventy-Fifth Anniversary Celebration of The Ohio State University on October 14–15, 1948; Professor T. R. Hollcroft at inauguration of Alan Willard Brown as President of Hobart and William Smith Colleges on October 22–23, 1948; Professor J. B. Rosenbach at inauguration of William Granger Ryan as President of Seton Hill College on November 11, 1948.

The following additional appointments by the President were reported: Professors J. H. Roberts (Chairman), F. G. Dressel, Tomlinson Fort, W. M. Whyburn as Committee on Arrangements for meeting at Duke University on April 1–2, 1949; Professors G. W. Smith

(Chairman), Wealthy Babcock, Paul Eberhart, G. B. Price, R. G. Sanger, J. W. T. Youngs as Committee on Arrangements for meeting at University of Kansas on April 29–30, 1949; Professor R. V. Churchill (Chairman), Dean Walter Bartky, Professors E. L. Eriksen, G. E. Hay, D. L. Holl, J. W. T. Youngs as Committee on Arrangements for Third Annual Symposium in Applied Mathematics at University of Michigan in June, 1949; Professors J. R. Kline (Chairman), E. F. Beckenbach, G. T. Whyburn as a committee to nominate representatives of Society on Policy Committee for Mathematics; Professors C. C. MacDuffee (Chairman), A. A. Albert, Emil Artin as committee on award of Cole Prize in Algebra, to be awarded at 1949 Annual Meeting; Professors S. S. Cairns (Chairman), J. R. Kline, P. A. Smith as committee to study questions raised by Committee on Revision of By-Laws; Dr. E. R. Kolchin, Professors C. E. Sealander and Walter Strodt as tellers for 1948 annual election; Professor G. A. Hedlund as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for period 1949–1950 (committee now consists of Professors J. R. Kline, Chairman, G. A. Hedlund, and T. H. Hildebrandt); Professor D. V. Widder as a member of the Committee to Select Hour Speakers for Eastern Sectional Meetings for period 1949–1950 (committee now consists of Professors T. R. Hollcroft, Chairman, Deane Montgomery, and D. V. Widder); Professor Ralph Hull as a member of the Committee to Select Hour Speakers for Western Sectional Meetings for period 1949–1950 (committee now consists of Professors J. W. T. Youngs, Chairman, Marshall Hall, and Ralph Hull); Professor E. F. Beckenbach as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for period 1949–1950 and Professor D. C. Spencer as a member for 1949 (committee now consists of Professors J. W. Green, Chairman, E. F. Beckenbach, and D. C. Spencer); Professor J. M. Thomas as a member of Committee on Printing Contracts for period 1949–1951 (committee now consists of Professor C. J. Rees, Chairman, Dean M. H. Ingraham, and Professor J. M. Thomas); Professor Richard Courant as a member of Committee on Visiting Lectureship for period 1949–1951 (committee now consists of Professors Hassler Whitney, Chairman, Richard Courant, and Dean R. G. D. Richardson); Professors William Prager and Eric Reissner as members of Committee on Applied Mathematics for period 1949–1951 (committee now consists of Dean Walter Bartky, Chairman, Professors R. V. Churchill, G. C. Evans, John von Neumann, William Prager, and Eric Reissner).

It was reported that Professor Deane Montgomery of the Institute for Advanced Study had accepted the invitation to deliver a series of Colloquium Lectures at the first Summer Meeting following that of 1949 and that the title of the Lectures would be *The topology of groups.*

At the annual election which closed on December 29, and at which 801 votes were cast, the following officers were elected:

President, Professor J. L. Walsh.

Vice President, Professor W. T. Martin.

Secretary, Professor J. R. Kline.

Associate Secretary, Professor T. R. Hollcroft.

Treasurer, Dean A. E. Meder.

Member of the Editorial Committee of the Bulletin, Professor Deane Montgomery.

Member of the Editorial Committee of the Transactions, Professor Saunders MacLane.

Member of the Editorial Committee of the Colloquium Publications, Professor Einar Hille.

Member of the Editorial Committee of Mathematical Reviews, Professor Hassler Whitney.

Member of the Editorial Committee of Mathematical Surveys, Professor A. W. Tucker.

Representative on the Editorial Board of the American Journal of Mathematics, Professor L. M. Graves.

Members at large of the Council, Professors Emil Artin, Herbert Busemann, M. H. Heins, G. W. Mackey, and L. C. Young.

Members of the Board of Trustees, Professors T. H. Hildebrandt, W. R. Longley, Dean R. G. D. Richardson, Professors P. A. Smith and G. T. Whyburn.

The Council voted to approve June 14–16, 1949, as the dates for the Third Annual Symposium in Applied Mathematics to be held at the University of Michigan, the topic for the Symposium to be *Elasticity*. Dates of other meetings during 1949 were set as follows: August 30–September 2 at University of Colorado; October 29 in New York City; November 26 in Pasadena, California; December 27–29 in New York City.

The following resolution on the retirement of Professor B. P. Gill as Treasurer of the Society was adopted by the Council:

> It is with profound regret that the Council and the Board of Trustees have accepted the decision of Professor Bennington P. Gill to retire from the Treasurership of the American Mathematical Society on December 31, 1948. His careful study of modern accounting practices to discover the methods best adapted to the business of the Society has resulted in reports which have brought high praise from the auditors.

As Secretary of the Board of Trustees he has presented matters to be considered in a form which greatly facilitated the transaction of the business in hand. During the past ten years there has been a great increase in the labor involved in caring for the funds of the Society and in handling the business connected with its publications. For his efficiency and devotion in performing the tasks of his office Professor Gill deserves the appreciation and thanks of all members of the Society.

In an appendix to this report are excerpts from the report of the Treasurer for the fiscal year 1948 as verified by the auditors. A copy of the complete report will be sent, on request, to any member of the Society.

The American Journal of Mathematics, which is a joint enterprise of the American Mathematical Society and The Johns Hopkins University, reported that it had published 908 pages in 1948. The Society is also giving a subvention to the Canadian Journal of Mathematics, the first number of which was on display at the registration desk.

The Librarian reported the following additions to the Library: 122 volumes of periodicals, 68 books, 94 pamphlets (including 44 dissertations).

Certain invitations to give addresses in 1949 were announced: Professor Stefan Bergman for the February meeting in New York City; Professor J. H. Roberts for the April meeting at Duke University; Professors G. W. Mackey and Kurt Reidemeister for the April meeting in Philadelphia.

The Bulletin Editorial Committee reported the increasing backlog of manuscripts accepted for publication in the Bulletin which cannot be published promptly as was formerly the case. There was considerable discussion in the Council of the serious situation confronting the Society with respect to publication and the Council voted to authorize the appointment of a special committee to consider the emergency situation. Professors R. H. Bing and I. M. Sheffer were reported as new Assistant Editors for the Bulletin.

The Transactions Editorial Committee reported that a total of 1196 pages had been published in the 1948 volumes. Plans for the publication of three volumes in 1949 were reported: the first volume will consist of the January, March and May issues; the second volume will be an extra volume of 500 pages, to be bound in two parts; the third volume will be a two number volume. Professor Garrett Birkhoff was reported as a new Associate Editor and Professor G. T. Whyburn as Managing Editor.

The Mathematical Reviews Editorial Committee reported that the amount of material received had continued to increase, that 638 pages of reviews had been published in 1948 as against 616 for 1947 and 540 for 1946. During 1948 the Indian Mathematical Society was added to the list of sponsors of Mathematical Reviews. The subscription list, as of December 1, 1948, was 1,966.

Professors J. L. Doob, Norman Levinson, and R. L. Wilder were appointed as representatives of the Society on the Board of Editors of the Annals of Mathematics for three years beginning January 1, 1949.

The report of the Committee on Aid to Devastated Libraries, as presented to the Council, was published in the January issue of this Bulletin.

The Council approved a report from the Policy Committee for Mathematics which recommended that a Union Conference be called for the discussion of the formation of an International Mathematical Union, this Conference to be held just before the 1950 International Congress and to be attended by representatives of all leading mathematical groups.

Certain recommendations of the Committee on the Role of the Society in Mathematical Publication were referred to the various editorial committees for consideration.

The Council voted to invite Professor Norbert Wiener to deliver the Josiah Willard Gibbs Lecture at the 1949 Annual Meeting in New York City.

Professor Marston Morse was re-elected as a representative of the Society on the Policy Committee for Mathematics for a period of four years beginning January 1, 1949.

Abstracts of the papers read follow. Presiding officers at the sessions for contributed papers were Professors E. J. McShane, W. L. Ayres, Richard Brauer, Hassler Whitney, J. S. Frame, Saunders MacLane, R. V. Churchill, Herbert Federer, S. C. Kleene and Dr. R. P. Boas.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 99 was presented by Professor Blumenthal, 101 by Professor Brauer, 116 by Professor Mann, 131 by Professor Seidel, 133 by Mr. Calabi, 135 by Professor Martin, 136 by Professor Goffman, 146 by Professor Green, 173 by Mr. Bott, 202 by Professor Zorn. Papers numbered 196, 198, 201, and 202 were presented at the joint session with the association for symbolic logic.

ALGEBRA AND THEORY OF NUMBERS

99. L. M. Blumenthal and D. O. Ellis: *Notes on metric lattices.*

This paper is concerned with lattice formulations of metric properties of the metric space $D(L)$ obtained from a normed lattice L by defining dist. $(a, b) = |a \cup b| - |a \cap b|$.

Lattice characterizations are given of metric betweenness and pseudo linear quadruples. It is shown that $D(L)$ is congruently contained in a line if and only if it contains no pseudo linear quadruples. If f is a bi-uniform mapping of one normed lattice onto another, equivalences are established among the properties: f preserves (1) order, (2) meets and joins, (3) distances, (4) norms. Properties (1) and (2) are (obviously) equivalent, while if L has a first or last element, any two of the properties (2), (3), (4) implies the third. (Received November 15, 1948.)

100*t*. Bailey Brown and N. H. McCoy: *The maximal regular ideal of a ring.*

An element a of the ring R is *regular* if there exists an element x of R such that $axa=a$, and an ideal is regular if and only if each element of the ideal is regular. It is shown that an arbitrary ring R contains a unique maximal regular two-sided ideal $M(R)$. Various properties of $M(R)$ are established, among them that (i) $M(R/M(R))=0$; (ii) if $\mathfrak{b}$ is a two-sided ideal in R, $M(\mathfrak{b})=\mathfrak{b}\cap M(R)$; (iii) if R_n is the complete matrix ring of order n over R, then $M(R_n)=(M(R))_n$. The proofs are entirely elementary and hence, in particular, a simple proof is obtained that R_n is regular if and only if R is regular. This result has been proved by von Neumann for the case in which R has a unit element. If R satisfies the descending chain condition for right ideals, $R=M(R)+A$, where A is the annihilator of $M(R)$. This decomposition coincides with that of Marshall Hall (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 391–404). (Received November 15, 1948.)

101. A. T. Brauer and Nathaniel Macon: *On the approximation of irrational numbers by the convergents of their continued fractions.*

Let ξ be any positive irrational number, A_n and B_n the numerator and the denominator of the nth convergent in the expansion of ξ as a regular continued fraction. We set $|\xi-A_n/B_n|=\lambda_n^{-1}B_n^{-2}$. Hurwitz (Math. Ann. vol. 39 (1891)) proved that there exist infinitely many $\lambda_n>5^{1/2}$ for every ξ, Vahlen (J. Reine Angew. Math. vol. 115 (1895)) that $\max(\lambda_n, \lambda_{n+1})>2$, and Borel (J. Math. Pures Appl. (5) vol. 9 (1903)) that $\max(\lambda_n, \lambda_{n+1}, \lambda_{n+2})>5^{1/2}$ for every n. It is proved in this paper that of each five consecutive λ_n either at least two exceed $5^{1/2}$ or at least one of them exceeds 3. Similar results are obtained for any set of $3m+2$ consecutive λ_n for $m=2, 3, \cdots$. Moreover estimates for the sums and products of consecutive λ_n are obtained. In particular, it is shown that $\liminf\{(\lambda_1+\lambda_2+\cdots+\lambda_m)/m\}>2.0169$. (Received November 15, 1948.)

102*t*. Leonard Carlitz: *Finite sums and interpolation formulas over $GF[p^n, x]$.*

Sums and interpolation formulas connected with polynomials in $GF[p^n, x]$ are discussed. Among the applications may be mentioned (1) criteria for the vanishing of certain sums; (2) the theorem that a polynomial $g(t)$ of degree less than p^{nm} is integral-valued if and only if $g(M)$ is integral for all M of degree less than m. (Received October 22, 1948.)

103*t*. Leonard Carlitz: *q-Bernoulli numbers and polynomials.*

"Numbers" η_m are defined by means of the symbolic formula $(q\eta+1)^m=\eta_m$, $m>1$; $\eta_0=1$, $\eta_1=0$; polynomials $\eta_m(x)$ in q^x by means of $\eta_m(x+1)-\eta_m(x)=mq^x[x]^{m-1}$, $\eta_m(0)=\eta_m$. Next define $\beta_m=\eta_m+(q-1)\eta_{m+1}$ and $q^x\beta_m(x)=\eta_m(x)+(q-1)\eta_{m+1}(x)$,

$\beta_m(0)=\beta_m$. Various properties of the η's and β's are derived. The main result is a partial generalization of the Staudt-Clausen theorem: $\beta_m=\sum N_{m,k}(q)/F_k(q)$, where $F_k(q)$ denotes the cyclotomic polynomial and $N_{m,k}(q)$ is a polynomial with integral coefficients. (Received October 22, 1948.)

104*t*. S. D. Chowla: *On a certain exponential sum.*

Let $\rho=e^{2\pi i/p}$ where p is a prime $\equiv 3 \pmod 4$. Suppose that the $(p-1)/2$ numbers $x_1, \cdots, x_{(p-1)/2}$, incongruent (mod p), have the property that (A) $\left|\sum_j^{(p-1)/2} \rho^{x_j}\right|^2 =(p+1)/4$. From the theory of the Gaussian sum it is evident that (A) is true if the x's are the numbers ar_m+b $(1\leqq m\leqq (p-1)/2)$ where r's are the quadratic residues of p. If $p=7$ or 11 the converse is also true. We show that the converse is not true when $p=31$ by the following example. For the set of x's take 0, 1, 2, 3, 5, 6, 8, 11, 12, 18, 19, 20, 23, 27, and 29. It is easy to verify that (A) is satisfied. But there exist no integers a and b such that this set is a rearrangement (mod 31) of the numbers ar_m+b where $r_1, \cdots, r_{15}$ are quadratic residues of 31. (Received November 16, 1948.)

105*t*. S. D. Chowla: *On a theorem of Hecke and Siegel.*

Without using the theory of binary quadratic forms the author proves the following result due to Hecke and Siegel (see Acta Arithmetica vol. 1 (1936) pp. 83–86). Let $\chi(n)$ denote a real primitive character (mod k); if there exists an $a>0$ such that $L(s)=\sum_1^\infty \chi(n)\cdot n^{-s}\neq 0$ for $s\geqq 1-a/\log k$ $(k>1)$, then $L(1)>b/\log k$ where $b=b(a)>0$. Details of the proof will appear with the paper. (Received November 15, 1948.)

106*t*. S. D. Chowla: *On difference sets.*

Veblen (Amer. Math. Monthly vol. 13 (1906) p. 46) raised the problem of finding 7 different integers $d_1, d_2, \cdots, d_7$ such that the 42 differences d_i-d_j $(i\neq j)$ are incongruent (mod 43). Singer (Trans. Amer. Math. Soc. vol. 43 (1938) pp. 377–385) proved that if m is equal to a power of a prime, we can find $(m+1)$ integers $d_1, \cdots, d_{m+1}$ such that the m^2+m differences d_i-d_j $(i\neq j)$ are incongruent (mod m^2+m+1). His proof is based on ideas of finite projective geometry. It seems very likely that the following assertion (A) is true: If m is *not* a power of a prime, it is *impossible* to find $(m+1)$ integers d_i $(i\leqq m+1)$ such that the m^2+m differences d_i-d_j $(i\neq j)$ are incongruent (mod m^2+m+1). A recent announcement of Bruck and Ryser (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 649–650) implies the truth of (A) for infinitely many $m\equiv 1$ or 2 (mod 4). The author proves (A) for infinitely many $m\equiv 0$ or 3 (mod 4). As examples, the author proves (A) for $m=10$ (a case not covered by Bruck and Ryser) and when $m=159$. (Received November 15, 1948.)

107*t*. S. D. Chowla and K. G. Ramanathan: *On difference sets.*

Let $m\equiv 3$ (mod 4). The following problem is considered: What are the values of m for which there exist $(m-1)/2$ integers $d_1, d_2, \cdots, d_j$ $(j=(m-1)/2)$ such that the number of solutions of the congruence $d_i-d_j\equiv n$ (mod m) is independent of n? Such a set of d's is called a "difference set" (mod m). It is proved that: If $p\equiv 3$ (mod 4) is a prime and the set of numbers $d_1, d_2, \cdots, d_{(p-1)/2}$ is a difference set (mod p) and if there exists a k such that $d_i+d_j\equiv k$ (mod p) has no solution, then the d's are congruent (mod p) to the numbers ar_1+b, $ar_2+b, \cdots, ar_{(p-1)/2}+b$ in some order, where $r_1, \cdots, r_{(p-1)/2}$ are quadratic residues (mod p) and a and b are suitably chosen integers. (Received November 16, 1948.)

108*t*. S. D. Chowla and Paul Turán: *A congruence property of Ramanujan's τ-function.*

Ramanujan's function $\tau(n)$ is defined by $\sum_1^\infty \tau(n)x^n = x\prod_1^\infty (1-x^n)^{24}$ for $|x| < 1$. Sharpening known results, the authors prove that: For almost all $n \leqq x$, we have $\tau(n) \equiv 0 \pmod{691^t}$ where $t > (1/690)(1-\epsilon) \log \log x$, where ϵ is an arbitrary positive number. (Received November 17, 1948.)

109*t*. F. Marion Clarke: *On the factorization of polynomials in n variables.* I.

For $f(x_1, \cdots, x_n)$, a finite, separable polynomial in n variables over a coefficient field K of characteristic zero, (1) a theorem of van der Waerden on transcendental extensions and a theorem of E. Noether on relations between the coefficients of a factorable polynomial are used to prove that any nontrivial factorization of f which is possible will occur over a finite algebraic extension of K; (2) the Riemann genus theorem and a property of dimension in the definition of a general point of an algebraic variety are used to show that if f is completely factored into absolutely irreducible factors over an algebraic extension field L of K: $f(x) = p_1(x) \cdots p_r(x)$, if $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are prime ideals in $L[x_1, \cdots, x_n]$ generated by $p_i(x)$ and $p_j(x)$, $i \neq j$, and if $V\mathfrak{p}_i$ and $V\mathfrak{p}_j$ are the corresponding irreducible algebraic varieties $(i, j = 1, \cdots, r)$, then, in general, there exists no algebraic relation between the general points of $V\mathfrak{p}_i$ and the general points of $V\mathfrak{p}_j$. (Received November 16, 1948.)

110*t*. F. Marion Clarke: *On the factorization of polynomials in n variables.* II.

For a polynomial in n variables, $f(x_1, \cdots, x_n)$ over a coefficient field K of characteristic zero, a Σ_i-evaluation of f is defined and used to prove that when $n \geqq 3$ a necessary and sufficient condition for f to be irreducible over K is that there exist an infinite number of Σ_i-evaluations of f producing corresponding polynomials $\bar{f}(x_i, x_j)$ in the two variables x_i and x_j $(i \neq j)$, such that $\bar{f}$ is irreducible over K. It is shown that this property is stronger than is needed for the sufficiency condition, but that, according to a generalization of the Hilbert Irreducibility Theorem, it is minimal for the necessity condition. It is shown further that nonhomogeneous polynomials in two variables lend themselves to the methods of the theorem. (Received November 16, 1948.)

111*t*. Eckford Cohen: *An extension of Ramanujan's sum.*

This paper is concerned with the sum $c_q^s(n) = \sum_{(h,q^s)_s=1} e^{2n\pi hi/q^s}$ where q, s, n are positive integers and h ranges over non-negative integers $< q^s$ containing no sth power divisors in common with q^s other than 1. When $s = 1$ this sum reduces to Ramanujan's sum $c_q(n)$ and several properties corresponding to those of $c_q(n)$ are proved. An analogue of $c_q^s(n)$ in the polynomial ring $GF[p^n, x]$ is discussed and an application is made to the problem of finding the number of representations of a polynomial as a sum of products. A corresponding asymptotic result for the rational case is mentioned. (Received November 9, 1948.)

112*t*. Harvey Cohn: *On a disproof of one of Minkowski's conjectures.*

Minkowski's conjecture concerns the minimum constant c for which every lattice of unit density has a nontrivial point satisfying the inequality $|x|^p+|y|^p\leqq c$. The shape of the critical lattice is determined by the minimum parallelogram with one vertex at the origin and remaining vertices on the curve $|x|^p+|y|^p=1$. The conjecture was that the critical lattice would belong to the parallelogram with horizontal base if $1<p<2$ and to the one with a base at 45° if $p>2$. By using asymptotic expansions the author tests these two symmetrical parallelograms for relative maxima and minima, finding, for instance, that the symmetrical parallelograms do not always provide the relative maxima and minima. In so doing, the author verifies C. S. Davis' disproof of Minkowski's conjecture, and reveals complications in the behavior of the inscribed parallelogram viewed as a function of the slope of a side. (Received December 8, 1948.)

113. Melvin Dresher: *Continuous games with polynomial pay-off functions.*

Consider a two-person zero-sum game, in which each player chooses a real number between zero and one. Let $M(x, y)$ be the pay-off to the first player if he chooses x and the second player chooses y and let $F(x)$ and $G(y)$ be the cumulative distribution functions of the frequencies with which the players choose these numbers. Then $F^*(x)$, $G^*(y)$ is said to be a *solution* of the game if $\max_F \int_0^1\int_0^1 M(x, y)dF(x)dG^*(y) = \min_G \int_0^1\int_0^1 M(x, y)dF^*(x)dG(y) = \int_0^1\int_0^1 M(x, y)dF^*(x)dG^*(y)$. If $M(x, y) = \sum_{j=0}^{n}\sum_{i=0}^{m} a_{ij}x^iy^j$, and if neither $\int_0^1 M(x, y)dG^*(y)$ nor $\int_0^1 M(x, y)dF^*(x)$ is identically constant, then every solution $F^*(x)$, $G^*(y)$ is a pair of step-functions having at most $[m/2]+1$ and $[n/2]+1$ jumps respectively. The jumps in $F^*(x)$ occur at the maximum points of $\int_0^1 M(x, y)dG^*(y)$. If both $\int_0^1 M(x, y)dG^*(y)$ and $\int_0^1 M(x, y)dF^*(x)$ are constant then the solution of the game is in terms of m moments of $F^*(x)$ and n moments of $G^*(y)$. If $\int_0^1 M(x, y)dG^*(y)$ is constant and $\int_0^1 M(x, y)dF^*(x)$ is not, then the solution consists of $[m/2]+1$ jumps in $F^*(x)$ and n moments of $G^*(y)$. These results provide a method of computing all solutions of games with polynomial pay-off functions. (Received November 15, 1948.)

114. N. J. Fine: *Basic hypergeometric series and arithmetic applications.* Preliminary report.

Let $F(\alpha, \beta; t:x) = \sum_{n=0}^{\infty}([n;\alpha]/[n;\beta])t^n$, where $[0;z]=1$, $[n;z]=(1-xz)(1-x^2z)\cdots(1-x^nz)$ for $n>0$. By comparatively elementary methods it is shown that each of $F(\alpha x, \beta; t:x)$, $F(\alpha, \beta x; t:x)$, $F(\alpha, \beta; tx:x)$ can be expressed in terms of $F(\alpha, \beta; t:x)$. These functional equations lead to other transformations of F, such as $(1-t)F(\alpha, \beta; t:x)=(1-t')f(\alpha', \beta'; t':x)$, where $\alpha'=\alpha t/\beta$, $\beta'=t$, $t'=\beta$. Other transformations lead, on suitable specialization of the parameters, to classical identities of Euler, Jacobi, Gauss, and to results on the "false" ϑ-functions of Rogers, the "mock" ϑ-functions of Ramanujan, and the classical ϑ-functions. A number of these results are believed to be new, and some have simple arithmetic interpretations. For example, define the *rank* of a partition as the excess of the largest part over the number of parts. Let $D_r(n)$ be the number of partitions of n into *distinct* parts, the rank of each partition being r; let $U_{2r+1}(n)$ be the number of partitions of n into *odd* parts, the largest being $2r+1$. Then (*) $U_{2r+1}(n)=D_{2r+1}(n)+D_{2r}(n)$. Euler's well known theorem follows by summing (*) for all $r\geqq 0$. Also, let $L(n)$ be the number of partitions of n into distinct parts the least of which is odd. Then $L(n)$ is odd if and only if n is a square. (Received November 16, 1948.)

115. R. H. Fox: *Free differential calculus and the Jacobian matrices of a group.*

Consider the integral group ring P of a free group F and the endomorphism $u \to u^0$ of P which maps F into the identity 1. To each generator x_j of F there is a linear mapping $u \to \partial u/\partial x_j$ of P into itself which is determined uniquely by the conditions (1) $\partial(uv)/\partial x_j = (\partial u/\partial x_j) \cdot v^0 + u \cdot (\partial v/\partial x_j)$, (2) $\partial x_k/\partial x_j = \delta_{jk}$. The formula $u = u^0 + \sum_j (\partial u/\partial x_j) \cdot (x_j - 1)$ holds, so that u is determined up to an additive constant by its partial derivatives $\partial u/\partial x_j$. Let G be a group determined by a set $x_1, x_2, \cdots$ of generators and a set $u_1 = 1, u_2 = 1, \cdots$ of defining relations and denote by ϕ the natural homomorphism of F upon G (and also the induced homomorphism of P upon the integral group ring of G). The matrix $\|(\partial u_i/\partial x_j)^\phi\|$ is called a *Jacobian* of the group G. The Jacobian matrices of G all belong to a single equivalence class. From this group invariant the various previously known invariants may be derived by specialization. The differentiation concept may be generalized in several ways and there are various applications of this calculus, particularly to free groups. (Received November 18, 1948.)

116. Marshall Hall and H. B. Mann: *On a canonical form of the Euclidean algorithm.*

The Euclidean algorithm is said to be valid in a commutative ring R if there exists on R a function $E(a)$ with the following properties. (1) $E(a)$ is an element of a well ordered set M. (2) $E(a)$ is the first element of M if and only if $a = 0$. (3) To every a, b in R, $b \neq 0$, there exist elements $g, r \subset R$ such that $a = bg + r$, $E(0) \leqq E(r) < E(b)$. The algorithmic chain is a chain of subsets S_a of elements of R defined as follows. S_0 consists of the 0 of R. S_a consists of all $x \subset R$ such that every residue class mod x contains at least one element in S_b with $b < a$. The following theorem is proved. The Euclidean algorithm is valid in R if and only if the algorithmic chain contains all elements of R. The function $A(x) = a$ if $x \subset S_a$, $x \not\subset S_b$ with $b < a$, is called the canonical algorithm if it is defined for every x. The function $A(x)$ satisfies the inequality $A(xy) \geqq A(x)$ if $xy \neq 0$, $A(xy) > A(x)$ if $xy \neq 0$, $A(y) > 1$. Every element in S_a decomposes into at most $(a-1)$ prime factors. (Received November 10, 1948.)

117*t*. Frank Harary: *On atomic Boolean-like rings.*

A zero ring is a ring of characteristic two in which the product of any two elements is zero. Any two zero rings of the same cardinal power are isomorphic. Every zero ring N has a base B such that each element of N can be uniquely written as the sum of a finite number of elements of B. For brevity, let "BLR" denote "Boolean-like ring." An atomic BLR is a BLR whose Boolean subring is atomic. A nilpotent element η of an atomic BLR, H, is called nil if $b\eta = 0$, where b is any atom of the Boolean subring J of H. Similarly η is called simple if for some atom b of J, $b\eta = \eta$. Let H be an atomic BLR; N its nilpotent ideal; B a base for N, then H is called simple if each element of B is either nil or simple. Our principal theorem is that every atomic BLR is isomorphic to a simple one. The proof is by transfinite induction. (Received November 17, 1948.)

118*t*. Frank Harary: *On n-dimensional projective algebras.* Preliminary report.

n-dimensional projective algebras (PA's) are defined by a suitable modification of the Everett-Ulam postulates. The elementary theorems on n-dimensional PA's analogous to those for ordinary (that is, 2-dimensional) PA's follow at once. It is shown that an n-dimensional PA can be synthesized from any non-atomless Boolean algebra B. The isomorphism types of finite n-dimensional PA's are completely determined. (Received November 17, 1948.)

119. Frank Harary: *The structure of finite projective algebras.* Preliminary report.

The Everett-Ulam postulates for projective algebra (PA) immediately imply a duality theorem relating x and y projections. It is shown that a PA can be synthesized from any nonatomless Boolean algebra B, with a fixed atom p, by defining (1) $0_x=0_y=0$, (2) $a_x=p$ and $a_y=a$, when $a\in B$, $a\neq 0$, (3) $p\square a=a$ for all $a\in B$, (4) $a\square b=0$ if a or b is 0. The isomorphism types of finite PA's are completely determined. The number of isomorphism types of PA's of 2^m elements is equal to the number of distinct factorizations of m into 2 factors. (Received November 17, 1948.)

120*t*. G. B. Huff: *Tentative methods in diophantine equations.*

If a diophantine equation is equivalent to one of the form $w_1w_2=f(w_3, w_4, \cdots, w_n)$, where f is a polynomial with integer coefficients, it is said to be solvable by factoring. It is shown that this idea provides a complete theory for the system $z_0^2-z_1^2-z_2^2-\cdots-z_m^2=b$, $z_0-z_1-z_2-\cdots-z_m=a$. Applications are given, particularly to problems in algebraic geometry. (Received November 15, 1948.)

121. L. C. Hutchinson: *Notes on the number of leaves of an alternating tensor.*

It has long been known that the smallest number $m(r)$ of leaves (simple tensors) in terms of which a bivector, or alternating tensor of valence 2 (2 indices) and rank r can be expressed is $r/2$ (r is always even in this case). For alternating tensors of higher valence the classification is more complicated, and $m(r)$ becomes a multiple-valued function of r. Some bounds for $m(r)$ are given, with particular attention to the trivector (valence 3). The results for the trivector differ from those given by R. Weitzenböck (K. Akademie van Wetenschappen, Proceedings vol. 32 (1929) pp. 248–250, vol. 40 (1937) pp. 312–315; Monatshefte für Mathematik und Physik vol. 48 (1939) pp. 129–140). The best lower and upper bounds so far determined (in the real) are $r/3$ and $(r-5)(r+3)/4$ ($r>7$), respectively. (Up to $r=8$ the exact values are known.) If complex transformations are allowed, this result can be strengthened. (Received November 17, 1948.)

122. B. W. Jones: *On m-universal quadratic forms.*

If A is the nonsingular matrix of an n-ary form f in a field F and if for every m-rowed nonsingular symmetric matrix B in F there is a matrix T in F and of rank m such that $T^TAT=B$, we call f an "m-universal form." The form f is shown to be m-universal in the field $F(p)$ of p-adic numbers if $n>m+2$ and conditions are found for the cases $n=m+2$ and $n=m+1$. From these conditions follow, by Hasse's results, criteria that a form be m-universal in the field of rational numbers. In the field of real numbers the form f is m-universal if and only if there is a matrix T of rank m such that T^TAT is the zero matrix of order m. (Received November 13, 1948.)

123*t*. Irving Kaplansky: *Elementary divisors and modules.*

Conditions are found on a ring R in order that matrices over R (both finite and infinite) admit reduction to diagonal form. This is applied to the decomposition of an R-module into a direct sum of cyclic modules. The uniqueness of the decomposition is studied. (Received October 29, 1948.)

124. J. C. C. McKinsey: *Isomorphism of games and strategic equivalence.*

Two zero-sum n-person games, v and v', are called *S-equivalent* if there exists a positive number k, and n numbers $a_1, \cdots, a_n$, whose sum is zero, such that, for every subset T of the set of players, $v'(T)=k\cdot v(T)+\sum_{i\in T}a_i$. Games are called *isomorphic* if there exists a one-to-one correspondence between the sets of imputations which, for every subset T of the set of players, preserves dominance with respect to T. It is shown that games are S-equivalent if and only if they are isomorphic. The notion of strategic equivalence is an intuitive one, but it is sufficiently sharp that intuitive arguments can be presented to show that isomorphism is a necessary condition for strategic equivalence, and that S-equivalence is a sufficient condition. From the result mentioned above it then follows that games are strategically equivalent (in the intuitive sense) if and only if they are S-equivalent. (Received November 15, 1948.)

125. Gordon Pall: *Sums of two squares in a quadratic field.*

It is shown that the number of representations of a binary quadratic form as a sum $(a_1x+b_1y)^2+(a_2x+b_2y)^2$ of squares of two linear forms with integral coefficients is (when the obviously necessary conditions of having a square determinant, integral coefficients, and being non-negative are satisfied) equal to the number of representations of the divisor of the form as the sum of two squares of integers, or equal to double this number, according as the determinant is or is not zero. As an application, it is shown that simple formulae can be obtained for the number of representations of an integer in a quadratic field as the sum of squares of two integers in that field. The number of such representations is finite for all real quadratic fields, but infinite (or zero) for all imaginary quadratic fields, with the single exception of the ring of Gauss integers $a+bi$ $(i^2=-1)$ for which the number of representations of a nonzero number is finite. As an example, the number of integral solutions of $a+2ki=(a_1+b_1i)^2+(a_2+b_2i)^2$ is given by $4\epsilon_\alpha\prod(1+\beta_i)\prod(1+\gamma_j)(1+\gamma_j+\delta_j)$, where $a^2+4k^2=2^\alpha q_1^{2\beta_1}\cdots q_r^{2\beta_r}p_1^{2\gamma_1+\delta_1}\cdots p_s^{2\gamma_s+\delta_s}$; here $\alpha\geqq 0$; $q_1, \cdots, q_r$ denote distinct primes of the form $4n+3$, $r\geqq 0$; $p_1, \cdots, p_s$ denote distinct primes of the form $4n+1$, $s\geqq 0$, and $p_j^{\gamma_j}$ is the power of p_j dividing the g.c.d. (a, k); and $\epsilon_0=1$, $\epsilon_\alpha=|\alpha-3|$ if $\alpha\geqq 2$. (Received December 1, 1948.)

126*t*. Hugo Ribeiro: *A remark on "uniform" lattices.*

A finite modular lattice is called u-uniform if and only if each convex complemented sublattice of dimension 2 has $u+3$ elements. In an author's paper to appear in Comment. Math. Helv. the following facts are proved: Disregarding non-Desarguesian projective geometries, Boolean algebras and lattices of all subgroups of elementary prime-power abelian groups are the only complemented u-uniform lattices for which u is an indecomposable integer (1 in the first case, a prime in the second case). p-uniformity (p prime) together with other simple conditions such as the cyclic character of all join-irreducible elements characterize the lattice of all subgroups of the primary p-component of a finite abelian group. It is now a consequence of the above facts that: Each p-uniform lattice for which the following conditions are satisfied is an auto-dual lattice: (1) if $x \neq x \cup y \neq y$ then for each z there are u and v with $u \neq u \cup v = x \cup y \cup z \neq v$ (join-irreducible elements are cycles), (2) the number of elements following an element immediately is at most the number of atoms, (3) all maximal cycles are of the same dimension. (Received November 19, 1948.)

127*t*. C. E. Rickart: *The uniqueness of norm problem in Banach algebras.*

A B-algebra is said to possess a unique norm provided any two norms under which it is a B-algebra are necessarily equivalent. It is known that the norm is unique in a semi-simple, commutative B-algebra (Gelfand, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 1–23) and in the B-algebra of all bounded operators on a B-space (Eidelheit, Studia Mathematica vol. 9 (1940) pp. 97–105). It is shown here that the same is true for any B-algebra which possesses a family of minimal left ideals such that $xL = (0)$ for every L in the family implies $x = 0$. This includes the case of a primitive B-algebra with minimal ideals and hence the Eidelheit result. The norm is also unique in any B-algebra with an adjoint operation x^* which is either essential in the Raikov sense (C. R. (Doklady) Acad. Sci. URSS N.S. vol. 54 (1946) pp. 387–390) or satisfies a condition $\|x\|^2 \leqq k\|x^*x\|$. This includes the case of a B-algebra of bounded operators on Hilbert space where x^* is the operator adjoint but the norm need not be the bound. (Received November 10, 1948.)

128*t*. C. E. Rickart: *Homomorphisms of one Banach algebra into a second.*

Let $h(x)$ denote an arbitrary homomorphism of one B-algebra B *into* a second B-algebra A. It is shown that $h(x)$ is automatically continuous in each of the following cases: (1) $h(x)$ is *onto* and A is a semi-simple B-algebra whose norm is uniquely determined up to equivalence, (2) A is semi-simple and commutative, (3) A possesses an adjoint operation x^*, which is either essential in the sense of Raikov (C. R. (Doklady) Acad. Sci. URSS N.S. vol. 54 (1946) pp. 387–390) or satisfies the condition $\|x\|^2 \leqq k\|x^*x\|$, provided $h(B)^* = h(B)$. (Received November 10, 1948.)

ANALYSIS

129. Nachman Aronszajn: *Hilbert spaces and conformal mappings.*

A formula for the projection on the sum of two closed linear subspaces of a Hilbert space in terms of projections on the subspaces is applied to the computation of conformal mappings of simply and multiply connected domains. The link between the Hilbert space theory and the conformal mapping is found in the theory of reproducing kernels, especially Bergman's kernels which are the reproducing kernels of classes of analytic functions (of one or more complex variables) regular and square integrable in a domain. (Received November 18, 1948.)

130*t*. Frederick Bagemihl: *A theorem on infinite products of transfinite cardinal numbers.*

Let α be a transfinite limit-number, and let its normal form be $\sum_{1 \leqq i \leqq n} \omega^{\delta_i} = \alpha$, where n is a natural number and $\delta_1 \geqq \delta_2 \geqq \cdots \geqq \delta_n > 0$. Let $\{\sigma_\xi\}_{\xi<\alpha}$ be an increasing sequence of ordinals such that $\lim_{\xi<\alpha} \sigma_\xi = \lambda$, and subject, if $n > 1$, to the following condition: if $\beta=\sum_{1\leqq k\leqq n-1}\omega^{\delta_k}$, and $\lim_{\xi<\beta}\sigma_\xi=\eta$, then $-\overline{\eta+\lambda}\leqq\overline{\omega^{\delta_n}}$. Then $\prod_{\xi<\alpha}\aleph_{\sigma_\xi}=\aleph_\lambda^{\overline{\alpha}}$. In particular, if α is a transfinite limit-number, and μ is an arbitrary ordinal number, then $\prod_{\xi<\alpha}\aleph_{\mu+\xi}=\aleph_{\mu+\alpha}^{\overline{\alpha}}$. The method of proof is essentially one due to Tarski (Fund. Math. vol. 7 (1925) pp. 1–14). (Received November 16, 1948.)

131. E. F. Beckenbach, Wladimir Seidel, and Otto Szasz: *Recurrent determinants of orthogonal polynomials.*

A study is made of recurrent determinants $D_n=[c_{\nu+\mu}]$ and $E_n=[c_{\nu+\mu+1}]$, ν, $\mu=0, 1, 2, \cdots, n-1$; n a positive integer, where $c_m=Q_m(x)$ is a polynomial of degree m of one of the following classes: Legendre, Hermite, Laguerre, and ultraspherical, suitably normalized. Explicit formulas for the determinants are obtained in a variety of cases. It is shown, for instance, that for Legendre polynomials $D_n\equiv[P_{\nu+\mu}(x)]=2^{-(n-1)^2}(x^2-1)^{(n-1)n/2}$ and $E_n\equiv[P_{\nu+\mu+1}(x)]=[(x^2-1)/4]^{(n-1)n/2}\cdot[(x+1)^n+(x-1)^n]/2$. Furthermore, all minors of D_n are positive for $x>1$. This is applied to the Stieltjes and Hamburger moment problems. (Received November 15, 1948.)

132. H. D. Block: *Explicit solution of certain singular integral equations.*

The equation $f(x)=\lambda\int_a^\infty K(|x-y|)g(y)dy$ is solved by a simple method for the unknown function $g(y)$ in certain special cases. $K(|x-y|)$ as a function of x is assumed to satisfy the same linear homogeneous differential equation in both the regions $a\leqq x\leqq y$; $y\leqq x$. The method of solution consists in constructing a linear differential equation with constant coefficient which $g(y)$ must satisfy. Necessary and sufficient conditions are then found for the cases when $K(|x-y|)=e^{-k|x-y|}$; and $K(|x-y|)=|x-y|e^{-k|x-y|}$; for $-\infty\leqq a<\infty$. The solution of the equation of the second kind $f(x)=g(x)-\lambda\int_a^\infty K(|x-y|)\,g(y)dy$ is also included for these cases. For the homogeneous equation, $g(x)=\lambda\int_a^\infty K(|x-y|)g(y)dy$, the complete solutions are found, and the characteristic values of λ are shown to form a continuous spectrum. (Received November 9, 1948.)

133. E. Calabi and Aryeh Dvoretzky: *Convergence and sum factors for series of complex numbers.*

Let Z be a given bounded set, and denote by $\tilde{Z}$ the closure of the convex set spanned by Z. Theorem 1: $0\in\tilde{Z}$ is the necessary and sufficient condition that given any null sequence $(a_n)_{n=1}^\infty$ of complex numbers, there exist $\zeta_n\in Z$ so that (I) $\sum_{n=1}^\infty\zeta_n a_n$ converges. Theorem 2: If, in addition, $\sum_{n=1}^\infty|a_n|=\infty$ the necessary and sufficient condition that $\zeta_n\in Z$ may be chosen so that (I) converges to any preassigned limit is that 0 be an interior point of $\tilde{Z}$. In this case one can also choose $\zeta_n\in Z$ so that the derived set of $\sum_{n=1}^N\zeta_n a_n$ is any preassigned closed, connected set. The case $Z=\{1, -1\}$ of Theorem 1 was proved by A. Dvoretzky and C. Hanani (C. R. Acad. Sci. Paris vol. 225 (1947) p. 516), while in the special case where Z consists of the kth ($k\geqq 3$) roots of unity Theorem 2 was established by H. Hornich (Monatshefte für Mathematik und Physik vol. 45 (1937–1938) p. 432, vol. 46 (1938–1939) p. 266). Generalizations to the case when ζ_n is to be chosen from a varying Z_n are also given. Applications to sequences of functions are included. The methods and results carry over to sequences of vectors in E^r and more general spaces. (Received November 13, 1948.)

134t. J. W. Calkin: *Spaces with inner products defined in terms of generalized limits.*

If $\mathfrak{H}$ is a Hilbert space, La_n a generalized (Banach) limit defined for all bounded sequences $\{a_n\}$ of complex numbers, we denote by $\mathfrak{H}_L$ the space of all sequences weakly convergent to zero with $(\{f_n\}, \{g_n\}) = L(f_n, g_n)$, sequences differing by a strongly convergent sequence being regarded as equal. This possibly incomplete Euclidean space, whose dimension number is c, was used in a previous paper (Ann. of Math. vol. 42 (1941) pp. 839–873) to study the ring of bounded operators on $\mathfrak{H}$ reduced modulo the totally continuous ones. If S denotes the Stone-Čech compactification of the integers I and $a(p)$ is the continuous function which $\{a_n\}$ induces on S, then for $p \in S-I$, $a(p) = La_n$ defines a generalized limit. If L is any generalized limit, then $La_n = \int_s a(p)d\mu$ where μ is a uniquely determined regular countably additive measure defined on Borel sets in S with $\mu(S) = 1$, $\mu(I) = 0$ (Nakamura and Kakutani, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 224–229). It is shown here the $\mathfrak{H}_L$ is complete if μ assumes only a finite number of values, but not otherwise. (Received October 21, 1948.)

135. R. H. Cameron and W. T. Martin: *Nonlinear integral equations.*

This paper deals with the integral equation (1) $y(t) = x(t) + \int_0^t F(t, s, x(s))ds$ on the interval I: $0 \leqq t \leqq 1$ and the corresponding equation (2) with fixed upper limit unity. Both of these equations are considered as special cases of the functional equation (3) $y(t) = x(t) + \Lambda(x|t)$, where the functional Λ satisfies certain smoothness conditions and (3) is a 1–1 transformation of the space C of continuous functions vanishing at the origin into itself. It is shown that under suitable restrictions on F or Λ the equations (1), (2), (3) have a solution of the form $x(t) = \text{L. I. M.}_{n\to\infty} \sum \Psi_m(y)A_m(t)$ where m represents an n-fold index $m_1, \cdots, m_n$, each component of which goes from 0 to n in the sum. If certain additional conditions are satisfied by F or Λ the expression for $x(t)$ can be replaced by a modified sum which is pointwise summable in the appropriate infinite-dimensional Abel sense. Thus absolute results as well as statistical results are obtained. (Part of the work done on this paper by one of the authors (R. H. C.) was carried on at the Institute for Numerical Analysis with the financial support of the Office of Naval Research of the Navy Department.) (Received October 25, 1948.)

136. L. W. Cohen and Casper Goffman: *Note on non-archimedean metrization.*

A non-archimedean metric is a function $\delta(x, y)$ on pairs of a space S to a Hahn field $\mathcal{F}$ such that (1) $\delta(x, y) \geqq 0$, $\delta(x, y) = 0$ if and only if $x = y$; (2) $\delta(x, y) = \delta(y, x)$; (3) $\delta(x, z) \leqq \delta(x, y) + \delta(y, z)$. Conditions are given under which a space S is homeomorphic to a non-archimedean metric space. These include conditions of the sort given previously by the authors (*A theory of transfinite convergence*, to appear in Trans. Amer. Math. Soc.) and the additional restriction that neighborhoods be closed sets. It follows that certain topological groups which do not satisfy the first denumerability axiom of Hausdorff may be metrized in the sense considered here. (Received November 18, 1948.)

137. V. F. Cowling: *Summability and analytic continuation.*

A new set of matrix methods of summability is introduced. The elements of the matrix (a_{nm}) are defined, for each real number r satisfying the inequality $0 \leqq r < 1$, as $a_{nm} = 0$, $m < n$; $a_{nm} = (1-r)^{n+1}(C_{m,n})r^{m-n}$, $m \geqq n$. It is shown that each matrix of this set determines a regular summability method. A sequence or series which is summable by one of these matrices will be said to be summable F_r (the real number r depending on the matrix employed). The question of consistency of the various methods F_r among themselves is discussed. A general theorem is obtained as to the regions in which these methods will sum a power series. We illustrate this theorem in the case of the series $\sum z^n$. The series $\sum z^n$ is summable F_r for each real number r satisfying the inequality $0 \leq r < 1/2$ to sum $1/(1-z)$ in any closed region contained in the region common to $|z+r/(1-2r)| < (1-r)/(1-2r)$ and $|z| < 1/r$. It is summable $F_{1/2}$ to sum $1/(1-z)$ in any closed region contained in the region common to $|z| < 2$ and $R(z) < 1$. Finally it is summable F_r for each real number r satisfying the inequality $1/2 < r < 1$ to sum $1/(1-z)$ in any closed region contained in the region common to $|z+r/(1-2r)| > (1-r)/(2r-1)$ and $|z| < 1/r$. (Received November 12, 1948.)

138*t*. R. J. Duffin: *Representation of Fourier integrals as sums*. II.

Let the function $\phi(x)$ have a derivative which has bounded variation in the interval $(1, \infty)$ and which has the limit zero at infinity. Let $\Phi(x) = \phi(1/x)/x$ have the same property. Under these hypotheses the following conclusions are drawn: For positive x a continuous function $g(x)$ is defined by the series $g(k/x)/x = \phi(x) - \phi(3x) + \phi(5x) - \cdots$. Here $k = (\pi/2)^{1/2}$. Likewise, a continuous function $G(x)$ is defined by the series $G(k/x)/x = \Phi(x) - \Phi(3x) + \Phi(5x) - \cdots$. These series are summable by the Cesaro mean. The functions $g(x)$ and $G(x)$ are a pair of Fourier sine transforms; that is, $kG(x) = \int_0^\infty \sin xtg(t)dt$ for positive x. The integral is summable by the Cesaro mean. Any function $\phi(z)$ which is analytic and uniformly bounded in the sector $-\epsilon < \arg z < \epsilon$ satisfies the conditions of this theorem. (Received November 16, 1948.)

139*t*. R. J. Duffin and A. C. Schaeffer: *Functions whose Fourier Stieltjes coefficients approach zero.*

Let $f(z) = \int_{-\pi}^{\pi} e^{izt} d\rho(t)$, where $\rho(t)$ is a function of bounded variation. Let k_n, $n = 1, 2, \cdots$, be a bounded sequence of real numbers, and suppose $f(n+ik_n) \to 0$ as $n \to \infty$. It is shown in this note that a necessary and sufficient condition that $f(x) \to 0$ as $x \to +\infty$ is that $\rho(t)$ be continuous at $t = \pi$. This is closely related to certain results of Rajchman (Math. Ann. vol. 101 (1929) pp. 686–700), and Verblunsky (Proc. London Math. Soc. vol. 34 (1932) pp. 526–560). Their theorems concern "n" rather than "$n+ik_n$." The theorem is proved by a simple application of a complex variable technique developed previously by the writers (Amer. J. Math. vol. 67 (1945) pp. 141–154). (Received November 18, 1948.)

140*t*. M. M. E. Eichler: *On the differential equation* $u_{xx} + u_{yy} + N(x)u = 0$.

This paper is closely connected with S. Bergman's recent work on differential equations. General solutions can be written as series $u = \sum_0^\infty p_\nu(x) f_\nu(z)$ where $f_\nu(z)$ are analytic functions of $z = x+iy$. Two such series developments are discussed: the "ascending series," where $f_\nu(z) = df_{\nu+1}(z)/dz$, and the "descending series," where $f_{\nu+1}(z) = df_\nu(z)/dz$. The ascending series can be written as an integral operator $u = f_0(z) - \int_{z_0}^{z} G(x, z-\zeta) f_0(\zeta) d\zeta$ working on $f_0(z)$. Specially distinguished operators of this type give insight into the analytic behavior of series $\sum q_\lambda(x) \exp(i\lambda y)$ of particular solutions, obtained by the method of separation of variables, outside of their domain of convergence. Examples of differential equations, important in the applications, are given where explicit expressions for the "generating solutions" G are known. Two methods to calculate G explicitly in other cases are explained. The convergence of the ascending and descending series is investigated, particularly in the neighborhood of a singular point of N where N is of the form x^{-1} times a power series in x^{ρ} $(\rho>0)$. A connection between both series is briefly touched in a special case. At last a class of differential equations in an arbitrary number of variables is mentioned to which methods and results of this paper can largely be extended. (Received October 22, 1948.)

141. H. Margaret Elliott: *On line and surface integral approximations to harmonic and analytic functions satisfying an integrated continuity condition.*

Results on approximation to harmonic and analytic functions are established when both the degree of approximation and the continuity properties of the function are measured by (1) a line (cf. E. S. Quade, Duke Math. J. vol. 3, pp. 529–543 for the trigonometric case) and (2) a surface integral. Let C be an analytic Jordan curve. Let $f(s)\in L^p$, $p\geqq 1$, s arc-length measured along C. The integral modulus of continuity $\omega_L^p(\delta)$ of $f(s)$ on C is defined as $\max_{0\leqq h\leqq\delta}\left[\int_0^{\sigma}|f(s+h)-f(s)|^p ds\right]^{1/p}$, where σ is the length of C. It is proved, for example, that if $f(z)$ is analytic interior to C and if $f^{(k)}(z)\in L^p$, $p\geqq 1$, on C and has integral modulus of continuity $\omega_L^p(\delta)$ on C, then for each n, $n=1, 2, \cdots$, there exists a polynomial $\pi_n(z)$ in z of degree n such that $\left[\int_C|f(z)-\pi_n(z)|^p ds\right]^{1/p}\leqq M\omega_L^p(1/n)/n^k$. Conversely if there exist polynomials $\pi_{n_j}(z)$ of respective degree n_j, $j=1, 2, \cdots$, such that $\left[\int_C|f(z)-\pi_{n_j}(z)|^p ds\right]^{1/p}\leqq\Omega(n_j)/n_j^k$, $p\geqq 1$, $k\geqq 1$, it is shown under suitable restrictions on $\Omega(x)$ and the sequence n_j that $f(z)$ is equal almost everywhere on C to a function $f_1(z)$ which is analytic interior to C; furthermore $f_1^{(k)}(z)$ has integral modulus of continuity on $C\omega_L^p(\delta)\leqq L\left[\delta\int_{n_1}^{1/\delta}\Omega(x)dx+\int_{1/\delta}^{\infty}(\Omega(x)/x)dx\right]$, $0<\delta\leqq 1/n_1$. Analogous results are obtained for the surface integral. (Received November 15, 1948.)

142*t*. Evelyn Frank: *Convergence of C-fractions.* Preliminary report.

Parabolic and other convergence regions of C-fractions $1+a_1z^{\alpha_1}/1+a_2z^{\alpha_2}/1+\cdots$ are obtained. Value regions for these continued fractions are also shown. (Received October 21, 1948.)

143. W. H. J. Fuchs: *On the growth of functions of mean type.*

$\{\lambda_n\}$ is a sequence of positive numbers, $\lambda_{n+1}-\lambda_n>c>0$. N. Levinson and others have discussed under which conditions on $\{\lambda_n\}$, $\limsup_{n\to\infty}\lambda_n^{-1}\log|f(\lambda_n)|=\limsup_{x\to\infty}x^{-1}\log|f(x)|$ (1) for every function $f(z)$ regular in $x\geqq 0$ and satisfying $|f(z)|=O(e^{\pi L|z|})$ in this half-plane. A refinement of Levinson's method shows that the following conditions are necessary and sufficient to ensure (1). (i) $\phi(r)=\sum_{\lambda_n<r}\lambda_n^{-1}-L\log r\to\infty$. (ii) For $Y>X\geqq N(\epsilon)$, $\phi(Y)-\phi(X)>-\epsilon$. (Received November 11, 1948.)

144. A. M. Gleason: *Subalgebras of a measure algebra.*

A measure algebra is a σ-Boolean algebra on which there can be defined a finite, non-negative, completely additive measure function μ, such that $\mu(a)=0$ only if a is the null element of M. Maharam (Proc. Nat. Acad. Sci. U. S. A. vol. 28 (1942) pp. 108–111) has characterized the structure of measure algebras. Using a similar technique, the author characterizes the structure of pairs consisting of a measure algebra M and a subalgebra L. As an example of such a pair consider the algebra M of all measurable sets, modulo null sets, of the direct product of two measure spaces X and Y, and let L consist of those sets of the form $E\times Y$ where E is a measurable subset of X. A more recondite example can be derived from the preceding by considering a principal ideal aM of M as a measure algebra and the set aL as a subalgebra. It is shown that the latter example is, indeed, the most general. (Received November 12, 1948.)

145. A. W. Goodman: *The Schwarz-Christoffel transformation and p-valent functions.* Preliminary report.

Subclasses $S(p)$ and $C(p)$ of the class of functions p-valent in the unit circle are introduced which are generalizations of the classes of univalent functions which map the unit circle on starlike and convex regions respectively. By using a general form of the Schwarz-Christoffel transformation which maps the unit circle onto a p-sheeted domain bounded by straight line segments, it is possible to prove sharp theorems for functions of class $S(p)$ or $C(p)$. In particular bounds are obtained for the coefficients and for $|f(z)|$. If $f(z)=a_1z+a_2z^2+a_3z^3+\cdots$ is in $S(2)$ and has all coefficients real, then $|a_3|\leqq 5|a_1|+4|a_2|$. This is a special case of an earlier conjecture. As a by-product of the methods two more proofs that $S(2)=\pi^2/6$ are given. (Received November 9, 1948.)

146. J. W. Green and William Gustin: *Quasiconvex sets.*

Let I be the closed unit real number interval. Any subset Δ of I containing at least one number interior to I is called a quasiconvexity generating set. A set Q in a finite-dimensional real vector space is Δ convex if for every real number α of Δ and every two points a and b lying in Q the point $\alpha a+\beta b$ $(\alpha+\beta=1)$ also lies in Q. The graph of a solution of the functional equation $\phi(x+y)=\phi(x)+\phi(y)$ is an example of a quasiconvex set. Such graphs are known to possess many interesting measure and topological properties. These known properties and other new properties as well follow from more general results on quasiconvex sets. Let Q be a quasiconvex set; and let P be the set of those points not in Q which are interior to the convex hull of Q. Results: Q is dense in its convex hull. If Q has positive outer measure, P has zero inner measure. If Q has positive inner measure, P is null. If Q contains a nonplanar continuum, P is null. If P contains a set which together with a plane bounds a bounded open set, then Q has topological dimension zero. (Received November 11, 1948.)

147. R. G. Helsel: *Convergence in area and convergence in volume.*

Using an enclosed volume based on topological considerations, it is shown that if a sequence of closed surfaces converges in the Fréchet sense to a closed surface which occupies a point set of three-dimensional measure zero, then convergence of the Lebesgue areas implies convergence of the enclosed volumes. This result is the extension to surfaces of a theorem which Radó (Fund. Math. vol. 27 (1936) pp. 212–225) established for plane curves. The methods used are analogous to those utilized by Radó. (Received November 10, 1948.)

148t. M. A. Hyman: *On partial differential equations of mixed type.* Preliminary report.

The equation (I) $y^{\lambda}\phi_{xx}+\phi_{yy}=0$ $(\lambda>0)$ may be transformed for $y>0$ into the Beltrami-Weinstein equation (II) $\phi_{\xi\xi}+\phi_{\eta\eta}+\rho\eta^{-1}\phi_{\eta}=0$ $(\rho>0)$ which has been investigated recently (A. Weinstein, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 342–354). A similar transformation into (II) is possible for the equation (III) $y\phi_{xx}+\phi_{yy}+\alpha y^{-1}\phi_{y}=0$ $(\alpha>-1/2)$. F. Tricomi (R. Academia Nazionale dei Lincei, Memorie (5) vol. 14 (1923)) has discussed equation (I) in detail when $\lambda=1$ (or $\rho=1/3$, $\alpha=0$). The present author considers the more general equations (I) and (III). Among the results obtained is a one-to-one correspondence between the family of solutions of (II) analytic across $\eta=0$ and a certain family of solutions of (I) analytic across $y=0$ (here λ is an integer); corresponding members of the two families transform into each other. A similar correspondence has been found between solutions of (II) and of (III) analytic across the respective singular lines of these equations. (Received November 19, 1948.)

149*t*. J. A. Jenkins: *Some problems in conformal mapping.*

As an extension of the concept of extremal length, introduced by Ahlfors and Beurling, modules are defined for the pentagon, hexagon and triply-connected domain. These modules are functions of several real variables and represent conformal invariants of the configurations. Various applications are made, in particular necessary and sufficient conditions are given that one triply-connected domain may be mapped conformally into another so as to have the same topological situation. These conditions relate only to the metric properties of the domains. (Received October 5, 1948.)

150*t*. R. E. Lane and H. S. Wall: *Continued fractions whose even and odd parts converge absolutely.*

A continued fraction with sequence of approximants $f_0, f_1, f_2, \cdots$ is said to converge absolutely if the series $f_0+\sum(f_p-f_{p-1})$ converges absolutely. It is shown that if the even and odd parts of $f=K_{p=0}^{\infty}(c_p/1)$ $(c_0=1)$ converge absolutely, then f converges if, and only if, the series $\sum|b_p|$ diverges, where $b_1=1$, $c_p=1/b_pb_{p+1}$, $p=1, 2, 3, \cdots$. The hypothesis holds, in particular, if there exist positive numbers r_p such that $r_1|1+c_1|>|c_1|$, $r_2|1+c_1+c_2|>|c_2|$, $r_p|1+c_{p-1}+c_p|\geqq r_pr_{p-2}|c_{p-1}|+|c_p|$, $p=3, 4, 5, \cdots$. Hence if these "fundamental inequalities" (Scott and Wall, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 155–172) are satisfied, f converges if, and only if, $\sum|b_p|$ diverges. (Received November 1, 1948.)

151. R. E. Langer: *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point.*

If in an equation (1) $d^2u/dx^2+[\lambda^2q_0(x)+\lambda q_1(x)+\sum_{\nu=0}^{\infty} q_{\nu+2}(x)\lambda^{-\nu}]u=0$, with $a\leqq x\leqq b$, and λ a large parameter, the coefficient $q_0(x)$ is bounded from zero, familiar procedures are available for the derivation of asymptotic solutions. If $q_0(x)$ vanishes at a point x_0, this is called a turning-point. For an interval (a, b), which contains a turning-point the asymptotic representation of solutions of (1) in terms of elementary functions involves the Stokes' phenomenon. This latter is avoidable if the restriction to elementary functions is dropped. The forms which have heretofore been given in such terms are, however, complicated to the extent that only their leading terms are directly available. This paper develops a procedure by which the asymptotic solutions are deducible by an easy formalism, and become available to the terms in any degree of $1/\lambda$. The method is applicable whenever the turning-point is a simple zero of $q_0(x)$, and requires no conditions upon $q_1(x)$. (Received November 13, 1948.)

152*t*. Joseph Lehner: *Generalized partial fraction decompositions of functions analytic in the unit circle.*

Let $G(x)=\sum_1^\infty a_n x^n$ be regular for $|x|<1$. The author proves that a necessary and sufficient condition that $G(x)$ have a generalized partial fraction decomposition, (1) $G(x)=\sum_{k=1}^\infty b_k \sum_{l=1}^\infty \alpha_l(1-x\exp(-i\lambda_k))^{-l}$, is that (2) $a_n=g(n)f(n)$, where $g(w)$ is a polynomial in the complex variable w, or an entire function of order at most one and minimum type; $f(n)$ be a uniformly almost periodic sequence, (3) $f(n)\sim\sum_{k=1}^\infty b_k \exp(-i\lambda_k n)$, $n=1, 2, 3, \cdots$; (4) the series in (3) converge absolutely. If these conditions are fulfilled, the series in (1) converges absolutely and uniformly in every closed region of the plane which does not intersect M, the closure of the set $\{\exp i\lambda_k\}$, $k=1, 2, \cdots$; it defines an analytic function singular on M and regular on each connected piece of the complement of M. If M is the entire circle $|x|=1$, $G(x)$ has a natural boundary, whose decomposition is given by the series (1). Explicit formulae are given for the coefficients α_l. The author treats also the case in which the series in (3) is not absolutely convergent. The function $G(x)$ is still regular in the same region as before; on M it is again singular, the type of singularity at $\exp(i\lambda_k)$ being described by the asymptotic value of G under radial approach. This paper extends results of Rademacher, *A convergent series for the partition function $p(n)$*, Proc. Nat. Acad. Sci. U. S. A. (1937), and of Bochner and Bohnenblust, *Analytic functions with almost periodic coefficients*, Ann. of Math. (1934). (Received November 15, 1948.)

153*t*. G. W. Mackey: *On a theorem of Stone and von Neumann.*

Let G be a separable locally compact group. Let R be the $*$ algebra of all Borel functions on G. Let $s\rightarrow U_s$ be a representation of G by unitary operators in a Hilbert space H. Let $g\rightarrow V_g$ be a representation of R by bounded linear operators in the same Hilbert space H. The principal theorem of this paper asserts that if these representations are suitably continuous and are so related that for all $s\in G$ and all $g\in R$, $U_sV_g=V_hU_s$ where $h(x)=g(xs)$, then they are simultaneously unitary equivalent to a direct sum of replicas of the "regular representations" obtained from the Hilbert space of functions on G, square summable with respect to right invariant Haar measure, by mapping $s\in G$ into the operator $f(x)\rightarrow f(xs)$ and $g\in R$ into the operator $f(x)\rightarrow g(x)f(x)$. This theorem combined with a known result on the unitary representations of locally compact Abelian groups implies a corresponding theorem in which G is Abelian and R is replaced by the character group of G. If G is taken to be the n-dimensional vector group this last theorem reduces to a theorem of Stone and von Neumann asserting the essential uniqueness of operators satisfying the Heisenberg commutation relations. (Received November 16, 1948.)

154*t*. G. W. Mackey: *On semi-direct products of locally compact Abelian groups.* I. Preliminary report.

Let G be a separable locally compact group. If G admits closed Abelian subgroups G_1 and G_2 such that G_1 is normal, $G_1\cap G_2=e$ (the identity) and $G_1\cdot G_2=G$ then G is said to be a semi-direct product of the locally compact Abelian groups G_1 and G_2. This paper is the first of a projected series in which it is proposed to study the properties of such semi-direct products in detail. In it the unitary representations of these groups are studied. All finite-dimensional irreducible unitary representations are explicitly determined. For the members of a certain class of semi-direct products (including the "$ax+b$ group" recently studied by Gelfand and Neumark) the infinite irreducible unitary representations are also determined. It is shown by several types of examples that this last determination does not apply to all semi-direct products. (Received November 16, 1948.)

155. G. R. MacLane: *Rational functions with all zeros and poles on a rectifiable Jordan curve.*

The following theorem is proved: let $f(z)$ be holomorphic and never zero in a simply-connected domain D of the z-plane bounded by a rectifiable Jordan curve C. There exists a sequence of rational functions $R_n(z)$, each of which has all of its zeros and all of its poles on C, and such that $\lim_{n\to\infty} R_n(z)=f(z)$ for z in D, uniformly in any closed subset of D. The condition on $R_n(z)$ is for the complete plane; that is, $R_n(z)$ is regular and doesn't vanish at $z=\infty$. (Received November 5, 1948.)

156. Szolem Mandelbrojt: *A converse theorem on approximation on the real axis.*

The author has proved a theorem giving conditions in order that, $F(x)>0$ and $\{K_n\}$ (positive integers) given, to each $\epsilon>0$ there correspond a polynomial $P(x)$ of the type $P(x)=a_0+a_1x^{K_1}+ \cdots +a_nx^{K_n}$ such that $|f(x)-P(x)|<\epsilon F(x)$ $(-\infty<x<\infty)$. This is a generalization of a theorem of S. Bernstein. The author proves now that the above-mentioned theorem cannot be, in a certain sense, much improved. (Received November 15, 1948.)

157*t*. F. I. Mautner: *On maximal Abelian subalgebras of operator algebras.* I.

The maximal comutative self-adjoint subalgebras of weakly closed self-adjoint operator algebras W in a Hilbert space H are of decisive importance for the decomposition and structure of W. In the present paper the following theorem is proved for separable Hilbert spaces H: Let C be any maximal self-adjoint commutative subalgebra of W and P any maximal self-adjoint commutative subalgebra of W', where W' consists of all those bounded operators in H which commute with every element of W. Assertion: The weakly closed self-adjoint operator algebra generated by C and P is a maximal self-adjoint commutative subalgebra of the algebra of all bounded operators in H. A similar property has been formulated by W. Ambrose in a Princeton seminar for slightly different algebraic systems, namely those arising from the L_2-systems of groups. This proof uses generalized decomposability of H into irreducible spaces and also von Neumann's generalized decomposability of W into factors. (Received November 12, 1948.)

158. C. N. Moore: *On the functional behavior of a function defined by a certain Dirichlet's series.* II.

The study initiated in a paper presented to the Society at the meeting in New York in October, 1948, is here continued. It is proved that the function $L^{(c)}(s)$, defined in the previous paper, has the asymptotic properties needed for its application to the problem of prime pairs. (Received November 16, 1948.)

159. O. M. Nikodým: *On the spectrum of measurable functions and on equimeasurable transformations.*

By a "soma" (Carathéodory's term) belonging to a measurable set a is understood the class of all measurable sets, b, which are almost equal to a, that is, meas. $(b-a)$ plus meas. $(a-b)=0$. We shall consider Boolean fields (measure algebras) whose elements are somata. To a given finite measurable function $f(x)$ defined on a set I there corresponds a Boolean field (T), the smallest one containing all the somata belonging to $[E(1)]$ where $[E(1)]$ is the set of all the x for which $f(x)$ does not exceed 1. (T) will be termed "spectral field of $f(x)$." Theorems on subfields of (T) in connection with functions $G(f(x))$ where $G(1)$ is a borelian function will be presented. The main question to be discussed is: whether $f(x)$ can be carried by an almost one-to-one and equimeasurable transformation into a monotonous function. Conclusive answer is given in the form of a general theorem. The proof is based on a deep lemma on Boolean fields connected with the known Maharam's Theorem, and on J. von Neumann's theorem on equimeasurable transformations. There are some relations to the multiplicity of the spectrum of Hermitian transformations in Hilbert space. (Received December 13, 1948.)

160. H. C. Parrish: *Note on certain decompositions of point sets.*

This paper concerns an elementary instance of "*methodical generalization*," that is, generalization by means of generally known cues. An immediate generalization of a well known theorem, namely, that a linear set is the disjoint sum of a dense-in-itself and a scattered set, leads to the problem of the genesis of an unconditionally additive set property α, that is, one such that the sum of any number of sets of property α has property α. Every such α yields an analogue of the particular case. A complete genesis of unconditionally additive set properties is obtained in terms of ascending set-point properties. (A set-point property $\beta=\beta(E, x)$, that is, a property of a set E in relation to a point x, is ascending if its validity for (E, x) implies its validity for (E_1, x) when $E_1\supset E$.) It is then shown that ascending set-point properties may be generated from "ascending set-interval properties," and the latter, in turn, from "ascending set properties"—the meaning of the latter terms being manifest. The results unify the literature analogues of the particular theorem, and add new cases. Each particular case gives a geometric property of unconditioned sets and of unconditioned functions. (Received December 9, 1948.)

161. M. O. Reade: *On p-analytic functions.*

Let $f(z)$ be defined in a bounded, simply-connected domain $\mathfrak{D}$. Kriszten has defined $f(z)$ to be p-analytic in $\mathfrak{D}$ if and only if $A^pf(z)=0$ holds in $\mathfrak{D}$; here p is a fixed positive integer and A is the complex differential operator $\partial/\partial x+i\partial/\partial y=A$ (Comment. Math. Helv. vol. 21 (1948) p. 73). In this paper, integral characterizations for p-analytic functions in general, and for certain p-analytic polynomials in particular, are derived; these results extend earlier results due to Féderoff (Rec. Math. (Mat. Sbornik) vol. 41 (1934) p. 92), Haskell (Bull. Amer. Math. Soc. vol. 52 (1946) p. 332), and Reade (Bull. Amer. Math. Soc. vol. 53 (1947) p. 98). The following results are typical ones. (I) Let $f(z)$ be continuous in $\mathfrak{D}$, let $C(z_0, r)$, for sufficiently small r, denote the circle in $\mathfrak{D}$ center at z_0 and radius r, and let $p\geqq 0$. A necessary and sufficient condition that $f(z)$ be $(p+2)$-analytic in $\mathfrak{D}$ is that $F_r(z)=\int_{C(z,r)}\zeta^pf(z+\zeta)d\zeta$ be 1-analytic for all sufficiently small r, in the subdomain $\mathfrak{D}_r$ where $F_r(z)$ is defined. (II) Let

Π_n denote a regular n-gon, and let n be a fixed positive integer, $n \geqq 2p+2$. A necessary and sufficient condition that (i) $f(z)$ be $(p+1)$-analytic in $\mathcal{D}$ and (ii) $\bar{A}^{(n-p-1)}f(z)=0$, where $\bar{A} \equiv \partial/\partial x - i\partial/\partial y$, is that $\int_{\Pi_n} \zeta^p f(\zeta)d\zeta = 0$ for all Π_n in $\mathcal{D}$. (Received November 16, 1948.)

162*t*. M. O. Reade: *Sufficient conditions that $f(z)$ be analytic.*

Let $D_1, D_2, \cdots, D_n, \cdots$ be a sequence of simply-connected domains, each bounded by a simple, closed, rectifiable curve, such that (i) $\iint_{D_n} z\,dxdy = \iint_{D_n} z^2 dxdy = 0$, $n=1, 2, 3, \cdots$, (ii) for each $\epsilon > 0$ there exists $m = m(\epsilon)$ such that D_n is contained in $|z| \leqq \epsilon$, for $n > m$. Now if $f(z)$ is continuous in a bounded, simply-connected domain $\mathcal{D}$, then a necessary and sufficient condition that $f(z)$ be analytic in $\mathcal{D}$ is that $\lim_{n\to\infty} [1/|D_n|^2 \iint_{D_n} \bar{z} f(z_0+z)dxdy] = 0$ for each z_0 in $\mathcal{D}$; here $|D_n|$ denotes the area of D_n. This generalizes a theorem due to Fédoroff (Rec. Math. (Mat. Sbornik) vol. 41 (1934) p. 92). (Received November 16, 1948.)

163. L. V. Robinson: *A new operational method for solving partial differential equations.* Preliminary report.

This method makes use of an operator, $\partial/\partial x + P(x, y)\partial/\partial y$, and its inverse, to solve linear partial differential equations, of the first and second orders, and some nonlinear equations. (Received November 16, 1948.)

164. P. C. Rosenbloom: *On the iteration of entire functions.*

By a fixed point of a function f we mean a root of the equation $f(z) = z$. If f is an entire function and if $f(f(z))$ has only a finite number of fixed points, then f is a polynomial. More generally, if f and $f(g(z))$ have only a finite number of fixed points and if f is a transcendental entire function, then $g \equiv z$ or $g \equiv$ constant. If f_n, the nth iterate of f, has only a finite number of fixed points, and $n > 1$, then f is a polynomial. The proofs are based on Nevanlinna's second fundamental inequality, on an inequality of Bohr in a precise form, and on Robinson's numerical estimate in Schottky's theorem. (Received December 2, 1948.)

165. Robert Schatten: "*Closing-up*" *of sequence spaces.*

Let $\mathfrak{L}$ denote the linear set of sequences of real numbers $(a_1, a_2, \cdots)$ having only a finite number of nonzero terms. A symmetric norm on $\mathfrak{L}$ is a function $\phi(a_1, a_2, \cdots)$ subject to conditions: (i) $\phi(a_1, a_2, \cdots) > 0$ unless $a_1 = a_2 = \cdots = 0$, (ii) $\phi(\alpha a_1, \alpha a_2, \cdots) = |\alpha| \phi(a_1, a_2, \cdots)$ for any constant α, (iii) $\phi(a_1+b_1, a_2+b_2, \cdots) \leqq \phi(a_1, a_2, \cdots) + \phi(b_1, b_2, \cdots)$, (iv) $\phi(a_1\ a_2, \cdots) = \phi(\epsilon_1 a_{1'}, \epsilon_2 a_{2'}, \cdots)$ where $\epsilon_i = \pm 1$ and $1', 2', \cdots$ denotes any permutation of $1, 2, \cdots$. It follows that $\rho(a_1, \cdots, a_n, 0, 0, \cdots)$ is a nondecreasing function of n. Consequently, $\phi(a_1, a_2\ \cdots)$ determines in the usual fashion a conjugate norm $\psi(a_1, a_2, \cdots)$. Thus, we obtain two normed linear spaces $\mathfrak{L}(\phi)$ and $\mathfrak{L}(\psi)$. We define $[[\mathfrak{L}(\phi)]]$ as the Banach space of all infinite sequences $(a_1, a_2, \cdots)$ of which an infinite number of a_i may be $\neq 0$, for which $\lim_{n\to\infty} \phi(a_1, \cdots, a_n, 0, 0, \cdots) < +\infty$, while $[\mathfrak{L}(\phi)]$ is defined as the Banach space of all fundamental sequences of elements in $\mathfrak{L}(\phi)$. The space $[\mathfrak{L}(\phi)]$ is quite often a *proper subset* of $[[\mathfrak{L}(\phi)]]$, as for instance, in the case $\phi(a_1, a_2, \cdots) = \max a_i$. The conjugate space of $[\mathfrak{L}(\phi)]$ may be characterized as $[[\mathfrak{L}(\psi)]]$. (Received November 10, 1948.)

166. H. M. Schwartz: *Notes on the differential equation $y''(x) - h(x, \lambda)y(x) = 0$. I.* Preliminary report.

The present note is concerned with the following question relating to the differential equation $y''(x) - h(x, \lambda) = 0$ defined in the x-interval $(0, \infty)$: If $h(x, \lambda)$ is for $x > 0$ a regular analytic function of λ in a given domain D of the complex λ-plane, but is not finite at $x = 0$, what conditions on h will insure the regularity in the domain D of the solution of the differential equation for which $y(0, \lambda) = 0$, $y'(0, \lambda) = 1$? By the use of standard methods a number of such conditions are established of which the following is typical: If the coefficient h is such that the solution in question exists, then a sufficient condition for the required result is the existence of the integral $\int_0^a (x \cdot h)^2 dx$ and its boundedness uniformly with respect to λ in D, for every fixed $a > 0$. (Received November 17, 1948.)

167. Seymour Sherman: *On a problem of Raikov.*

Raikov (*Harmonic analysis on commutative groups with the Haar measure and the theory of characters*, Trav. Inst. Math. Stekloff vol. 14 (1945) 86 pp.), for the purpose of harmonic analysis, presents a postulational development of Haar-Raikov measure on topological groups. The domain of this measure function is the σ-ring of all strict Borel sets. It is not obvious that continuous functions are measurable with respect to this σ-ring. Raikov raises the question of extending this invariant measure to all Borel sets. This is done here and in answer to another of Raikov's questions it is shown that the relation between the extended measure on the group and the extended measure on its completion is the natural one. It is also shown that the σ-ring of σ-bounded strict Borel sets is the same as the σ-ring of Baire sets. A fundamental tool used in settling Raikov's first question is the recently announced result of Halmos: Haar measure is completion regular. (Received November 4, 1948.)

168. G. E. Springer: *Distortion theorems in pseudoconformal mapping.*

In order to obtain new domains of comparison needed for the evaluation of bounds in pseudoconformal (p.c.) mapping, the p.c. properties of several special domains in the space of two complex variables are considered; namely, $E_\lambda: |z_1|^2 < e^{-\lambda|z_2|^2}$, $\lambda > 0$, and $C_A: |z_1|^2 < 1 - |z_2|^2 - A|z_2|^4$, $A > 0$. Complete systems of orthonormal functions $\{\phi_\nu(Z)\}$, $Z = (z_1, z_2)$, are determined for E_λ and C_A and are used to define the kernel function $K(Z, \bar{Z}) = \sum_\nu \phi_\nu(Z)\overline{\phi_\nu(Z)}$ and the p.c. invariant $I(Z) = (1/K(Z, \bar{Z})) \cdot (T_{1\bar{1}}T_{2\bar{2}} - |T_{1\bar{2}}|^2)$, where $T_{m\bar{n}} = \partial^2 \log K(Z, \bar{Z})/\partial z_m \partial \bar{z}_n$ (Bergman, Mémorial des Sci. Math. vol. 106 (1947)). 1. In any p.c. transformation of E onto itself each shell $|z_1|^2 e^{\lambda|z_2|^2} = \text{constant}$ is mapped into itself. 2. If E is mapped by a schlicht p.c. transformation onto a domain D such that $E_{\lambda_1} \subset D \subset E_{\lambda_2}$, the image W^0 of any point Z^0 of E_λ lies between certain shells $|z_1|^2 e^{\mu|z_2|^2} = c$, where μ and c depend only upon λ, λ_1, λ_2 and Z^0. 3. Each value of A determines a different p.c. class of which C_A is a representative. 4. In any map of C_A onto itself, the origin is a fixed point. 5. If C_A is mapped onto a domain B such that $C_{A_1} \subset B \subset C_{A_2}$, the image of the origin lies in a hypersphere with center at the origin and radius depending only upon A, A_1 and A_2. (Received November 12, 1948.)

169. Otto Szasz: *On some trigonometric transforms.*

In previous papers [see Ann. of Math. vols. 43 and 46; Trans. Amer. Math. Soc. vol.

[redacted] netric transforms
paper contains
nce transform $A_n = \sum_{\nu=1}^{n} U_\nu \sin \nu\theta/\nu\theta$
detailed discussion of this transform was stim
ing paper of C. Lanczos who came to consider the same transform
point of view. Furthermore, related transforms are discussed. The main
cern the comparison of the author's transform with the Cesàro scale. (Received November 13, 1948.)

170. H. P. Thielman: *On a class of singular integral equations occurring in physics.*

The Wiener-Hopf integral equation is solved by elementary means for the case when the kernel $K(x, y)$ is a function of the absolute value of $x-y$ only, and satisfies the same linear differential equation with constant coefficients in each of the regions $0 \leq y \leq x$, $0 \leq x \leq y$. (Received November 15, 1948.)

171. J. L. Ullman: *A simplified proof of Jentzsch's theorem.* Preliminary report.

Let $g(z) = \sum_0^\infty a_n z^{-n}$, $a_0 = 1$, $\limsup |a_n|^{1/n} = 1$, $g_k(z) = \sum_0^k a_n z^{-n} = \prod_1^k (1 - z_{ki} z^{-1})$, then Jentzsch's theorem states that if $|z_0| = 1$, z_0 is a limit point of the set $\{z_{ki}\}$, $i \leq k$, $k = 1, 2, \cdots$. The proof of the case where $g(z) \neq 0$ for $|z| > 1$ illustrates the method. The Cauchy-Hadamard formula applied to the expansion of $g(w)/(w-z)$ in powers of $1/w$ yields $\limsup |z^n g_n(z)|^{1/n} = 1$ if $|z| \leq 1$. A contradiction arises by assuming that a point of unit modulus, say 1 without loss of generality, is not a limit point of $\{z_{ki}\}$. It follows by Hurwitz' theorem that Re $(z_{ni}) < \rho < 1$ for some $\rho < 1$ and $n > N_\rho$. But then it follows for these values of n that $|b - z_{ni}| > \eta |a - z_{ni}|$ for some $\eta > 1$ and $\rho < a < b < 1$. Hence the contradiction $\limsup |b^n g_n(b)|^{1/n} > \limsup |a^n g_n(a)|^{1/n}$. The same method applied to the generating function of Faber polynomials leads to new results concerning the asymptotic distribution of the zeros of these polynomials. (Received November 12, 1948.)

APPLIED MATHEMATICS

172t. Milton Abramowitz: *Asymptotic solutions of the spheroidal wave equation.*

Solutions of the spheroidal wave equation $(1-z^2)w'' - 2(m+1)zw' + (b - c^2z^2)w = 0$ are obtained for the cases (1) m large and c small (2) m large and c large. In the first instance, the solutions are expressed in terms of Hermite polynomials while in the second they are expressed in terms of parabolic cylinder functions. In both cases approximations are obtained for the separation constants b and some numerical results are cited to illustrate the formulas. (Received October 13, 1948.)

173. R. Bott and R. J. Duffin: *On the algebra of networks.* Preliminary report.

An electrical or mechanical system is considered having a quadratic energy law $E = \sum g_s v_s^2 = \sum \sum A_{rs} q_r q_s$. The v_s are constrained coordinates, linear functions of unconstrained coordinates q_r. It is shown that $D = \det A_{rs}$ is a homogeneous multilinear form with positive coefficients as a function of the parameters g_s. For example, if the

ctivities of the w
e products. If $\chi = \log$
g_i. This suggests that χ be termed the
ve real part, so also do all χ_i. The second deriv
of rational functions R_{ij}. The transfer impedance matrix R_{ij} has
esting properties. The quantities derived from the impedance potential are invariants not depending on the choice of the coordinate system q_r. Conversely, it is shown that the logarithm of any homogeneous multilinear function is the impedance potential of some system if its second derivatives are negative perfect squares. This partially confirms a conjecture by Richard Cohn. (Received November 16, 1948.)

174. Y. W. Chen: *Initial value problems of a hyperbolic system of quasi-linear differential equations with singularity.*

When initial values on the line $x+y=0$ or on the two half-axes $x\geqq 0$, $y=0$; $y\geqq 0$, $x=0$ are properly given, solution of the equation $L(u; k, l) \equiv u_{xy}+k(x+y)^{-1}(u_x+u_y)+l(x+y)^{-2}u=0$ with constants, k, l, can be obtained by Riemann's method. With the help of Riemann function an iteration scheme can be set up to obtain solution of initial or characteristic initial value problem of equation $L(u; k, l)=f(x, y; u, u_x, u_y)$, assuming proper behavior of the function f on the right-hand side. Suppose we have a hyperbolic system of quasi-linear differential equations $\sum_{j=1}^{n} a_{i,j} u_x^j=0$, $\sum_{j=1}^{n} a_{k,j} u_y^j=0$, $i=1, 2, \cdots, n'$ $(<n)$, $k=n'+1, \cdots, n$. The values of u^j assigned on $x+y=0$ or on the two half-axes $x\geqq 0$, $y=0$; $y\geqq 0$, $x=0$ have the property that after substituting them into the coefficients $a_{s,j}$, $s=1, 2, \cdots, n$; $j=1, 2, \cdots, n$, the determinant $\|a_{s,j}\|$ becomes zero on the initial line or at the vertex $(x, y)=(0, 0)$. Under certain conditions such a system of equations can be reduced to another system of n equations of the following form: $j=1, 2, \cdots, r$, $L(u^j; k_j, l_j)=f_j(x, y; u^1 \cdots u^n, u_x^1 \cdots u_x^n, u_y^1 \cdots u_y^n)$, where k_j and l_j are constants. The initial value problems can then be treated by iteration with the help of Riemann functions. In particular this method can be applied to certain problems of supersonic axial symmetric flows, which were mathematically formulated and investigated by K. O. Friedrichs. (Received November 16, 1948.)

175*t*. R. J. Duffin: *Nonlinear networks.* IV.

The networks considered in this paper consist of transformers with ferromagnetic cores (assumed to have negligible hysteresis) and Ohmic resistors. It is found that as in the case of linear networks a given electromotive force gives rise to a unique state of currents after sufficient time has elapsed. Moreover, if the electromotive force is periodic, a unique periodic current flow can exist. The proof depends upon transforming the network equations into equations for a network containing nonlinear resistors. The differential permeability of a ferromagnetic lies between positive limits. In the transformed equations this is imaged by the condition that the differential resistance lies between positive limits. The results stated then follow from theorems previously developed for networks containing such quasi-linear resistors. (Received November 16, 1948.)

176*t*. W. F. Eberlein: *On the convergence of certain perturbation series.* Preliminary report.

Despite decades of numerical evidence suggesting that the radii of convergence ρ_n of the perturbation series $\lambda_n(\epsilon)=\lambda_{0n}+a_{1n}\epsilon+a_{2n}\epsilon^2+\cdots$ for the characteristic values λ_n

of the Mathieu equation $u''(x)+(\lambda+\epsilon\cos 2x)u(x)=0$, $u(0)=u(2\pi)$, and related wave equations are nonzero and increase with the natural index n, the only published result appears to be that of G. N. Watson (1915). The latter obtained a positive lower bound for ρ_0 of the Mathiew equation by a classical method which did not extend to higher characteristic values. It is shown that the modern perturbation theory of operators in Hilbert space developed by Rellich and de Sz. Nagy (Comment. Math. Helv. vol. 19 (1947)) yields explicit lower bounds of the form $\rho_n \geqq Kn$ $(K>0)$ in problems of this type. (Received November 18, 1948.)

177. G. E. Forsythe: *A solution of the telegrapher's equation with initial conditions on only one characteristic.* Preliminary report.

When conditions (c) and (ii) below are satisfied, the future motion in a certain linearized model (due to Rossby) of non-divergent, plane atmospheric flow is uniquely determined by the present motion. We have the following unusual boundary-value problem with initial conditions on one characteristic: Let $f(x)$ be a real-valued function of bounded variation for $-\pi \leqq x \leqq \pi$, such that: (a) $f(-\pi)=f(\pi)$, (b) $f(x)$, $=[f(x+0)+f(x-0)]/2$, (c) $\int_{-\pi}^{\pi} f(x)dx=0$. The problem is to define $v(x, t)$ for $-\pi \leqq x \leqq \pi$ and $0 \leqq t < \infty$ such that: (i) the telegrapher's equation $\partial^2 v/\partial x \partial t + v/4 = 0$ is satisfied for all t and almost all x; (ii) $v(-\pi, t)=v(\pi, t)$; (iii) $v(x, 0)=f(x)$. It is shown that the unique solution of the problem is given by $v(x, t)=\sum_{n=1}^{\infty}(a_n \cos nz_n + b_n \sin nz_n)$, where $z_n = x+(t/4n^2)$, and where a_n, b_n are the Fourier's coefficients of $f(x)$. A Green's function is defined, $G(x, t)=\sum_{n=1}^{\infty}(1/n)\sin nz_n$, and it is shown that v can always be represented by a Lebesgue-Stieltjes integral of G: $v(x, t)=(1/\pi)\int_{-\pi}^{\pi} G(x-u, t)df(u)$. A program is outlined to compute $G(x, t)$ to three decimal places for $x=-\pi(\pi/36)\pi$ and $t=0(36)144$, using 10-decimal-place calculating equipment. Computing G from the defining series would use about 92,000 terms and require about 26,500,000 multiplications. Improvement of the convergence along lines suggested by Tamarkin reduces the number of multiplications to about 12,000. The hardest feature of the program is to avoid exceeding 10-place machine capacity. (Received November 15, 1948.)

178. Herbert Jehle: *Nonlinear resonance and self-duplication of genes.*

The current view about self-duplication is that the macromolecules which a gene is composed of attract from the surrounding medium specifically those macromolecules which happen to be identical with the former ones. This selection is extremely accurate, and this phenomenon's versatility which allows all kinds of genes and their mutants to self-duplicate is equally remarkable. The macromolecules are known to be rigid. Most of their thermal vibrations have so densely spaced quantum levels that they can be treated classically. Let "modes" be vibrations which are normal for very small amplitudes. Nonlinearity, that is, cubic and quartic coupling terms in the potential energy, will make synchronization possible for sets of almost commensurable modes of a molecule. In other words, librational motion about definite phaserelations of a set of commensurable frequencies can occur, and therefore there will be a statistical preference for certain phaserelations. A pair of macromolecules interacts through electric dipole vibrations accompanying their mechanical ones. If the pair of fundamental modes of a pair of identical macromolecules is inphase, then also the pairs of other corresponding modes are mostly inphase. This means a total interaction energy which can become larger than kT, and therefore almost exclusively inphase vibration, and sufficient attraction even at long range. (Received November 16, 1948.)

179*t*. R. E. Langer: *A note on the theory of vibration-rotation interaction.*

The equation for the radial component F of the *eigenfunction* of a rotating harmonic oscillator is of the form $d^2F/dr^2-[(r-1)^2/4\alpha^2+E/\alpha\omega_e+l(l+1)/r^2]F=0$. The constant α is small, so that $1/\alpha$ appears as a large parameter. l is an integer. The problem is to determine the characteristic values of E for which the equation admits of a solution that vanishes both at $r=0$ and at $r=\infty$. This requires special considerations of the solutions of the equation at the singular point $r=0$ and around the turning-point $r=1$. The characteristic E-values are determined to within terms of the order of $\alpha \log \alpha$, and the characteristic solutions are described in the various intervals of the range $(0, \infty)$. (Received November 13, 1948.)

180. C. G. Maple: *The Dirichlet problem: bounds at a point for the solution and its derivatives.*

Synge has shown that the vector representation in function space of the solution of the Dirichlet problem must lie on a hypercircle in the function space. For a suitably defined free Green's vector, bounds for the scalar product of the solution vector and the Green's vector are established by making use of this result and the fact that the metric of the function space is positive definite. This same scalar product is then expressed in terms of certain calculable integrals and the value of the solution at a point by use of a well known harmonic mean value theorem. When these results are combined, bounds at a point interior to the region of definition of the problem for the solution are obtained. By a change in the definition of the Green's vector, bounds at an interior point are established for the derivatives of the solution. When the region of definition of the problem is such that the bounding surface contains a plane portion, bounds at a point on this plane portion for the normal derivative are established. (Received November 15, 1948.)

181*t*. George Pólya and Gabor Szegö: *Approximations and bounds for the electrostatic capacity and for similar physical quantities.*

This paper consists of eight chapters: 1. Definitions, methods and results. 2. The principles of Dirichlet and Thomson. 3. Applications of the principles of Dirichlet and Thomson to estimation of the capacity. 4. Circular plate condenser. 5. Torsional rigidity and principal frequency. 6. Nearly circular and nearly spherical domains. 7. On symmetrization. 8. On ellipsoid and lens. Besides bringing proofs for, and complements to, some results announced elsewhere (especially Amer. Math. Monthly vol. 54 (1947) pp. 201–206) the paper presents in a new form the second variation of certain important functionals when the domain varied is a circle or a sphere (chap. 6), gives new inequalities between torsional rigidity, principal frequency and the maximum inner radius of simply connected plane domains (chap. 5), brings extensive surveys of solved and open problems (chap. 1), and so on. (Received November 1, 1948.)

182*t*. George Pólya and Alexander Weinstein: *On the torsional rigidity of multiply connected cross sections.*

Of all multiply connected cross sections with given area and with given joint area of the holes, the ring bounded by two concentric circles has the maximum torsional rigidity. This theorem generalizes a statement due to de Saint-Venant, which has been proved recently by one of the authors jointly with G. Szegö. The same method is used in proving the present theorem in conjunction with a reshaping of the theory of torsion for multiply connected cross sections. (Received November 1, 1948.)

183t. Eric Reissner: *Note on compressibility corrections for subsonic flow past slender bodies of revolution.*

The relation between two-dimensional and axi-symmetrical compressibility corrections for flow past slender bodies, according to the linear perturbation theory, is examined for the problem of flow past a corrugated cylinder. A solution in terms of modified Bessel functions is obtained which contains as limiting cases both the Prandtl-Glauert correction and the Göthert correction and which moreover shows the nature of the transition from one to the other of the two limiting cases. In the present problem the form of the Göthert correction which previously has caused some discussion (see for instance W. R. Sears, Quarterly of Applied Mathematics vol. 5 (1947) pp. 89–91) is obtained in particularly simple fashion. (Received November 5, 1948.)

184t. Eric Reissner: *On the theory of thin elastic shells.*

The present paper gives a self-contained formulation of the problem of *finite* symmetrical deflections of shells of revolution. From this there is obtained by specialization the small-deflection (linearized) theory. In the reduction of this latter theory to two simultaneous second order differential equations slight modifications of the known results are introduced, the advantages of which are indicated. From the equations of the small-deflection theory there is derived a simplified system of equations which applies for *shallow* shells. It is shown that the solutions of these simplified equations can be expressed in terms of Bessel functions for the class of paraboloidal shells of constant thickness and that solutions in terms of elementary functions are obtained for a practically interesting class of shallow shells of varying thickness. (Received November 15, 1948.)

185t. H. E. Salzer: *Formulas for numerical differentiation in the complex plane.*

Formulas are given for numerical differentiation of a complex analytic function $f(z)$, where $z=x+iy$. If h is the length of the square in the Cartesian grid, $z_j=z_0+jh$, differentiation of Lagrange's n-point formula yields $h^\nu f^{(\nu)}(z_j)\sim\sum_k L_k^{(\nu)}(j)f(z_k)$. (All expressions, such as $L_k^{(\nu)}(j)$, are understood to depend upon n.) The summation is over the n fixed points z_k which are chosen to lie, for each n, in a close configuration in the z-plane, resulting in better approximation to $f^{(\nu)}(z)$ than the simpler method of choosing z_j to lie equidistant on a straight line parallel to either the x- or y-axis. Exact values of $L_k^{(\nu)}(j)$ are given for the three- to nine-point cases, for obtaining all the derivatives $f^{(\nu)}(z_j)$, $\nu=1, 2, \cdots, n-1$, at each of the n points z_j. The computation was expedited by two new formulas for $L_k^{(\nu)}(j)$, namely the recursion formula $L_k^{(\nu)}(j)=\nu\left[A_kL^{(\nu)}(j)/\nu-L_k^{(\nu-1)}(j)\right]/(j-k)$, for $k\neq j$, and $L_k^{(\nu)}(j)=A_jL^{(\nu+1)}(j)/(\nu+1)$, for $k=j$. Here $L^{(\nu)}(j)=d^\nu L(p)/dP^\nu\big|_{p=j}$ where $L(p)=\prod_j(P-j)$ and $A_k=1/\prod_j'(k-j)$ ($\prod_j$ for all points j; $\prod_j'$ for all points except $j=k$.) For configurations that are symmetric about the 45° ray, $L_{b+ia}^{(\nu)}(d+ic)\overline{L_{a+ib}^{(\nu)}(c+id)}/i^\nu$. (Received November 10, 1948.)

186t. B. R. Seth: *Bending of an elliptic plate with a confocal hole.*

The bending of an elliptic plate with a confocal hole subjected to uniform pressure and clamped at the edges is discussed. The numerical results obtained are compared with those for a complete plate. It is found that the maximum deflection, W_m, occurs near the inner boundary. For the plate bounded by ellipses whose semi-axes are (1.6c, 1.249c) and (1.14c, 0.548c), it is found that W_m is almost one-fifth the value for the plate complete up to the outer boundary. By making the minor axis of the hole vanishingly small, the interesting case of an elliptic plate with a crack can be treated. (Received November 16, 1948.)

187. A. C. Sugar: *An axiomatic development of classical mechanics.* Preliminary report.

In an earlier paper the writer gave a rigorous definition of dimensional number by set theoretic methods and, in terms of this concept, defined the notion of physical function. Employing this theory of physical number and function an axiomatic reconstruction of classical mechanics is made without a direct or indirect use of the notion of an actual infinitesimal. The laws of motion are formulated for bodies rather than particles thereby avoiding the usual pseudo-derivation of body dynamics from particle dynamics. (Received December 16, 1948.)

188. C. A. Truesdell: *The kinematics of vorticity.* I.

This paper contains a general treatment of those aspects of the kinematics of continuous media which bear on the vorticity of fluid motions. The presentation is in terms of vectorial and dyadic formulae, and full historical references are given. Section I contains necessary preliminaries regarding Lagrangian and Eulerian descriptions of the motion; rate of change formulae; line, surface, and volume integrals; trajectories and tubes. Section II contains an analysis of a general motion of a continuous medium, in which the vorticity represents one part; vortex-lines, vortex-tubes, and various vorticity integrals are introduced. Section III is devoted to convection and diffusion of vorticity. These concepts are first introduced from a Eulerian viewpoint; then it is shown that their meaning can be fully clarified by a Lagrangian analysis, generalizing a result of Cauchy. The effect of the mechanism of diffusion upon the circulation, the vortex-lines, and the strength of the vortex-tubes is analyzed in detail, both from a Eulerian and from a Lagrangian point of view. (Received November 15, 1948.)

189*t*. Arthur Zeichner: *Certain singularities of the compressibility equation.*

The author considers solutions of the compressibility equation, $\psi_{\eta\eta}+l(\eta)\psi_{\theta\theta}=0$, $l(\eta)=-\eta+\sum_{n=2}^{\infty}c_n\eta^n$, $\eta(M)$ being a regular function of the Mach number M which is regular at $M=1$; $\eta(\lambda)=0$. With the use of the integral operator method (see Bergman, Amer. J. Math. vol. 70 (1948)), the behavior of multivalued solutions which possess a singularity on the line $M=1$ is investigated. Singularities at the point $M=1$, $\theta=0$ for the simplified equation, that is, where $l(\eta)=l^*(\eta)=-\eta$ (Tricomi equation), are also considered. By setting $f=\zeta^{1/2}$, in the operator $\Pi(f)$ introduced by Bergman ((4.11), loc. cit.) and applying the Kummer transformation for hypergeometric functions, the solution $\psi=A\eta(\theta^2-4\eta^3/9)^{-1/6}F(1/6, 2/3, 4/3; (-4\eta^3/9)(\theta^2-4\eta^3/9)^{-1})$ $+B(\theta^2-4\eta^3/9)^{1/6}F(-1/6, 1/3, 2/3; (-4\eta^3/9)(\theta^2-4\eta^3/9)^{-1})$ is obtained. Here A and B are constants, and $F(\cdots)$ denotes the hypergometric function. Analogous expressions are obtained for $f=\zeta^{\nu/2}$, ν positive integers. The fundamental solution, with a logarithmic singularity at M_0, θ_0, is derived for $M_0\leqq 1$. (Received November 18, 1948.)

GEOMETRY

190. N. A. Court: *Semi-inverse tetrahedrons.*

Two tetrahedrons $(T) = ABCD$, $(T') = A'B'C'D'$ whose corresponding vertices A, A'; $\cdots$ are pairs of inverse points with respect to a sphere (M) are said to be "semi-inverse" for (M). Among the properties of two such tetrahedrons the following may be mentioned. The two tetrahedrons correspond to each other in the homology $(M, \mu, -1/c)$ having for center the center M of (M), for plane of homology the common radical plane μ of the sphere (M) and the circumspheres (O), (O') of (T), (T'), and for constant of homology the negative reciprocal of the ratio c of the power of M for (O) to the square of the radius of (M). The circumcircles of two corresponding faces of (T), (T') lie on a sphere whose center is the mid-point of the segment determined by the poles, for (M), of the two faces considered. The tetrahedral polar planes of M for (T), (T') are coaxal with the plane μ. The pedal tetrahedron of M for one of the two tetrahedrons (T), (T') and the antipedal tetrahedron of M for the other are two semi-inverse tetrahedrons with respect to (M). (Received November 15, 1948.)

191. Michael Goldberg: *Rotors in a regular tetrahedron.*

A rotor of a polyhedron is defined here as a convex surface (not necessarily spherical) which may be rotated through every possible orientation while keeping in continual contact with all the faces of the fixed polyhedron. The well known surfaces of constant width are rotors for the parallelepipeds which include the cube as a special case. This paper exhibits a class of rotors for the regular tetrahedron. These rotors are described by the polar tangential equation $p = a + bl^2 + cm^2$ where p is the distance from the origin to the tangent plane and l, m are the direction cosines of the normal to the tangent plane. Other geometric properties of these rotors are derived. (Received October 25, 1948.)

192. A. J. Hoffman: *On the foundations of inversion geometry.*

The author gives an autonomous development of inversion geometry over certain kinds of fields F, including the case in which F is the real numbers. The methods are lattice-theoretic, and the only undefined relations are incidence relations. Three equivalent forms of the "fundamental axiom" are considered (analogous, in projective geometry, to the various equivalent statements of the fundamental theorem). Co-ordinates and the equations of circles and spheres are introduced synthetically, and the associated number field of the geometry is shown to be an ordered field F in which every positive number is a square. Contact is made with work of Gorn, Izumi, Pieri, and van der Waerden and Smid. (Received November 15, 1948.)

193t. M. O. Reade: *Generalizations to space of a theorem of Fédoroff.*

Fédoroff has shown that if $f(z)$ is continuous in a bounded, simply-connected domain $\mathfrak{D}$, then a necessary and sufficient condition that $f(z)$ be analytic in $\mathfrak{D}$ is that $\iint (z-z_0)f(z)dxdy=0$, the integration being over any disc $D(z_0, r)$, with center z_0 and radius r in $\mathfrak{D}$ (Rec. Math. (Mat. Sbornik) vol 41 (1934) p. 92). Generalizations of the following type are proved in this paper. Let $U_j(z) \equiv U_j(x, y)$, $j=1, 2, 3$, be real-valued and have continuous partial derivatives of the third order in $\mathfrak{D}$. Then a necessary and sufficient condition that the functions $U_j(z)$ either (1) map $\mathfrak{D}$ isothermically on a surface that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or (2) be the coordinate functions of a minimal surface in isothermic representation, is that $\sum_{j=1}^{3}[\iint(z-z_0)U_j(z)dxdy]^2=0$, the integration being over any disc $D(z_0, r)$ in $\mathcal{D}$. This result is analogous to a result obtained by Reade and Beckenbach (Trans. Amer. Math. Soc. vol. 49 (1941) p. 354). (Received November 16, 1948.)

194. T. G. Room: *Arithmetic of curves on certain surfaces.*

The transversal lines to a line, l, lying on a cubic surface, Π, and a line, v, not on Π, determine an involutory self-transformation $\mathcal{L}$ of Π. The curve $\mathcal{L}l$, together with v, determines a new involutory transformation of Π, and the 27 lines on Π determine an enumerable family of transformations. These form a groupoid in which the product of any three members is a member. If curves $c_1, \cdots, c_7$ are taken as a base on Π, then under $\mathcal{L}$ the curves c become curves $\mathcal{L}c$, expressible as Lc, where L is an involutory unitary matrix of integers. The matrices L may be expressed both as products formed from 8 basic matrices, and as sums of integer multiples of these 8, the integers being quadratic functions of 7 parameters. The matrices L may be interpreted as involutory collineations in [7], in which the base elements are a family of lines through a point W, and their polar [5]'s with regard to a quadric through W, and may thence be represented by an enumerable system of points in [6]. The figure formed by these points is an expansion of the figure of two perspective simplexes in [5] by which the 27 lines are ordinarily represented. (Received November 22, 1948.)

195*t*. W. R. Utz: *Almost periodic geodesics on manifolds of hyperbolic type.*

The manifolds considered are secured by the identification of congruent points (under a Fuchsian group) of certain Riemannian manifolds whose fundamental forms are defined on the interior of the unit sphere of Euclidean n-space. The principal theorem shows that almost periodic, nonperiodic geodesics exist on these manifolds when such geodesics exist on a companion manifold of constant negative curvature. In the case $n=2$, the almost periodic, nonperiodic geodesics on a class of manifolds of constant negative curvature are shown to be everywhere dense when the group employed is of the first kind. The latter result is an improvement on an unpublished result due to Marston Morse. (Received November 15, 1948.)

LOGIC AND FOUNDATIONS

196. Henry Blumberg: *Conception of set; elimination of the paradoxes of set theory.*

This paper sets forth a conception of set—termed "genetic"—which, as the author submits, constitutes an appropriate solution of the problem of the set-theoretic paradoxes. This conception is close to mathematical experience, requires no special devices or new constructions, and permits a logically unimpeachable development of the salient points of Cantor's discoveries—in particular, of Zermelo's theorem on the normal order. The principal idea is to validate the concepts and modes of conceptual derivation which mathematicians had no hesitation in accrediting before the phenomenon of the paradoxes, and developing the implications of such validation. This ideas is supported by the fact that the set-theoretic paradoxes may be eliminated frontally; in other words, as the author shows, the paradox maker, in every case, commits an error in the argument allegedly establishing the paradox. Such direct refutation is facilitated by the proposed genetic conception of set. This conception does not fix once for all what a set is; a satisfactory conception of set cannot be expected to do this. Legitimate sets are derived from accredited sets by accredited associations; but the phrases "all legitimate sets" and "there exists a legitimate set" have no meaning. There is clarification, too, of the deficiencies of other proposed conceptions of set. (Received November 12, 1948.)

197t. H. B. Curry: *A theory of formal deducibility.*

This is a revision of a paper read in 1937 (Bull. Amer Math. Soc. Abstract 43-9-325). The fundamental idea is to apply the inferential-rule methods of Gentzen to propositions relating to a formal system, so as to make a semantic analysis of the "compound" propositions formed from the elementary propositions of the system by the connectives of propositional algebra and predicate calculus. The new results relate chiefly to extending these methods to include the classical as well as the intuitionistic approaches. When negation is introduced four types of system are considered, namely: M, the minimal system (Johansson); J, the intuitionistic system (Heyting); K, the classical system; and D, a minimal system with excluded middle (Johansson), applicable when the underlying formal system is decidable. For each type of system there are three types of formulation called respectively L, T, H; the L formulation is like that so-called by Gentzen, T is Gentzen's "natural" formulation, and H is a more orthodox calculus! Relations between these types of system and formulation are considered, including a generalization of the Glivenko theorem. The paper will be published in booklet form by the University of Notre Dame. (Received November 16, 1948.)

198. H. B. Curry: *The elimination theorem when necessity is present.*

In the Notre Dame lectures on formal deducibility (see the preceding abstract) a proof of the fundamental theorem, called the elimination theorem (Genzen's "Hauptsatz"), was lacking for the case of systems involving necessity. This hiatus is filled in the present paper. This enables the treatment of modal systems to be completed; and we now have relations between L, T, and H, formulations of propositional algebra just like those for non-modal systems. In particular the procedure gives a decision process for the Lewis system S4; however the relation of this procedure to those previously known for this system has not been investigated. The generalized form of the elimination theorem so obtained leads to some simplifications in the previous theory, notably the connections between the LC (classical positive) and LA (intuitionistic positive) systems. (Received November 16, 1948.)

199t. H. B. Curry: *The permutability of rules in the classical inferential calculus.*

Suppose we have a rule theoretic system à la Gentzen with elementary statements of the form $X_1, X_2, \cdots, X_m \rightarrow Y_1, Y_2, \cdots, Y_n$. With reference to the rules of such a system we distinguish as parameters those X_i, Y_j which go over unchanged from premises to conclusion, as components those which appear in the premises but not in the conclusion, and as principal constituents those which occur in the conclusion only. Suppose the rules are such that the same parameters appear in all the premises, and that parameters can be added to and deleted from all premises and conclusion simultaneously without destroying the validity of the inference. Then the following theorem is true: if a rule R_1 is followed by a rule R_2 in such a way that the principal constituents for R_1 are parameters for R_2, then the rules can be interchanged. This simple observation includes the strong form of the elimination theorem ("Hauptsatz") which is valid in Gentzen's system LK. In the system LJ it is not possible to add parameters on the right, and consequently a proof of the elimination theorem by this method breaks down. (Received November 15, 1948.)

200. Marshall Hall: *The decision problem for semigroups with two generators.*

Emil Post has shown (Journal of Symbolic Logic vol. 12 (1947) pp. 1–11) that the decision problem for semigroups is unsolvable. It is shown here that the decision problem for an arbitrary semigroup may be reduced to that for an appropriate semigroup with two generators. Hence it follows that the decision problem is unsolvable for semigroups with two generators. (Received December 14, 1948.)

201. Ilse L. Novak: *The relative consistency of von Neumann's and Zermelo's axioms for set theory.*

The system of axioms for class and set theory which was adapted from von Neumann's by Bernays (Journal of Symbolic Logic vol. 2 (1937) p. 65) and Gödel (*Consistency of the continuum hypothesis*, Princeton, 1940) differs from Zermelo's (including the Aussonderungs- and Ersetzungs-axiome) by admitting classes (non-elements) as well as sets. In the present paper a model of the von Neumann-Bernays-Gödel system is constructed in the syntax of Zermelo's. The syntax employed has axioms stating (i) certain basic signs and their combinations exist, (ii) quantification theory, (iii) induction on the length of formulas and (iv) identity theory may be used to derive metatheorems, (v) Zermelo's system is consistent. The "ϵ" of Zermelo's system is reinterpreted as a syntactic relation between names of sets with help of a syntactically defined predicate which proves true of all Zermelo's theorems and true of any two formulas if and only if true of their conjunction, and true of any formula if and only if not true of its denial. The construction turns on the fact that the well-ordering hypothesis is consistent with Zermelo's system. The existence of this model shows von Neumann's system consistent relative to this syntax of Zermelo's system. (Received November 13, 1948.)

202. Ernst Snapper and M. A. Zorn: *On transfinite induction.*

A new mathematical proof for a variant of transfinite induction up to the first epsilon-number is given. The proof is (i) capable of generalization, (ii) of metamathematical interest. (Received November 16, 1948.)

STATISTICS AND PROBABILITY

203*t*. D. A. Darling: *Note on a distribution.*

If $X_1, X_2, \cdots, X_n$ are positive independent identically distributed random variables with the common continuous distribution $F(x)$ and if $X^* = \max(X_j)$ we define $Z_n = (X_1 + X_2 + \cdots + X_n)/X^*$. If $\phi(x)$ is the density of X_j, the characteristic function of Z_n is $E(\exp(itZ_n)) = n(\exp(it))\int_0^\infty z^{n-1}\phi(z)(\int_0^1 \exp(ity)\phi(yz)dy)^{n-1}dz$. The cases of practical importance where the X's are distributed as χ^2 with m degrees of freedom or as the range of m independent variables appear difficult to calculate explicitly. Certain limiting distributions are considered. (Received November 15, 1948.)

TOPOLOGY

204. R. D. Anderson: *A characterization of a certain class of subcontinua of the sphere.*

Let M be a subcontinuum of a two-dimensional sphere Σ. The author shows that in order that there exist an upper semicontinuous collection G of mutually exclusive continua filling up M such that G with respect to its elements as points is topologically equivalent to Σ, it is necessary and sufficient that there exist two uncountable collections E and F of mutually exclusive subcontinua of M such that each element of E intersects each element of F. Other results of a similar nature are obtained by the author. The above theorem can be applied to certain sums of two pseudo arcs in the plane. (Received November 15, 1948.)

205*t*. R. D. Anderson: *Concerning upper semicontinuous collections of continua in the sphere.*

A collection F of continua will be said to have the W property provided that if H_1 is any element of F and x is any point of F^*-H_1 there exists a continuum H_2 of F containing x and lying in F^*-H_1 and if y is any point of $F^*-H_1-H_2$ there exists a continuum H_3 of F containing y and lying in $F^*-H_1-H_2$ and if z is any point of $F^*-H_1-H_2-H_3$ there exists a continuum H_4 of F containing z and lying in $F^*-H_1-H_2-H_3$ and there exists a set E of six continua of F each containing exactly two of the continua H_1, H_2, H_3, H_4 such that the common part of any two continua of E if it exists is one continuum of the set H_1, H_2, H_3 and H_4. Let M be a subcontinuum of a two-dimensional sphere Σ. The author shows that in order that there exist an upper semicontinuous collection G of mutually exclusive continua filling up M such that G with respect to its elements as points is topologically equivalent to Σ it is necessary and sufficient that there exist a nondegenerate collection F of subcontinua of M covering M having the W property. A V property is also defined with similar application. (Received November 15, 1948.)

206. S. S. Cairns: *An elementary proof of the Jordan curve theorem.*

An elementary and relatively brief proof is given that a simple closed curve in the plane determines two regions, one of which is a 2-cell. The argument is better adapted for presentation to a beginning graduate topology class than are other proofs known to the author. (Received November 9, 1948.)

207. R. P. Dilworth: *The space of normal functions.*

If f is a bounded real function on a compact Hausdorff space S, let f^* and f_* denote the upper and lower limit functions of f respectively. If $(f^*)_*=f_*$ and $(f_*)^*=f^*$, then f is said to be ***normal***. Let $N(S)$ denote the set of all normal functions on S. $N(S)$ is a complete lattice under the partial ordering $f^*\geqq g_*$. The following results are obtained: (1) $N(S)$ is lattice isomorphic to the lattice of all real continuous functions on a compact Hausdorff space T_S; (2) T_S is extremally disconnected; (3) S is homeomorphic to a factor space of T_S; (4) T_S is essentially the minimal compact Housdorff space having properties (2) and (3). (Received November 18, 1948.)

208. C. H. Dowker: *Čech cohomology groups and the axioms.*

A definition is given for the Čech cohomology groups $H^n(X, A)$ of a topological space X relative to a not necessarily closed subset A. Infinite coverings by open sets are used. With this definition it is shown that the Čech cohomology theory satisfies the Eilenberg-Steenrod axioms, including the homotopy axiom, for arbitrary pairs (X, A). (Received November 12, 1948.)

209*t*. Marianne R. Freundlich: *Banach spaces and group duality.*

It is shown that the apparent similarity of the Pontrjagin duality theorem for locally compact groups to the duality existing in reflexive Banach spaces is more than superficial, in the following sense: for reflexive Banach spaces considered as topological groups the Pontrjagin theorem holds verbatim; that is, the character group of the character group of a reflexive Banach space B is topologically and algebraically isomorphic with B. The standard topologies are assigned to character groups. The proof makes use of results of Arens (*Duality in linear spaces*, Duke Math. J. vol. 14 (1947) pp. 784–794) and Myers (*Equicontinuous sets of mappings*, Ann. of Math. vol. 47 (1946) pp. 496–502). (Received November 15, 1948.)

210*t*. R. E. Fullerton: *On a semi-group of subsets of a linear space.*

A semi-group of subsets of a space is defined to be a family $\mathcal{J}$ of subsets such that if S_1 and S_2 are in $\mathcal{J}$, then $S_1 \cap S_2 \in F$. It is proved that if S is a subset of a linear space, closed in a certain weak sense and possessing an extreme point, then all translates of S form a semi-group only if S is a convex cone. (Received November 16, 1948.)

211. V. G. Gorciu: *Interior transformations in the case of uniform, non-metrizable spaces.*

Two uniform, non-metrizable compact spaces, A and B, are defined as neighborhood-spaces by assigning an indexed set of neighborhoods to every point. The set I of indices is directed, non-denumerable and has no last element. The following definitions are introduced: A property P, depending on an index α, is "of non-cofinal character" if and only if the set of indices satisfying that property is a non-cofinal subset of I. A point x is a "limit point" of a directed set $\{x_\alpha\}$ if and only if every neighborhood of x contains a cofinal subset of $\{x_\alpha\}$. Denote by $P(\alpha)$ the following property: For any β there exists an α such that for any $y \in B$ and any $x \in T^{-1}(y)$, $T[V_\beta(x)] \supset V_\alpha(y)$. The following theorem can now be stated: In order that the continuous mapping $T(A) = B$ be an interior transformation, it is necessary and sufficient that $\sim P$ be of non-cofinal character. This theorem can be regarded as an extension of a result of G. T. Whyburn. (Received December 9, 1948.)

212*t*. J. W. Green and William Gustin: *On the direct sum of continua.*

Let X be a real vector space of finite dimension n; and let a_ν (ν denotes a variable index ranging over the n indices $1, \cdots, n$) be n linearly independent vectors in X. Consider n continua Q_ν in X, the continuum Q_ν passing through the origin and through a_ν. It is shown that every point x in the space X is of the form $\sum(\alpha_\nu a_\nu + q_\nu)$ where q_ν lies in Q_ν and the α_ν are integers. (Received November 16, 1948.)

213. O. G. Harrold: *Euclidean domains with uniformly abelian local fundamental groups.*

Let A be an arcwise connected subset of a topological space X. A group $C(A, X)$

is defined as follows. Closed paths in A are considered based at $p\in A$ that are homotopic in X to a commutator of paths in X. Classes of paths and multiplication of classes are defined as for the fundamental group. A local analogue of the fact that the commutator-quotient group of the fundamental group is the 1-dimensional homology group over the integers is given. If for each $p\in X$ and each neighborhood U of p rel. A there is a neighborhood V of p rel. A such that for each component V_1 of V, $C(V_1, U)=0$, A is said to have uniformly abelian local fundamental groups. Included in the results are the following. Let X be spherical n-space and A the complement of a closed, topological k-cell, $k=1, 2, \cdots, n$. If A has the above defined property, $\pi_1(A)$ is trivial. If A is the complement of a closed, totally disconnected subset and A has the above property, $\pi_1(A)$ is trivial. (Received November 15, 1948.)

214. Edwin Hewitt: *Functionals related to normal spaces.*

Results of F. Riesz, Kakutani, and Markov are extended to obtain the following theorem. Let X be any normal space and let $C(X, R)$ be the space of all continuous real-valued functions defined on X. Let ϕ be any real-valued function defined on $C(X, R)$ such that (1) $\phi(f+g)=\phi(f)+\phi(g)$; (2) $\phi(\alpha f)=\alpha\phi(f)$, where α is any real number; (3) $f\geqq 0$ implies $\phi(f)\geqq 0$. Then there exists a measure σ on X such that all Borel sets are measurable and such that $\phi(f)=\int_X f d\sigma$. If X admits unbounded continuous real-valued functions, then the measure σ is subject to certain strong restrictions, which are described. (Received November 12, 1948.)

215t. E. J. McShane: *Images of sets satisfying the conditions of Baire.*

Assume $S\subset X_1$, a topological space; F a family of functions f with open domain D_f containing S and mapping open subsets of D_f on open subsets of X_2, a topological space. If S is of second category and satisfies the condition of Baire, and for each $f_0\in F$ and each $x_0\in D_f$ at which S is of second category the aggregate of all inverse images of $f_0(x_0)$ under all $f\in F$ is of second category at x_0, then there is a set S_0 consisting of all of S except a set of first category such that the union of the $f(S_0)$ for $f\in F$ is open. The corollaries include and extend a number of known theorems on subgroups and semi-groups of topological groups, on additive mappings and on "mid-point-convex" sets and functions. (Received November 19, 1948.)

216. W. S. Massey: *Classification of mappings of an $(n+1)$-dimensional space into an n-sphere.*

The problem of determining all homotopy classes of continuous maps of an $(n+1)$-dimensional complex K into an n-sphere S^n was solved by Steenrod (Ann. of Math. vol. 48 (1947) pp. 290–320). This solution made essential use of the fact that the homotopy group $\pi_{n+1}(S^n)$ is cyclic of order two, a result due to Freudenthal and Pontrjagin. In this paper another proof is given of this homotopy classification theorem which does not require a knowledge of $\pi_{n+1}(S^n)$. The fact that $\pi_{n+1}(S^n)$ is cyclic of order two then follows as a corollary. The proof is based on an extension theorem of the following kind. Let X be a compact metric space of dimension $\leqq n+2$, let A be a closed subset of X, and $f:A\rightarrow S^n$ a continuous map. New operations are introduced which enable one to state necessary and sufficient conditions for the existence of a continuous map $F:X\rightarrow S^n$ which is an extension of f. In case (X, A) is a simplicial pair, and f is a simplicial map, these new operations are effectively calculable. The extension theorem is proved by an inductive procedure similar to that used in the proof of the Hopf Extension Theorem by Hurewicz and Wallman (*Dimension theory*, p. 142). (Received November 13, 1948.)

217. G. E. Schweigert: *Basic invariance theorems for onto-transformations.*

It is well known that, for homeomorphisms, the closure (more generally P-hull), complement, union, intersection and certain limits of invariant sets are themselves invariant. These are investigated in the setting $f(M)=M$ (where $f^{-1}(x)$ is not necessarily a point); the theorems show a wide divergence in the assumptions needed to reach the desired conclusions. Some sample theorems: The P-hull of an invariant set is invariant provided f and f^{-1} preserve property P; $A\subset f(A)$ implies the union of $f^n(A)$ is invariant; f continuous, B closed in a compact Hausdorff space and $f(B)\subset B$ implies $f(\Pi)=\Pi$ where Π is the intersection; $K=\bigcap K_\alpha$, K_α invariant, implies $f(K)\subset K$, but $K\subset f(K)$ does not follow when f is interior. If L and l are limits superior and inferior for collections of invariant sets, then $f(L)=L$ and $f(l)\subset l$ follow the usual assumptions, but $l\subset f(l)$ does not. The corresponding statements for strong invariance, namely $f^{-1}(A)=A$, show better resemblance to the homeomorphism case. The results are, in general, rich enough to invite an interesting "orbit-theory." (Received November 6, 1948.)

218. G. T. Whyburn: *Expansive mappings.*

Let A and B be locally connected generalized continua and let f be a mapping (=continuous transformation) of A into B. Then f is *expansive* provided A is the union of a strictly monotone increasing sequence $[R_n]$ of conditionally compact regions, that is, $\overline{R}_n\subset R_{n+1}$, such that if $F_n=f[F(R_n)]$ where $F(R_n)$ denotes the boundary of R_n, any compact set in B intersects at most a finite number of the sets F_n. Also f is *strongly interior* or open provided the image of every open set in A is open in B. A non-constant entire analytic function (which is always strongly interior) is expansive provided that for some sequence of circles $|z_n|=r_n$ with $r_n\rightarrow\infty$ we have $\min_{|z|=r_n}|f(z_n)|\rightarrow\infty$ and a converse holds if we use simple closed curves. Non-constant entire functions of order less than 1/2 are thus expansive. If f is strongly interior and expansive, it maps A *onto* B; and, indeed, either for each $z\in B$, $f^{-1}(z)$ is compact and nonempty or else for each $z\in B$, $f^{-1}(z)$ is non-compact and thus infinite. A mapping of A onto B is expansive if and only if each component of the inverse of a continuum in B is compact. (Received October 1, 1948.)

219*t*. G. T. Whyburn: *Quasi-interior mappings.*

A mapping f of one locally connected generalized continuum A into another one B is *quasi-interior* provided that for any $y\in f(A)$ and any open set U in A containing a compact component of $f^{-1}(y)$, y is interior rel. B to $f(U)$. A mapping is quasi-interior if and only if for each region R in B, each conditionally compact non-empty component of $f^{-1}(R)$ maps onto R under f. There exist quasi-interior mappings which preserve neither local compactness nor local connectedness. If the sequence of quasi-interior mappings of A into B converges to the mapping f on A, the convergence being uniform on each compact set in A, then f is quasi-interior. Product and factor theorems are also established. Clearly every strongly interior mapping is quasi-interior and every light quasi-interior mapping is strongly interior. Also, in case A is compact as well as locally connected and connected, the property of quasi-interiority of a mapping turns out to be the same as quasi-monotoneity as defined originally by A. D. Wallace. (Received October 1, 1948.)

220t. G. T. Whyburn: *On compact mappings.*

A study was made of mappings $f(A)=B$ (A and B separable metric) having the property that the inverse of any compact set in B is compact. This property is equivalent to the mapping being closed and having compact point-inverses. Upon completion we learned that Vainstein had recently studied mappings having the latter property, calling them *compact* mappings. Thus we adopt his term and report some of our results that do not seem to duplicate his. If $f(A)=B$ is compact, for any $y\in B$ and any open set U in A containing $f^{-1}(y)$, y is interior to $f(U)$. Thus local compactness and local connectedness are invariant under compact mappings. If f is compact and monotone, the inverse of every connected set is connected; thus f is *strongly monotone*, that is, the inverse of a continuum is a continuum. If A is a locally connected generalized continuum, for a monotone mapping on A, the properties of compactness, strong monotoneity and quasi-interiority are equivalent. Compact mappings have the group property and however a compact mapping be factored continuously: $f=f_2f_1$, *both* factors are necessarily compact. Any two monotone light factorizations of a compact mapping are topologically equivalent. Such factorizations always exist (as remarked also by Vainstein). Other product and factor theorems are obtained. (Received October 1, 1948.)

J. W. T. Youngs,
Associate Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER

December 22, 1948

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1948, with certain pertinent comments.

Investment Portfolio

On November 30, 1948, the market value of securities held for Invested Funds exceeded book value by $2,789, but the market value of securities held for Current Funds was less than book value by $1,626. On the whole portfolio, the market value therefore exceeds book value by $1,163. This showing is less favorable than in recent years because security prices have declined considerably in the last quarter of the fiscal year. Nevertheless, reserves held in accounts "Reserve for Investment Losses" ($4,386) and "Profit on Sales of Securities" ($18,174) may still be considered adequate protection against contingent depreciation in market value.

The following is a summary of the changes in security holdings made during the year.

Acquired	
$5,000	U. S. Treasury 2s 1954–52
75 shares	Consumers Power Co. com. pfd. $4.52
100 shares	Continental Oil Co. com.
3 shares	Texas Co. com.
200 shares	Union Carbide and Carbon Corp. cap.
1 share	Standard Oil Co. of New Jersey cap.
Sold	
$5,000	U. S. Treasury 2 1/2s 1972–67
100 shares	Union Carbide and Carbon Corp. cap.
200 shares	Consolidated Edison Co. of New York Inc. com.
20 shares	Texas Co. com.
50 shares	Timken Roller Bearing Co. cap.
1 share	Standard Oil Co. of New Jersey cap.

The investment portfolio, valued at market November 30, 1948, now includes Government bonds 36.3 per cent, other bonds 8.1 per cent, preferred stock 15.4 per cent, common stock 35.0 per cent, cash in savings banks 5.2 per cent.

Income from Investments

Income received during the year from investment of Current Funds amounted to $2,717. This represents a return of 3.0 per cent computed on average book value of investments. Income on Invested Funds amounted to $8,437, representing a return of 4.3 per cent. Total investment income from all sources was $11,154, representing a return over 3.9 per cent. These rates of return are higher than in 1947.

Income from the Henderson Estate was $4,990; in 1947 it was $4,850.

Changes in Net Assets

Net assets increased by $3,756 during the year. That this showing is somewhat better than had been anticipated is to be attributed principally to record-breaking sales of publications, to receipts from dues and initiation fees amounting to $3,000 more than estimates, and to increased yield from investments. Included in receipts is the sum of $1,000 from the estate of J. K. Whittemore. Navy Contract N7 onr 429 provided $22,500 to cover editorial expenses of Mathematical Reviews for June 20, 1947 through October 20, 1948. About $7,000 more will be received under this contract before it expires in March, 1949. Without this item, net assets would have shown a decrease of nearly $19,000.

Included in assets is the sum of $480 due the Society under Navy Contract N8 onr 553 for translation of papers from the Russian and other unfamiliar languages. It has seemed advisable thus to treat expenses paid by the Society and billed under this contract, although it is not our present practice to list other accounts receivable as assets nor accounts payable as liabilities.

During the year the account for International Congress was restored to active status by direction of the Board.

Decrease in Surplus

Surplus account shows a decrease of $5,349, somewhat less than in 1947. During the year it was necessary to transfer $10,000 from Surplus to Colloquium account in order to pay for the heavy volume of printing and reprinting currently under way.

It should be noted also that Surplus account reflects the dues and initiation fee income resulting from continued rapid growth of membership, as well as the additional income resulting from the new dues rates that became effective on January 1, 1948.

Increase in Inventory

In interpreting the report and the preceding remarks, cognizance should be taken of the unusual situation with regard to inventory. Because change in inventory tends in normal years to be relatively small and quite steady, it has not been the practice of the Society to include an inventory account in financial statements. On December 1, 1947, stocks of books were unusually low. During the year, printing expenses for Colloquium account amounted to $25,500. For 1941–1947 inclusive they totalled only about $14,000. A study has therefore been made of the change in inventory of books, not including periodicals, resulting from the year's activities. It appears that, valuing inventory at prices to members and allowing for handling charges, there was an increase in inventory from about $11,000 at the beginning of the year to about $37,000 at the end. No corresponding study was made for periodicals, but it is probable that there was some increase here also. This increase to a considerable extent balances the above mentioned somewhat disturbing change in net assets. For, while to some extent the financial soundness of the Society depends on the time required to reconvert inventory into cash, the current volume of sales is, as already noted, very high, so that there is no immediate cause for anxiety in the increased inventory.

At the same time it is necessary to be watchful that suitable sources of income are obtained to cover the steadily rising costs for the Bulletin, Transactions, and Mathematical Reviews, and for salaries.

In concluding this last letter of transmission which I shall have the privilege of writing, it may be helpful to give a brief indication of the tremendous growth of the Society during the past eleven years as reflected in its finances by citing side by side, without comment, a few figures from the Treasurer's reports for 1937 and for 1948.

	1937	1948
Income		
Dues and Initiation Fees—Individuals	$14,500	$31,700
Dues—Institutions	6,400	9,600
Investment Income	4,300	16,100
. . .	. . .	. . .
Total Income Excluding Income from Sales of Publications	$26,800	$ 84,200
Sales of Publications	9,700	51,400
Total Income	$36,500	$135,600

Expenses		
General Expenses—Offices, Officers, Salaries, Etc.	$ 14,400	$ 54,300
Printing	14,200	75,200
. . .	. . .	. . .
Total Expenses	$ 31,200	$132,000
Total Assets (Not Including Inventories)	$130,800	$286,900

Finally, may I be permitted to express here the pleasure and satisfaction which have accompanied the work of the treasurership during my rather long tenure, and which have resulted from the unfailing and cordial cooperation of the other officers and of the employees of the Society. Especially has the work of Miss Evelyn M. Hull as Office Manager been indispensable, for without her excellent judgment and thorough familiarity with all the details of the finances, the difficulties of the treasurer would have been nearly insuperable.

And to the members of the Board, past and present, I offer sincere gratitude not only for the joy of working together but for their repeated expressions of personal appreciation.

Respectfully submitted,
BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

	November 30, 1948	November 30, 1947
Assets		
CURRENT FUNDS:		
Cash	$ 16,292.01	$ 16,315.50
Account Receivable from United States Government	480.21	
Investments	77,500.19	70,559.50
	$ 94,272.41	$ 86,875.00
INVESTED FUNDS:		
Cash	$ 531.01	$ 1,287.90
Investments	192,102.21	194,987.15
	$192,633.22	$196,275.05
TOTAL ASSETS	$286,905.63	$283,150.05
Liabilities		
CURRENT FUNDS:		
Mathematical Reviews	$ 13,992.32	$ 1,897.50
Colloquium	10,813.01	11,708.12
Mathematical Surveys	3,787.21	3,734.70
Symposia on Applied Mathematics	2,595.25	2,700.00
Birkhoff Memorial Project	3,131.68	3,061.68
International Congress	5,484.84	6,413.49
Committee on Aid to Devastated Libraries	128.96	1,320.50
Policy Committee	97.64	
Prize Funds and Other Special Funds Accumulated Income	8,238.39	6,848.12
Reprinting Funds	6,847.71	6,165.74
Sinking Fund	1,156.05	1,044.54
Profit on Sales of Securities	2,032.98	982.60
Miscellaneous	617.50	300.45
Surplus	35,348.87	40,697.56
	$ 94,272.41	$ 86,875.00
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 71,000.00
Prize Funds and Other Special Funds	33,033.22	32,033.22
Life Membership and Subscription Reserve	3,073.09	3,407.61
Mathematical Reviews	65,000.00	65,000.00
Colloquium		5,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	16,141.02	15,448.33
	$192,633.22	$196,275.05
TOTAL LIABILITIES	$286,905.63	$283,150.05

SUMMARY STATEMENT OF INCOME AND EXPENDITURES
1947–1948

	1948		1947	
	Receipts	Disbursements	Receipts	Disbursements
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$29,120		$21,628	
Dues—Contributing Memberships	712		932	
Dues—Institutional Memberships	9,582		8,012	
Initiation Fees	1,880		1,195	
Investment Income	11,953		11,056	
Miscellaneous	191		288	
GENERAL DISBURSEMENTS:				
Secretaries		$ 13,395		$11,301
Treasurer		1,182		1,470
Librarian		2,132		1,478
Committee Expense		831		
Editorial Office Expense		736		305
Office Remodeling		55		
Furniture and Fixtures		750		609
Miscellaneous		103		65
Total	$53,438	$19,184	$43,111	$15,228
Excess of General Receipts	$34,254		$27,883	
PUBLICATIONS:				
Bulletin	$ 4,318	$ 23,950	$ 3,180	$21,940
Transactions	8,647	15,490	7,398	13,065
Bulletin and Transactions—Back Volumes	6,156	5,474	2,363	1,978
Mathematical Reviews	46,641	34,546	19,818	25,310
Colloquium Publications	22,471	28,366	8,135	5,923
Mathematical Surveys	263	211	2,489	551
Semicentennial Publications	114	2	93	9
Proceedings Symposia on Applied Mathematics		105	2,700	
Birkhoff Papers	70		3,225	164
Annals of Mathematics				1,000
American Journal				2,250
Total	$88,680	$110,144	$49,401	$72,190
Excess Cost of Publications		$ 21,464		$22,789
OTHER:				
Policy Committee for Mathematics	$ 675	$ 577	$ 1,500	$ 2,452
International Congress		929	95	4
Prize and Other Special Funds	2,390		2,257	
Committee on Aid to Devastated Libraries	20	1,212	2,068	748
Profit on Sales of Securities	1,743		1,904	
Miscellaneous	317		141	
Appropriations:				
Policy Committee	39	500		
Mathematical Reviews		1,000		1,000
Colloquium		10,000		
Symposia on Applied Mathematics				2,700
Committee on Aid to Devastated Libraries				1,500
Mathematical Surveys				2,000
Total	$ 5,184	$ 14,218	$ 7,965	$10,404
Difference		$ 9,034		$ 2,439
Net Change in Assets	$ 3,756		$ 2,655	
ASSETS BEGINNING OF YEAR	$283,150		$280,495	
ASSETS END OF YEAR	$286,906		$283,150	

BOOK REVIEWS

Finite differences and difference equations in the real domain. By Tomlinson Fort. London, Oxford University Press, 1948. 7+251 pp. $8.00.

The subject-matter of this book, as the title indicates, falls into two general categories; roughly half of the book is devoted to each of them.

In the first category, *finite differences*, are the following topics: difference operators and their elementary properties; the problem (in its most elementary form) of the *sum* of a function; the Bernoulli polynomials and numbers, and the Euler-Maclaurin and Euler summation formulas, and generalizations of all these concepts (the method of generalization is a brief and very inclusive one due to the author); numerical differentiation; interpolation formulas; numerical integration.

The second category, *difference equations in the real domain*, is made up almost exclusively of topics in the theory of the linear *recurrent relation*, that is, of the linear difference equation where the domain of the independent variable is a set of *integers*. The section of the book devoted to these topics consists in large part of published and unpublished research work of the author. For the homogeneous nth-order linear recurrent relation the concept of *fundamental system* of solutions is studied, and for the nonhomogeneous equation the method of variation of constants is described; special techniques for the case of constant coefficients are treated. Study is made of the question of the number of linearly independent solutions of that difference system which consists of a linear recurrent relation together with n linear boundary conditions, and the concept of a *Green's function* for an incompatible system is introduced.

For the *second order* linear recurrent relation, with coefficients dependent on a parameter, a Sturm-Liouville theory is developed. The manner in which the rapidity of oscillation of the solutions varies with the parameter is studied; the existence of characteristic values and characteristic functions is proved; the orthogonality of characteristic functions (orthogonality defined in terms of finite sums instead of integrals) is shown; and the expansion of an arbitrary function in terms of characteristic functions is obtained. Application is made to the problem of the vibration of a mass-less string loaded with n particles, and the vibration of a material string is studied as a limiting case. For second-order recurrent relations with periodic coefficients the question of the existence of periodic solutions is considered in itself and also in connection with the Sturm-Liouville theory. In addition the problem of the existence of unbounded solutions is treated.

One chapter of the book is devoted to the finite analogue of the calculus of variations; a necessary condition analogous to the Euler equation is derived, and sufficient conditions are obtained also.

The final chapter provides a very brief summary (totalling only 14 pages) of some topics in the theory of linear difference equations with *continuous* independent variable. Among these topics are the gamma, digamma, trigamma functions, and so on, the Nörlund sum, and the adjoint equation.

This book is a welcome addition to the literature. In the first place, its presentation of the elements of the calculus of finite differences, being compact and clean-cut, and augmented by numerous exercises, makes it a serviceable and attractive text for a course in finite differences on about the first-year graduate level. In the second place, its treatment of the theory of linear difference equations emphasizes various interesting aspects of which there has been hitherto no unified account.

WALTER STRODT

The differential geometry of ruled surfaces. By Ram Behari. (Lucknow University Studies, no. 18.) Lucknow University, 1946. 94 pp.

This book is the basis of a series of Extension Lectures on *The differential geometry of ruled surfaces in euclidean space of three dimensions*, delivered by the author at the Lucknow University during 1942.

It is a concise and comprehensive survey of the differential geometry of ruled surfaces including ample references to the literature of the subject. The notation adopted is that of ordinary cartesian coordinates as used by Forsyth in his *Lectures on the differential geometry of curves and surfaces*, and by L. P. Eisenhart in his *Differential geometry*, Boston, 1909.

The book consists of six chapters. The first three form an introduction to the subject and the last three contain the investigations of the author.

The first three chapters are concerned with the standard theory of ruled surfaces. Chapter IV is devoted in the first part to the determination of ruled surfaces whose curved asymptotic lines can be found by quadratures. This problem has also been studied by Picard, Buhl, Goursat, Srinivasiengar, Hayashi. A systematic study of the osculating quadrics of a ruled surface is developed. The following new geometric meaning of Laguerre's function L' (see Weatherburn, *Differential geometry of three dimensions*, vol. 2, p. 139) is given: The vanishing of L' along a curve on a surface is the condition that the osculating quadrics Q of the ruled surface formed by drawing normals to the given surface along the curve be equilateral (three generators of Q are mutually orthogonal). This fourth chapter is concluded with the discussion of four new theorems concerning ruled surfaces with equilateral osculating quadrics.

Chapter V begins with the theory of applicability and continuous deformations of general surfaces. The famous theorem of Gauss and the problem of Minding are discussed. The general problem of infinitesimal deformations of surfaces, which has been developed by Goursat, Darboux, Haag, is considered together with applications to ruled surfaces. Some theorems of Bonnet and Beltrami relating to the applicability of ruled surfaces are proved. A new invariant is obtained under any deformation of a ruled surface into a ruled surface.

Chapter VI deals in the first part with the general theory of a rectilinear congruence. The differential geometry of the ruled surfaces of the congruence through a fixed ray L is developed. Various proofs are given of the theorem of K. Ogura that repeated applications of forming the jacobian with respect to the differentials, starting with Sannia's two quadratic forms f and ϕ, lead only to five distinct families of ruled surfaces through a line L of the congruence.

In the second part of Chapter VI applications are made to normal congruences of lines. In a general rectilinear congruence there are ∞^2 ruled surfaces with equilateral osculating quadrics, but only ∞^1 in a normal congruence.

In the third and final part is studied the pitch p at a ray of a pencil of the congruence. This is the curvilinear integral $p=\int_C R dr$, where R is the unit vector parallel to the ray through the point of a simple closed curve C whose position vector is r and where C is on the director surface S through the ray L. This was first studied by E. Cartan. The following three theorems which reduce to the corresponding theorems of Malus-Dupin, Beltrami, Ribaucour by letting $p=0$, so that the congruence becomes normal, are established.

I. The pitch p at a ray of a pencil of the congruence formed by the incident rays remains unaltered by refraction except for a factor which is equal to the ratio of the refracting indices of the two media. The pitch associated with the congruence formed by incident rays remains unaltered by reflection.

II. If the surface of reference S be deformed in such a way that the direction of the lines of the congruence with respect to S be unchanged, then the pitch p is unaltered.

III. If tangent planes be drawn through the rays of a congruence of any surface, the pitch p remains unaltered if the surface be deformed in any manner carrying the rays in its tangent planes.

Also is discussed the limiting value of the pitch, $dp/d\sigma$, which has interesting applications, in particular to the Anormalita of Levi-Civita.

This book is very readable and can be easily understood by any student who has had a first year course in Differential Geometry. In the opinion of the reviewer, this is a worthwhile addition to the library of any one who is interested in the classical theory of surfaces.

JOHN DECICCO

Methods of algebraic geometry. By W. V. D. Hodge and Daniel Pedoe. Cambridge University Press, 1947. 8+440 pp. $6.50.

This work by two disciples of H. F. Baker naturally retains some of the flavor of the latter's *Principles of geometry*; but in keeping with the modern trend it is more algebraic and less geometrical. The spirit of the book is indicated by the fact that there is no mention of order or continuity. The first four of the nine chapters are concerned with algebraic preliminaries, chiefly in preparation for vol. II, and are so clear and concise that they would serve very well as an introduction to modern algebra, quite apart from their application to geometry. The topics treated in this part include groups, rings, integral domains, fields, matrices, determinants, algebraic extensions, and resultant-forms. The theory of linear dependence is developed without assuming commutativity of multiplication, and there is a neat algebraic treatment of partial derivatives and Jacobians.

Analytic geometry of projective n-space is taken up in Chapter V. A point of right-hand projective number space is defined as a set of right-hand equivalent $(n+1)$-tuples of elements of a given field, not necessarily commutative; and right-hand projective space is defined as a set of elements which can be put in one-to-one correspondence with the points of such a number space by means of any one of a certain set of "allowable" coordinate systems. A linear subspace is defined as the set of points which are linearly dependent on $k+1$ linearly independent points; and the Propositions of Incidence follow readily. The notation of Möbius' barycentric calculus, as developed by Baker, arises naturally at this stage, and is used in proving Desargues' Theorem for coplanar triangles. Quadrangular constructions are given for the points $O+U(\alpha+\beta)$ and $O+U\alpha\beta$ as derived from points O, U, $O+U$, $O+U\alpha$ and $O+U\beta$, and it is shown that Pappus' Theorem is equivalent to the commutative law $\alpha\beta=\beta\alpha$. In all such work, great care is taken to deal with all the special cases that may arise when points with different names coincide.

Chapter VI is an interlude on synthetic geometry, where linear spaces and incidence are primitive concepts, and the Propositions of Incidence are axioms. After constructing F. R. Moulton's non-Desarguesian plane geometry, the authors introduce Desargues' Theorem as an extra axiom to be used when the number of dimensions is only 2. A projectivity is defined as a product of perspectivities, and it is proved that any such chain of perspectivities can be reduced to as few as three. This is always an awkward piece of work, and the present version is as neat as any. (§§4 and 5 could perhaps have been shortened by first proving the uniqueness of the sixth point of the general quadrangular set.) The algebra of points on the projective line is developed in the manner of Veblen and Young, but with some improvements of detail. Homogeneous coordinates in n dimensions are introduced in a way that most ingeniously avoids any appeal to the Fundamental Theorem. Then various restrictions are considered. Pappus' Theorem is taken as a ninth axiom, and Desargues' Theorem is deduced from it as in Baker's *Introduction to plane geometry* (Cambridge, 1942, p. 26). Finite geometries are ruled out by a tenth axiom to the effect that a parabolic projectivity cannot be periodic.

Chapter VII contains an elegant exposition of Grassmann coordinates, generalizing the familiar properties of line-coordinates in 3-space. In the two final chapters, collineations and correlations are defined as linear transformations and are classified with characteristic thoroughness. The chapter on correlations is particularly valuable, as the authors have not shirked the formidable task of enumerating the various canonical forms for n dimensions. This last chapter also brings into prominence the skill of the compositors of the Cambridge University Press, which is to be congratulated on producing such a fine book in these difficult times.

H. S. M. COXETER

Set functions. By Hans Hahn and Arthur Rosenthal. The University of New Mexico Press, 1948. 9+324 pp. $12.50.

When Hans Hahn died in 1934, he left manuscripts for the second volume of his treatise on real function theory. Arthur Rosenthal has now completed and edited that part of the work which deals with the theory of measure. This book gives a scholarly presentation of the foundations of the subject, taking account both of the theory of completely additive functions on σ-fields and of the theory of Carathéodory.

The first chapter deals with the basic properties of additive and totally additive set functions, zero sets and complete fields, and the decomposition into regular and singular parts. Chapter II is devoted to the Carathéodory theory of measure. Special attention is paid to regular measures, and to n-dimensional Lebesgue measure. In the third chapter the author discusses the properties of measurable functions and sequences of such functions. The theory of integration is developed in Chapter IV by characterizing the indefinite integral as a new measure satisfying certain inequalities. A discussion of the approximation of integrals by sums, some mean value theorems, and convergence theorems, is followed by a section on product measures and the Fubini theorem. The last chapter deals with the Vitali covering theorem, the differentiation of measures and interval functions, and some applications to density and approximate continuity.

The book is carefully written and systematic. The proofs are given in great detail, a fact which may help many who wish to become acquainted with the fundamentals of measure theory.

HERBERT FEDERER

Sur les groupes classiques. By Jean Dieudonné. (Actualités scientifiques et industrielles, no. 1040; Publications de l'Institut de Mathématiques de l'Université de Strasbourg. VI.) Paris, Hermann, 1948. 82 pp.

The main purpose of this little book is to obtain the structural properties of the classical groups which can at present be obtained by purely algebraic methods. By skillful organization, complete mastery of his subject, and constant adherence to the "conceptual" point of view so fruitfully introduced into linear algebra in modern times, the author achieves this purpose with simplicity, efficiency, and elegance. The results are, with some exceptions, either old ones (to be found in the pioneering works of L. E. Dickson), or extensions of old ones to more general situations. But the long complicated matrix computations of the older literature, in which the ideas are frequently buried beyond recall, are here almost entirely replaced by conceptual arguments expressed in geometric language which brings out for inspection the intuitive geometric motivation in the proofs.

The classical groups are the full linear groups $GL_n(K)$, the symplectic groups $Sp_n(K)$, the orthogonal groups $O_n(K, f)$, and the unitary groups $U_n(K, f)$. The full linear groups over an arbitrary skew field K have been previously treated by the author (Bull. Soc. Math. France vol. 71 (1943) pp. 27–45), and therefore are not considered in the present book. The symplectic and orthogonal groups are defined over a completely arbitrary field K, and the unitary groups either over a separable extension K of degree 2 over an arbitrary field or over a non-commutative skew field K possessing an involutorial anti-automorphism. Most of this great generality comes gratis with the proofs; but a large proportion of pages is required to clear up certain "marginal" cases (characteristic 2, fields with 3 elements, and so on).

The symplectic groups prove to be the easiest. A bilinear form f on a vector space E of dimension n over K is *alternating* if $f(x, x)=0$ identically; the rank of such an f is always even. Given such a form of rank n (so that $n=2m$), a linear transformation u of E onto E is *symplectic* if $f(ux, uy)=f(x, y)$ identically; the group of all symplectic transformations of E is $Sp_n(K)$. It is always possible to reduce to the case $f(x, y)=\sum_{i=1}^{m}(\xi_i\eta_{m+i}-\xi_{m+i}\eta_i)$ by choosing a suitable base of E; therefore up to isomorphism $Sp_n(K)$ does not depend on the particular choice of the alternating form f. The structure of $Sp_n(K)$ is determined by proving: (a) the center of $Sp_n(K)$ consists of the identity transformation I and $-I$ (easy); (b) except when K has 2 elements and $n=2$ or 4, or when K has 3 elements and $n=2$, the factor group of $Sp_n(K)$ by its center is simple. The proof of (b) largely depends on the one hand on the fact that $Sp_2(K)$ is identical with the unimodular group $SL_2(K)$ which, by the author's above-mentioned paper on $GL_2(K)$, is known to be simple modulo its center except when the number of elements in K is 2 or 3, and on the other hand on the author's theorem that every symplectic transformation is a product of *symplectic transvections*, that is, mappings of the form $x\rightarrow x+\lambda f(x, a)$ (where $\lambda\in K$, $a\in E$, $f(a, a)=0$).

The structure of the orthogonal groups proves to be more complicated. In the case of field characteristic other than 2, a symmetric bilinear form f on E uniquely determines a quadratic form g, and conversely. A transformation u of E onto E is said to be *orthogonal* if $g(ux)=g(x)$ (or equivalently $f(ux, uy)=f(x, y)$). The group of all such orthogonal transformations is $O_n(K, f)$. In general, two quadratic forms are not equivalent, so that it is necessary to specify the particular f used. A vector subspace V of E is *totally isotropic* if $f(x, y)=0$ for all x and y in V. The maximum dimension ν of a totally isotropic subspace is the *index* of f. (This index, introduced by E. Witt (J. Reine Angew. Math. vol. 176 (1937) pp. 31–44), in an ordered field becomes the smaller of the two numbers s and $n-s$, where s is the index of inertia of g.) In the case of $Sp_n(K)$ the integer analogous to ν is $n/2$; much of the complication surrounding $O_n(K,f)$ seems to arise from the fact that ν can assume any value subject to $0 \leqq \nu \leqq n/2$. When $n \geqq 5$ and $\nu \geqq 1$ the structure can be described as follows: (a) the group $O_n^+(K, f)$ of all orthogonal transformations with determinant unity is a normal subgroup of $O_n(K,f)$ of index 2; (b) the commutator group $\Omega_n(K,f)$ of $O_n(K,f)$ is a normal subgroup of $O_n^+(K,f)$ with factor group isomorphic with a subgroup of K^*/Q, where K^* is the multiplicative group of nonzero elements of K and Q is the group of all squares of elements of K^*; (c) the factor group $\Omega_n(K,f)/Z_n \cap \Omega_n(K,f)$, where Z_n is the center of $O_n(K,f)$, is simple; (d) $Z_n = \{I, \ -I\}$. The proof depends on a theorem on quadratic forms due to Witt (loc. cit.), on certain detailed information concerning $O_n(K, f)$ for $2 \leqq n \leqq 6$ quoted from B. L. van der Waerden's *Gruppen von linearen transformationen* (Berlin, 1935), and on a sequence of interesting propositions concerning orthogonal equivalence of subspaces of E and sets of generators for $O_n(K, f)$, $O_n^+(K, f)$, and $\Omega_n(K, f)$. When $\nu = 0$, however, the structure of $O_n(K, f)$ evidently depends in a deep way on the nature of the particular field K, and the author confines himself to examples illustrating this fact. When K is the field of real numbers and f is definite (so that $\nu = 0$) then (for $n > 2$) $\Omega_n(K, f) = O_n^+(K, f)$ and $O_n^+(K, f)/Z_n$ is known to be simple; but when K is the field of formal power series $\sum_{k=h}^{\infty} \lambda_k t^k$ with real coefficients then the author shows that $O_n(K,f)$ contains a sequence $O_n(K,f) = G_0 \supset G_1 \supset \cdots$ of normal subgroups such that $\bigcap_{i \geqq 0} G_i = \{I\}$ and each factor group G_{i-1}/G_i is abelian.

The case of field characteristic 2 presents special problems. A quadratic form g is defined as a mapping g of E into K such that $g(\lambda x + \mu y) = \lambda^2 g(x) + \mu^2 g(y) + \lambda\mu f(x, y)$ identically, where f is a bilinear form on E. f is then alternating and therefore is of even rank $2p \leqq n$; the set E^* of all vectors x such that $f(x, y) = 0$ for all y in E is a vector space of dimension $n - 2p$; the set E_0 of all vectors x in E^* such that $g(x) = 0$ is a vector space of dimension say $q \leqq n - 2p$. g is called *regular* if $q = 0$; only regular quadratic forms are considered. The index ν is now defined as the maximum dimension of a totally singular subspace of E, a vector subspace V being totally singular if $g(x) = 0$ for all x in V. $O_n(K, g)$ is now defined as the group of all linear transformations u of E onto E such that $g(ux) = g(x)$ identically. When $n - 2p = 0$ (*defect* 0), $n \geqq 6$, and $\nu \geqq 1$, and when $n - 2p > 0$ (*defect*>0), $2p \geqq 2$, $\nu \geqq 1$ (with the possible exception of the case $2p = 4$, $\nu = 2$), the author proves that the commutator group $\Omega_n(K, g)$ is simple. Moreover, he shows that $O_n(K, g)/\Omega_n(K, g)$ is, in the case of defect 0, isomorphic with a subgroup of K^*/Q, as in the case of characteristic not 2, or with the product of such a subgroup and a cyclic group of order 2. Examples are given which show that when $\nu=0$ the structure need not be so simple.

Turning to the unitary groups the author considers two very general types. To define the first type, consider a separable extension K of degree 2 over an arbitrary field K_0. By means of the unique automorphism $\xi\to\bar{\xi}$ of K over K_0, the concept of ***hermitian symmetric form*** is introduced: A mapping f of $E\times E$ into K is an hermitian symmetric form if $f(x+x', y)=f(x, y)+f(x', y)$, $f(x, y+y')=f(x, y)+f(x, y')$, $f(\lambda x, \mu y)=\bar{\lambda}\mu f(x, y)$, $f(y, x)=f(x, y)$ identically. Given a symmetric hermitian form f of rank n, a linear transformation u of E onto E is ***unitary*** if $f(ux, uy)=f(x, y)$ identically, and the group of all such unitary transformations is $U_n(K, f)$. For the second type of unitary group the starting point is a ***reflexive*** non-commutative skew field K, that is, a skew field K distinct from its center K_0 for which there exists an involutorial antiautomorphism $\xi\to\bar{\xi}$ relative to K_0 such that $\xi+\bar{\xi}$ and $\xi\bar{\xi}$ belong to K_0 for every ξ in K. A reflexive skew field is always of rank 4 over its center (for characteristic not 2 is a generalized quaternion skew field). If E is a right vector space over K then a mapping f of $E\times E$ into K is an ***hermitian symmetric form*** on E if $f(x+x', y)=f(x, y)+f(x', y)$, $f(x, y+y')=f(x, y)+f(x, y')$, $f(x\lambda, y\mu)=\bar{\lambda}f(x, y)\mu$, $f(y, x)=\overline{f(x, y)}$ identically, and $f(x, x)$ always belongs to the center K_0. The definition of $U_n(K, f)$ then proceeds as in the commutative case. For both types the theory develops much as for $O_n(K, f)$, except that characteristic 2 does not cause so much difficulty as before. The index ν is defined in the expected way. In the commutative case, when $n\geqq 2$ and $\nu\geqq 1$, the structure can be described as follows: (a) the set $U_n^+(K, f)$ of unitary transformations with determinant unity is a normal subgroup of $U_n(K, f)$ with factor group isomorphic with the multiplicative group of all elements $\lambda\in K$ such that $\lambda\bar{\lambda}=1$; (b) except when K_0 has 3 elements and $n=2$ and when K_0 has 2 elements and $n=2$ or 3, the factor group $U_n^+(K, f)/Z_n$, where Z_n is the center of $U_n^+(K, f)$, is simple; (c) Z_n consists of those mappings $x\to\lambda x$ for which $\lambda^n=1$ and $\lambda\bar{\lambda}=1$. In the non-commutative case for $n\geqq 2$ and $\nu\geqq 1$, the factor group of $U_n(K, f)$ by its center Z_n is simple, and $Z_n=\{I, -I\}$. In both cases the structure is irregular for $\nu=0$.

E. R. KOLCHIN

NOTES

A statement of editorial policy. Because of the large accumulation of manuscripts and the heavy demand for space and in order to avoid long delay in publication, the editors of the Transactions may require substantial condensation of manuscripts before they are accepted. Authors must present their results in as compact a form as possible and should avoid space consuming exposition and discussion. Long manuscripts will be subject to especial scrutiny on these points by the editors.

The French Academy of Sciences has announced the following awards in mathematics for 1948: Poncelet Prize to Georges Valiron of the University of Paris; Carrière Prize to P. J. Dubreil of the University of Paris; Dickson Prize to Julien Kravtchenko of the University of Grenoble; Grand Prize to Henri Milloux of the University of Bordeaux; Albert I. de Monaco Prize to Jacques Hadamard of the College of France and Polytechnic School; Laplace Prize to François Morin of the Polytechnic School; Becquerel Foundation Prize to André Bloch of the Polytechnic School.

The Naval Ordnance Laboratory aeroballistics facilities at White Oak, Maryland, will be dedicated on June 27, 1949. The dedication will be followed on June 27 to June 29 by technical sessions. These sessions will, in turn, be followed by a meeting of the Fluid Dynamics Division of the American Physical Society on June 30 and July 1.

Professor R. E. Gaines of the University of Richmond has retired with the title emeritus.

Associate Professor E. H. C. Hildebrandt of Northwestern University has been elected president of the National Council of Teachers of Mathematics.

Professor S. W. Reaves of the University of Oklahoma has retired with the title emeritus.

Professor K. Ananda-Rau of Presidency College, Madras, India, has retired.

Professor C. L. da Silva Dias of the University of São Paulo, who is on leave of absence, has been appointed research associate at the University of Chicago.

Dr. Joseph Gillis of Sunderland, England, has been appointed senior assistant at the Weizmann Institute, Rehovot, Israel.

Assistant Professor Leopoldo Nachbin of the University of Brazil, who is on leave of absence, has been appointed research associate at the University of Chicago.

Professor H. A. Pérsico of the University of La Plata has been appointed to a professorship at the University of Cuyo, San Luis, Argentina.

Mr. Å. V. C. Pleijel of Lund University has been appointed to a professorship at the Royal Institute of Technology, Stockholm, Sweden.

Dr. C. J. Blackall has been appointed to an associate professorship at the University of Toledo.

Mr. S. K. Bright of Vanderbilt University has been appointed to a professorship at Austin Peay State College, Clarksville, Tennessee.

Associate Professor Jack Britton of the University of Colorado has been appointed to a visiting professorship at the University of Michigan.

Mr. K. V. Casey of Brown University has accepted a position as statistician with the Engineer Analysis and Fiscal Office, Wright-Patterson Air Force Base, Wright Field, Ohio.

Professor H. S. M. Coxeter of the University of Toronto, who is on leave of absence, has been appointed to a visiting professorship at Barnard College, Columbia University.

Professor M. W. Dehn of Black Mountain College has been appointed to a visiting professorship at the University of Wisconsin.

Mr. H. E. Ellingson of the State University of Iowa has accepted the position of physicist at the Rosemount Research Center of the University of Minnesota.

Assistant Professor H. J. Greenberg of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Mr. R. T. Gregory of the Iowa State College of Agriculture and Mechanic Arts has accepted a position as mathematician at the United States Naval Proving Ground, Dahlgren, Virginia.

Mr. D. L. Herr of the Allen Bradley Company has accepted a position as project engineer with the Reeves Instrument Corporation, New York, New York.

Dr. L. G. F. Jones of the General Precision Laboratory, Incorporated, Pleasantville, N. Y., has accepted a position as engineer with the Eckert-Mauchly Computer Corporation, Philadelphia, Pennsylvania.

Assistant Professor J. A. Krumhansl of Brown University has been appointed to an assistant professorship at Cornell University.

Dr. Cornelius Lanczos of Boeing Aircraft Company has accepted an appointment as staff mathematician with the National Bureau of Standards, Washington, D. C.

Professor W. G. Madow of the University of North Carolina has been appointed to a professorship of mathematical statistics at the University of Illinois.

Dr. R. J. Marcou of Raytheon Manufacturing Co., Waltham, Massachusetts, has been appointed to a professorship at Boston College.

Dr. W. L. Mitchell of the Bell Aircraft Corporation has accepted the position of senior aerodynamicist in the Research and Development Division of New Mexico School of Mines.

Associate Professor E. W. Montroll of the University of Pittsburgh, who is on leave of absence, has been appointed head of the Physics Branch, Office of Naval Research, Washington, D. C.

Dr. T. F. Morris of the University of Toronto has been appointed lecturer at Carleton College, Ottawa, Ontario, Canada.

Professor Tibor Rado of the Ohio State University has been appointed Research Professor in the first appointment to this position since its creation at Ohio State University in 1946.

Mr. A. E. Roberts has accepted a position as mathematician with Engineering Research Associates, Inc., Arlington, Virginia.

Miss Vivian Spurgeon of Southwest Baptist College has been appointed to an assistant professorship at Ouachita College, Arkadelphia, Arkansas.

Mr. F. W. Thalgott of the Clinton Laboratories, Oak Ridge, Tennessee, has accepted a position as associate mechanical engineer with the Argonne National Laboratory, Chicago, Illinois.

Professor H. S. Tsien of the Massachusetts Institute of Technology has been appointed to a Goddard professorship at the California Institute of Technology to head a jet propulsion center.

Associate Professor E. W. Vickery of Louisiana State University and Agricultural and Mechanical College has accepted a position as senior mathematician at the Naval Ordnance Test Station, China Lake, California.

Mr. N. W. Wells of the Agricultural and Mechanical College of Texas has been appointed to an assistant professorship at Sam Houston State Teachers College.

Professor L. C. Young of the University of Cape Town has been appointed to a professorship at the University of Wisconsin.

The following promotions are announced:

T. M. Apostol, University of California, to the position of lecturer.

Samuel Borofsky, Brooklyn College, to an associate professorship.

J. L. Hodges, University of California, to the position of lecturer.

Banesh Hoffmann, Queens College, to an associate professorship.

W. C. Hubert, City College, New York City, to a professorship.

Jennie P. Kormes, Brooklyn College, to an assistant professorship.

S. S. McNeary, Drexel Institute of Technology, to an associate professorship.

The following appointments to instructorships are announced: Duke University: Mr. M. P. Jarnagin; Georgetown University: Mr. M. W. Oliphant; University of Illinois, Chicago Undergraduate Division: Miss Lona L. Turner; University of Missouri: Mr. R. A. Mauller; University of Virginia: Mr. R. R. Bernard, Mr. H. F. DeFrancesco, Mr. W. E. Pace.

Associate Professor R. E. Byrne of the California Institute of Technology died September 17, 1948 at the age of thirty-seven years.

Associate Professor Emeritus Frank Irwin of the University of California died December 25, 1948, at the age of eighty years. He had been a member of the Society for forty years.

Associate Professor Emeritus J. W. A. Young of the University of Chicago died October 26, 1948. He had been a member of the Society for more than fifty years.

NEW PUBLICATIONS

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BOOLE, G. *The mathematical analysis of logic.* Oxford, Blackwell, 1948. 82 pp. 7s. 6d.

CHURCHILL, R. V. *Introduction to complex variables and applications.* New York, McGraw-Hill, 1948. 6+216 pp. $3.50.

KNOPP, K. *Problem book in the theory of functions.* Vol. I. Trans. by L. Bers. New York, Dover, 1948. 8+126 pp. $1.85.

STREETER, V. L. *Fluid dynamics.* New York, McGraw-Hill, 1948. 11+263 pp. $5.00.

WILLERS, F. A. *Practical analysis.* Trans. by R. T. Beyer. New York, Dover, 1948. 10+422 pp. $6.00.

BULLETIN

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BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, a President Elect (during the even-numbered years only), an Ex-president (during the odd-numbered years only), three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer and a Librarian.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of five Trustees elected by the Society in accordance with Article VII.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the financial business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

EDITORIAL COMMITTEES

Section 1. There shall be six Editorial Committees—one of four members for the *Bulletin*, one of three members for the *Transactions*, one of three members for the *Mathematical Reviews*, one of three members for the *Colloquium Publications*, one of three members for the *Mathematical Surveys*, and one consisting of three representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large and the following ex officio members: the officers of the Society specified in Article I, the members of the Editorial Committees specified in Article III, any former Secretary for a period of two years following his term of office, and members of the Executive Committee who remain on the Council by the operation of Article VII, Section 4. Former Presidents and former Secretaries belonging to the Council in 1948 shall continue as members of the Council during the full terms specified in the By-Laws in effect on January 1, 1948, namely, life membership for former Secretaries who have served ten years or more and six years for former Presidents after completion of their terms of office.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 4. All members of the Council shall be voting members. The method for settling matters before the Council at any meeting shall be by majority vote of the members present. If the result of a vote is challenged, it shall be the duty of the presiding officer to determine the true vote by a roll call. In a roll call vote, each Council member shall vote only once (although he may be a member of the Council in several capacities), and he shall state before the vote in which capacity he votes. The group consisting of the four Associate Secretaries shall have one vote, and it shall be divided equally among those who vote as Associate Secretaries. Each of the six Editorial Committees shall have one vote, and it shall be divided equally among those who vote as members of the respective Editorial Committees. All other members of the Council shall have one vote each. Fractional votes shall be counted.

Section 5. Any group of members of the Council who have a total of five votes as defined in Section 4 of this Article shall constitute a quorum for the transaction of business at any meeting of the Council.

Section 6. Between meetings of the Council, business may be transacted by a mail vote. Votes shall be counted as in the case of a roll call vote as specified in Section 4 of this Article, "members present" being replaced by "members voting." An affirmative vote by mail on any proposal shall be declared if and only if (a) more than half of the total number of possible votes is received by the time announced for the closing of the polls, and (b) at least three-quarters of the votes received by then are affirmative. If members who have a total of five or more votes request postponement at the time of voting; action on the matter at issue shall be postponed until the next meeting of the Council, unless either (1) at the discretion of the Secretary, the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated or (2) the Council places the matter at issue before the Executive Committee for action.

Section 7. The Council may delegate to the Executive Committee (Article V) certain of its duties and powers. Between meetings of the Council the Executive Committee shall act for the Council on such matters and in such ways as the Council may specify. Nothing herein contained shall be construed as empowering the Council to divest itself of responsibility for formulating and administering the scientific policies of the Society.

ARTICLE V

EXECUTIVE COMMITTEE

Section 1. There shall be an Executive Committee of the Council, consisting of four elected members and the following ex officio members: the President, the Secretary, the President Elect (during even-numbered years), and the Ex-president (during odd-numbered years).

Section 2. The Executive Committee of the Council shall be empowered to act for the Council on matters which have been delegated to the Executive Committee by the Council. If three members of the Executive Committee request that any matter be referred to the Council, the matter shall be so referred. The Executive Committee shall be responsible to the Council and shall report its actions to the Council. It may consider the agenda for meetings of the Council and may make recommendations to the Council.

Section 3. Each member of the Executive Committee shall have one vote. An affirmative vote on any proposal before the Executive Committee shall be declared if and only if at least four affirmative votes are cast for the proposal. A vote on any proposal may be determined at a meeting of the Executive Committee, but it shall not be necessary to hold a meeting to determine a vote.

ARTICLE VI

EXECUTIVE DIRECTOR

Section 1. There shall be an Executive Director who shall be a paid employee of the Society. He shall have charge of the central office of the Society, and he shall be responsible for the general administration of the affairs of the Society in accordance with the policies that are set by the Board of Trustees and by the Council.

Section 2. The Executive Director shall be elected by the Council for a specified term and approved by the Board of Trustees. The terms and conditions of his employment shall be fixed by the Board of Trustees.

Section 3. The Executive Director shall work under the immediate direction of the Board of Trustees and of the Council, and in cooperation with the Secretary. He shall attend meetings of the Board of Trustees, of the Council, and of the Executive Committee, but he shall not be a member of any of these bodies.

ARTICLE VII

ELECTION AND TERMS OF OFFICERS

Section 1. The term of office shall be one year in the case of the President Elect and the Ex-president; two years in the case of the Trustees (with the exceptions stated in the next paragraph of this Section), the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. The term of office for elected members of the Executive Committee shall be two years (with the exceptions stated in the last paragraph of this Section), two of the elected members retiring annually. In every case, however, the officials specified in Articles I, II, III, IV, and V, with the exception of the President Elect and the Ex-president, shall continue to serve until their successors have been duly elected and qualified.

The term of office in the case of the Trustees to be elected at the Annual Meeting to be held between the fifteenth of December, 1948, and the fifteenth of January, 1949, shall be three years in the case of two of the Trustees to be elected and shall be two years in the case of three of the Trustees to be elected. Upon the expiration of the terms of office of the Trustees elected at the Annual Meeting to be held between the fifteenth of December, 1948, and the fifteenth of January, 1949, the term of office of all Trustees shall be two years.

In the case of members elected to the Executive Committee at the first election following adoption of these By-Laws the term of office shall be one year for two of the members so elected and two years for the other two. After the expiration of the one-year terms of office, the term of office for all elected members of the Executive Committee shall be two years as specified in the first paragraph of this Section.

Section 2. Election of officers specified in Article I (with the exception of the President and the Ex-president), Trustees, members of the Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member of the Society by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot among the members having the highest number of votes.

Section 3. At the end of his term of office, the President Elect shall become the President. At the end of his term of office, the President shall become the Ex-president.

Section 4. Within fourteen days after the election of officers at the Annual Meeting the Secretary shall send to all members of the Council for a mail vote a ballot containing two names for each place to be filled on the Executive Committee. The nominees shall be chosen by a committee appointed by the President. Members of the Council may vote for persons not nominated. Any member of the Council who is not an ex officio member of the Executive Committee (see Article V, Section 1) shall be eligible for election to the Executive Committee. In case a member is elected to the Executive Committee for a term extending beyond his regular term on the Council, he shall automatically continue as a member of the Council during his term on the Executive Committee.

Section 5. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an ex officio member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 6. If the President of the Society die or resign while a President Elect is in office, the President Elect shall serve as President for the remainder of the year and thereafter shall serve his regular two-year term. If the President of the Society die or resign when no President Elect is in office, the Council, with the approval of the Board of Trustees, shall designate one of the Vice Presidents to serve as President for the balance of the regular presidential term. If the President Elect of the Society die or resign before becoming President, his office shall remain vacant until the next regular election of a President Elect, and the Society shall, at the next Annual Meeting, elect a President for a two-year term. If the Ex-president die or resign before the expiration of his term of office, the Council, with the approval of the Board of Trustees, shall designate a former President of the Society to serve as Ex-president during the remainder of the regular term of the Ex-presidency. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees shall be filled by the Council with the approval of the Board of Trustees. The Council shall fill from its own membership any vacancy in the elected membership of the Executive Committee.

Section 7. If any Trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 8. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VIII

MEMBERS AND THEIR ELECTION

Section 1. Election of members shall be by vote of the Council or of its Executive Committee.

Section 2. There shall be three classes of members, namely, ordinary, contributing and institutional.

Section 3. Application for admission to ordinary membership shall be made by the applicant on a blank provided by the Secretary, and shall be approved either by two ordinary members of the Society or by the institutional member whose nominee he is (see Article IX, Section 6). Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council approved by the Board of Trustees.

Section 4. An ordinary member may become a contributing member by paying the dues for such membership. (See Article IX, Section 3.)

Section 5. A university or college, or a firm, corporation, or association interested in the support of mathematics may be elected an institutional member.

ARTICLE IX

DUES AND PRIVILEGES OF MEMBERS

Section 1. Any person elected to ordinary membership in the Society under the provisions of Article VIII, Section 3, who is not a nominee of an institutional member (See Section 6 of this Article) shall be admitted to membership upon payment, within sixty days of the date of his election, of an initiation fee of five dollars and such dues as are payable under Section 2 of this Article.

Section 2. The annual dues of a person, not a nominee of an institutional member, who is elected to ordinary membership under the provisions of Article VIII, Section 3, shall be ten dollars, with the following exceptions: (1) during the first and second years of membership, the annual dues shall be six dollars; during the third and fourth years of membership, the annual dues shall be eight dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The minimum dues for an institutional member shall be twenty-five dollars. Institutions may pay larger dues.

Section 5. Any member whose annual dues are at least one hundred dollars shall be designated as a Sustainer of the Society; any member whose annual dues are at least five hundred dollars shall be designated as a Patron of the Society.

Section 6. An institutional member which pays less than one hundred dollars dues annually shall, for each twenty-five dollars of dues, receive the *Bulletin* or have the privilege of nominating one person for ordinary membership. An institutional member which pays one hundred dollars or more annual dues shall receive the *Bulletin*, the *Transactions*, and *Mathematical Reviews*, and also shall have the privilege of nominating five persons for ordinary membership for each one hundred dollars of dues and one additional person for each additional twenty-five dollars of dues.

Section 7. A person elected to ordinary membership as a nominee of an institutional member under the provisions of Section 6 of this Article and of Article VIII, Section 3, shall be admitted to membership immediately and shall not be required to pay an initiation fee. He shall not be required to pay dues so long as it is agreed that he is designated as a nominee of the institutional member under these provisions. If a nominee of an institutional member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first two years of membership, eight dollars a year for the remainder (if any) of his third and fourth years of membership, and ten dollars a year thereafter.

Section 8. After retirement from active service on account of age, any ordinary or contributing member who is not in arrears of dues and with membership extending over at least twenty years may, by giving proper notification to the Secretary, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings but not the *Bulletin*.

Section 9. An ordinary member or a contributing member shall receive the *Bulletin*, except in the case of members whose dues are remitted under Section 8 of this Article.

Section 10. Five dollars and fifty cents of the annual dues of those who receive the *Bulletin* under the provisions of this Article shall be allocated in payment therefor.

Section 11. The annual dues of ordinary and contributing members shall be due and payable at the beginning of each calendar year, at which time the Society shall submit a bill for these dues. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice. If a member wishes to discontinue his membership at any time, he shall send a written resignation to the Society.

Section 12. Any member who became a life member before October 25, 1941, by the payment of a sum determined in accordance with actuarial principles shall have for life the status and privileges of an ordinary member without further payment of dues. No additional applications for life memberships will be accepted.

ARTICLE X

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. Meetings of the Executive Committee may be called by the President; he shall call a meeting at any time upon the written request of two of its members.

Section 3. The Council shall meet at the Annual Meeting of the Society. Special meetings of the Council may be called by the President; he shall call a special meeting at any time upon the written request of five of its members. No special meeting of the Council shall be held unless written notice of it shall have been sent to all members of the Council at least ten days before the day set for the meeting.

Section 4. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each Trustee. He shall call a meeting upon the receipt of a written request of two of the Trustees. Meetings for the transaction of business may also be held by common consent of all the Trustees.

Section 5. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE XI

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish a journal called the *Transactions of the American Mathematical Society*, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called *Mathematical Reviews* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and the results of particular fields of mathematical research. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the *Bulletin*, *Transactions*, *Mathematical Reviews*, *Colloquium Publications*, and *Mathematical Surveys*, and the participation of the Society in the management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

ARTICLE XII

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

A SUBSTITUTE FOR THE PICONE FORMULA

WALTER LEIGHTON

It has been known for a number of years that the calculus of variations affords a powerful tool for the study of the oscillation of solutions of self-adjoint differential systems.[1] It is the purpose of the present paper to demonstrate how it can be used to replace and extend the Picone theorem.[2]

Consider the pair of self-adjoint differential equations

$$\frac{d}{dx}(ru') + pu = 0, \tag{1.1}$$

$$\frac{d}{dx}(r_1u') + p_1u = 0, \tag{1.2}$$

where, for definiteness, it is assumed that $r(x)$, $r_1(x)$, $r'(x)$, $r_1'(x)$, $p(x)$, $p_1(x)$ are continuous with r and r_1 positive on the interval $a \leqq x \leqq b$. With these equations we associate the functional identity

$$\begin{aligned}&\int_a^b [(r - r_1)u'^2 + (p_1 - p)u^2]dx \\ &\qquad = ruu'\Big|_a^b - \int_a^b u[(ru')' + pu]dx - \int_a^b [r_1u'^2 - p_1u^2]dx.\end{aligned} \tag{2}$$

The proof of (2) requires simply the integration by parts of the term ru'^2 in its left-hand member. In what follows it is convenient to *admit* to our discussion functions[3] $u(x)$ of class C' on the interval $a \leqq x \leqq b$ which vanish at a and b.

Repeated use is made of the following well known result.[4]

LEMMA. *If there exists an admissible curve $y = y(x)$ along which*

$$J = \int_a^b (Ry'^2 - Py^2)dx < 0, \tag{3}$$

Presented to the Society, September 10, 1948; received by the editors April 15, 1948.

[1] See, for example, Morse [2, Chap. IV]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Cf. Bôcher [1, p. 54], Ince [1, p. 225].

[3] A function is said to be of class C' on an interval $a \leqq x \leqq b$ if it is continuous and has a continuous derivative on $a \leqq x \leqq b$.

[4] Cf. Morse [2, chap. 2].

a solution $\bar{y}(x) \not\equiv 0$ *of the differential system*

$$(Ry')' + Py = 0, \quad y(a) = 0 \tag{4}$$

will have a zero on the interval $a < x < b$.

We return now to equations (1.1) and (1.2) and suppose that $u(x) \not\equiv 0$ is a solution of (1.1) having x_1 and x_2 as consecutive zeros $(x_1 < x_2)$. *We seek sufficient conditions that every solution* $u_1(x)$ *of* (1.2) *shall have at least one zero on the interval* $x_1 < x < x_2$. To that end we evaluate (2) with u, a, b replaced respectively by $u(x)$, x_1, x_2, obtaining the fundamental relationship

$$\int_{x_1}^{x_2} [(r_1 u'^2(x) - p_1 u^2(x)]dx = -\int_{x_1}^{x_2} [(r - r_1)u'^2(x) + (p_1 - p)u^2(x)]dx, \tag{5}$$

which for simplicity we write as $I_1 = -I$. It is clear that if we prescribe conditions which will insure that I is positive, I_1 will then be negative. It will follow from the lemma that every solution $u_1(x)$ of (1.2) which vanishes at x_1 will vanish again on the interval $(x_1 < x < x_2)$. The elementary Sturm separation theorem then insures that every solution of (1.2) will then vanish at least once on the interval $x_1 < x < x_2$.

The Sturm-Picone (SP) conditions provide that $r \geqq r_1$ and $p_1 \geqq p$, with the latter inequality holding in the strict sense in at least one point of the interval of integration. Clearly these conditions will insure that $I > 0$, and hence that all solutions of (1.2) will have a zero on $x_1 < x < x_2$.

The SP conditions are clearly not very delicate. One does not need to require that the coefficients of u'^2 and u^2 be essentially positive to insure that I is positive. Indeed, we may apply the Lemma to I. We obtain at once the following result.

THEOREM.[5] *In equations* (1.1) *and* (1.2) *let* $r > r_1$ *and suppose that a solution* $u(x)$ *of* (1.1) *has* $x = x_1$ *and* $x = x_2$ *as consecutive zeros. If* $p(x)$ *and* $p_1(x)$ *are such that a solution* $w(x) \not\equiv 0$ *of the differential system*

$$\frac{d}{dx}[(r - r_1)w'] + (p - p_1)w = 0, \tag{6.1}$$

---

[5] It should be pointed out that this result can be derived as a consequence of a theorem due to Morse [**2**, p. 102].

$$w(x_1) = 0 \tag{6.2}$$

has the property that $w(x) \neq 0$ *on the interval* $x_1 < x \leqq x_2$, *every solution of* (1.2) *has at least one zero on the interval* $x_1 < x < x_2$. *The conclusion of this theorem will remain valid if* $w(x) \neq 0$ *on* $x_1 < x < x_2$ *and* $w(x_2) = 0$, *provided the solution* $u(x)$ *does not satisfy equation* (6.1).[6]

The condition $r > r_1$ which is desirable in order that the functional I shall be nonsingular may appear at first glance to be unduly restrictive. It is easily seen however that the theorem permits the study of the oscillation of solutions of (1.1) and (1.2) even when $r \leqq r_1$. We may simply replace $r(x)$ and $p(x)$ by $kr(x)$ and $kp(x)$, respectively, where k is any positive constant such that $kr(x) > r_1(x)$. Such a determination of k is possible since $r(x)$ is positive throughout its domain of definition.

The observation above suggests incidentally the following simple extension of the SP theorem.

COROLLARY. *Let* $u(x) \neq 0$ *be a solution of* (1.1) *which has zeros at* $x = x_1$ *and* $x = x_2$. *If there exists a positive constant* k *such that both the conditions*

$$kr - r_1 \geqq 0, \qquad p_1 - kp \geqq 0 \tag{7}$$

hold, every solution $u_1(x)$ *of* (1.2) *has a zero on the interval* $x_1 < x < x_2$, *provided that at some point of this interval the latter inequality holds in the strict sense.*

The following example demonstrates that the theorem includes cases not covered by the SP theory even in the extended form suggested by the corollary.

EXAMPLE. The Legendre equation

$$\frac{d}{dx}\left[(1 - x^2)y'\right] + 12y = 0$$

has the solution $y = P_3(x) = (5x^3 - 3x)/2$ which vanishes at $x = 0$ and at $x = (3/5)^{1/2}$. We shall show that every solution of the differential equation

$$\frac{d}{dx}\left[(3/4 - x^2)y'\right] + 8y = 0$$

---

[6] A complete bibliography of oscillation and comparison theorems for Sturm-Liouville systems cannot be given here. Closely related papers are Bliss and Schoenberg [1] and Morse [1] to which the reader's attention is directed.

has a zero on the interval $0<x<(3/5)^{1/2}$. Here the analogues of equations (6.1) and (6.2) are

$$\frac{d}{dx}(w'/4) + 4w = 0, \qquad w(0) = 0.$$

It is clear that $w(x)$ may be taken as $\sin 4x$ which is positive on the interval $0<x<(3/5)^{1/2}$. An appeal to the theorem completes the proof. It is readily verified that the SP theory does not apply even in the extended form suggested by the corollary.

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MAXIME BÔCHER

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ÜBER DIE LAGE DER NULLSTELLEN EINES ABSTANDSPOLYNOMS UND SEINER DERIVIERTEN

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1. Der Begriff eines Abstandspolynoms. *Ein Abstandspolynom* $F(x, y, z)$ *n-ten Grades* in dem rechtwinkligen Koordinatensystem x, y, z ist ein reelles Polynom der drei Veränderlichen von der Form

$$F(x, y, z) = C\prod_{k=1}^{n} d_k(x, y, z), \tag{1}$$
$$d_k(x, y, z) = (x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2, \qquad C > 0.$$

Das Abstandspolynom $F(x, y, z)$ ist also eine nichtnegative reelle Funktion von der Form

$$F(x, y, z) = C(x^{2n} + y^{2n} + z^{2n}) + \Phi(x, y, z), \tag{2}$$

wo Φ ein reelles Polynom ist, das für die drei Veränderlichen bzw. für die einzelnen Veränderlichen höchstens den Grad $2n-1$ bzw. $2n-2$ besitzt.

Das Abstandspolynom $F(z, y, z)$ besitzt n Nullstellen, und zwar die Punkte

$$Q_k = (x_k, y_k, z_k) \qquad (k = 1, 2, \cdots, n). \tag{3}$$

Ausser den Nullstellen nimmt das Abstandspolynom $F(x, y, z)$ in jedem (reellen) Punkte des Raumes einen positiven Wert an.

Das Polynom $d_k(x, y, z) = (x-x_k)^2 + (y-y_k)^2 + (z-z_k)^2$ ist ein Abstandspolynom ersten Grades und ist ein *Wurzelfaktor* des Abstandspolynoms $F(x, y, z)$ n-ten Grades. F ist also ein Produkt von n Wurzelfaktoren. Kommt der Wurzelfaktor d_k unter den n Wurzelfaktoren p-mal vor ($p \geqq 1$), so ist Q_k eine p-fache Nullstelle von F. Dann ist

$$F(x, y, z) = d_k^p(x, y, z)\cdot G(x, y, z), \tag{4}$$

wo $G(x, y, z)$ ein Abstandspolynom $(n-p)$-ten Grades ist, für welches die Ungleichung $G(x_k, y_k, z_k) \neq 0$ besteht.

Jedes Abstandspolynom $f(x, y, z) = d_k(x, y, z)$ ersten Grades genügt offenbar der Identät

$$f_x^2 + f_y^2 + f_z^2 = 4\cdot f. \tag{5}$$

Hat eine ganze rationale Funktion $2n$-ten Grades von der Form (2)

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einen Faktor zweiten Grades, so hat dieser Faktor leicht ersichtlich die Form $f(x, y, z) = c(x^2+y^2+z^2+a_1x+a_2y+a_3z+a_4)$, $c \neq 0$. Das Abstandspolynom $F(x, y, z)$ n-ten Grades ist eine Verallgemeinerung eines Polynoms $f(z)$ der komplexen Veränderlichen z.

Sind nämlich

$$\begin{aligned} f(z) &= (z - z_1)(z - z_2) \cdots (z - z_n), \\ z &= x + iy, \ z_k = x_k + iy_k \quad (k = 1, 2, \cdots, n), \\ f(x + iy) &= U(x, y) + iV(x, y), \\ \bar{f}(x - iy) &= U(x, y) - iV(x, y), \end{aligned} \tag{6}$$

so ist

$$\begin{aligned} F(x, y) &= U^2(x, y) + V^2(x, y) = | f(z) |^2 \\ &= \prod_{k=1}^{n} | z - z_k |^2 = \prod \left[(x - x_k)^2 + (y - y_k)^2\right] \end{aligned} \tag{7}$$

ein Abstandspolynom n-ten Grades in der x, y Koordinatenebene.

Ein Abstandspolynom $F(x, y, z)$ geht durch eine orthogonale Transformation der Veränderlichen x, y, z offenbar wieder in ein Abstandspolynom über. Daraus folgt, dass ein Abstandspolynom, dessen Nullstellen in einer Ebene liegen, durch eine lineare Transformation sich auf die Form

$$F(x, y, z) = C \prod_{k=1}^{n} \left[(x - x_k)^2 + (y - y_k)^2 + z^2\right] \tag{8}$$

bringen lässt.

2. Die Derivierte eines Abstandspolynoms. Ist $F(x, y, z)$ ein Abstandspolynom n-ten Grades, so ist das Polynom $(4n-2)$-ten Grades

$$H(x, y, z) = F_x^2(x, y, z) + F_y^2(x, y, z) + F_z^2(x, y, z)$$

durch $4 \cdot F(x, y, z)$ teilbar, sodass die Funktion

$$\begin{aligned} F'(x, y, z) &= \frac{H(x, y, z)}{4F(x, y, z)} \\ &= \frac{F_x^2(x, y, z) + F_y^2(x, y, z) + F_z^2(x, y, z)}{4 \cdot F(x, y, z)} \end{aligned} \tag{9}$$

ein Polynom $2(n-1)$-ten Grades von den Veränderlichen x, y, z ist.

Diese Funktion $F'(x, y, z)$ wird *als Derivierte des Abstandspolynoms* $F(x, y, z)$ genannt.

Ist

$$F(x, y, z) = f^p(x, y, z)\cdot G(x, y, z),$$
$$f(x, y, z) = d_k(x, y, z), \qquad p \geqq 1, \qquad G(x_k, y_k, z_k) \neq 0$$

so ist

$$\begin{aligned} H(x, y, z) &= F_x^2 + F_y^2 + F_z^2 = p^2 f^{2p-2} G^2 (f_x^2 + f_y^2 + f_z^2) \\ &\quad + 2pf^{2p-1}(f_xG_x + f_yG_y + f_zG_z) + f^{2p}(G_x^2 + G_y^2 + G_z^2) \\ &= f^{2p-1}[4p^2G^2 + 2pG(f_xG_x + f_yG_y + f_zG_z) \\ &\quad + f(G_x^2 + G_y^2 + G_z^2)]. \end{aligned}$$

Daraus folgt, dass H durch die $(2p-1)$-te Potenz des Wurzelfaktors $f=d_k$ teilbar ist, durch eine höhere Potenz von f aber nicht, weil die zwischen den eckigen Klammerzeichen stehende Funktion im Punkte $Q_k=(x_k, y_k, z_k)$ den Wert $4p^2G^2(x_k, y_k, z_k)\neq 0$ annimmt.

Hieraus folgt, dass H durch jeden Wurzelfaktor von F und deshalb auch durch F teilbar ist und dass eine p-fache Nullstelle Q_k des Abstandspolynoms F eine $2p-1-p=(p-1)$-fache Nullstelle der Derivierten ist, weil F' durch die $(p-1)$-te Potenz von d_k teilbar ist.

Nach der Regel der logarithmischen Differentiation ist

$$\frac{F_x}{F} = \frac{\partial}{\partial x}\log F = \frac{\partial}{\partial x}\sum_{k=1}^{n}\log d_k = \sum_{k=1}^{n}\frac{1}{d_k}\cdot\frac{\partial d_k}{\partial x} = 2\cdot\sum_{k=1}^{n}\frac{x-x_k}{d_k}\,.$$

Es gelten also für die Nullstellen (x, y, z) der Derivierten $F'(x, y, z)$, die von den Nullstellen des Abstandspolynoms $F(x, y, z)$ abweichen, die Gleichungen

$$\text{(10)} \qquad \begin{gathered} \frac{F_x}{2\cdot F} = \sum_{k=1}^{n}\frac{x-x_k}{d_k} = 0, \qquad \frac{F_y}{2\cdot F} = \sum_{k=1}^{n}\frac{y-y_k}{d_k} = 0, \\ \frac{F_z}{2\cdot F} = \sum_{k=1}^{n}\frac{z-z_k}{d_k} = 0. \end{gathered}$$

Auf Grund der Bezeichnungen (6) und (7) ist

$$\text{(11)} \qquad F'(x, y) = \frac{F_x^2(x, y) + F_y^2(x, y)}{4\cdot F(x, y)} = |\, f'(z)\,|^2,$$

weil

$$F_x = 2(UU_x + VV_x), \qquad F_y = 2(UU_y + VV_y),$$

$$f'(z) = U_x + iV_x = V_y - iU_y, \qquad U_x = V_y, \qquad U_y = -V_x.$$

und deshalb

$$\begin{aligned} F_x^2 + F_y^2 &= 4[(UU_x + VV_x)^2 + (UU_y + VV_y)^2] \\ &= 4(U^2 + V^2)(U_x^2 + V_x^2) = 4 \cdot F(x, y) \cdot F'(x, y) \end{aligned}$$

sind.

Die Derivierte des Abstandspolynoms $F(x, y)$ ist also wieder ein Abstandspolynom der Veränderlichen x, y, falls $n > 1$ ist.

Sind $z_k = 0$ ($k = 1, 2, \cdots, n$), so erhält man für die von den Nullstellen des Abstandspolynoms $F(x, y, z)$ abweichenden Nullstellen der Derivierten $F'(x, y, z)$ aus (10) die Gleichungen

$$z = 0, \qquad \sum_{k=1}^{n} \frac{x - x_k}{(x - x_k)^2 + (y - y_k)^2} = 0,$$
$$\sum_{k=1}^{n} \frac{y - y_k}{(x - x_k)^2 + (y - y_k)^2} = 0.$$

Die Nullstellen der Derivierten $F'(x, y, z)$ stimmen also im Falle $z_k = 0$ ($k = 1, 2, \cdots, n$) mit den Nullstellen der Funktion $F'(x, y)$ von (11) überein. Ist

$$f'(z) = n(z - z_1')(z - z_2') \cdots (z - z_{n-1}'),$$
$$z = x + iy, \qquad z_k' = x_k' + iy_k' \qquad (k = 1, 2, \cdots, n - 1)$$

so ist

$$F'(x, y) = n^2 \prod_{k=1}^{n-1} [(x - x_k')^2 + (y - y_k')^2] = F_1(x, y, 0)$$

und deshalb ist auch die Derivierte

$$F'(x, y, z) = n^2 \prod_{k=1}^{n-1} [(x - x_k')^2 + (y - y_k')^2 + z^2] = F_1(x, y, z)$$

ein Abstandspolynom.

Naheliegend ist die Vermutung, das die Derivierte jedes Abstandspolynoms $F(x, y, z)$ n-ten Grades ($n \geqq 2$) stets ein Abstandspolynom ist.

3. Der Gauss-Lucas-sche Satz über Abstandspolynome.

I. *Die (reellen) Nullstellen der Derivierten eines Abstandspolynoms* $F(x, y, z)$ *fallen in die konvexe Hülle der Nullstellen von* $F(x, y, z)$. *Die Randfläche dieser konvexen Hülle kann nur dann eine Nullstelle*

P_0 der Derivierten enthalten, wenn P_0 mit einer Nullstelle von $F(x, y, z)$ zusammenfällt.

Bedeutet nämlich $P_0=(x_0, y_0, z_0)$ eine Nullstelle der Derivierten des Abstandspolynoms $F(x, y, z)$, für welche $F(x_0, y_0, z_0)\neq 0$ gilt und sind

$$m_k = \frac{1}{(x_0 - x_k)^2 + (y_0 - y_k)^2 + (z_0 - z_k)^2}, \qquad M = \sum_{k=1}^{n} m_k,$$

so lassen sich die Gleichungen (10) in der Form

$$\sum_{k=1}^{n} m_k(x_0 - x_k) = 0, \quad \sum_{k=1}^{n} m_k(y_0 - y_k) = 0, \quad \sum_{k=1}^{n} m_k(z_0 - z_k) = 0 \tag{12}$$

oder

$$Mx_0 = \sum_{k=1}^{n} m_k x_k, \qquad My_0 = \sum_{k=1}^{n} m_k y_k, \qquad Mz_0 = \sum_{k=1}^{n} m_k z_k$$

schreiben.

Daraus folgt der Satz, weil P_0 der Schwerpunkt der Punkte Q_k von den Massen m_k $(k=1, 2, \cdots, n)$ ist.

Aus den Gleichungen (12) folgt auch die folgende Form des Satzes I.

II. *Jede Nullstelle $P_0=(x_0, y_0, z_0)$ der Derivierten des Abstandspolynoms $F(x, y, z)$ für welche $F(x_0, y_0, z_0)\neq 0$ gilt, ist eine Gleichgewichtslage eines materiellen Punktes P des Raumes, auf welchen die Nullstellen Q_k von $F(x, y, z)$ dem Abstand umgekehrt proportionale abstossende Kräfte ausüben.*

Bei geeignet gewähltem Masssystem sind nämlich die linken Seiten der Gleichungen (12) den Komponenten der Resultante dieser Kräfte in bezug auf den Punkt P_0 gleich.

Bezeichnet $Q_k' = (x_k', y_k', z_k')$ den Spiegelpunkt von $Q_k=(x_k, y_k, z_k)$ in bezug auf die Kugel vom Mittelpunkt P_0 und vom Einheitsradius, so sind

$$x_k' - x_0 = m_k(x_k - x_0), \qquad y_k' - y_0 = m_k(y_k - y_0),$$
$$z_k' - z_0 = m_k(z_k - z_0).$$

Für diese Punkte Q_k' bestehen also die Gleichungen

$$\sum (x_k' - x_0) = 0, \qquad \sum (y_k' - y_0) = 0, \qquad \sum (z_k' - z_0) = 0, \tag{13}$$
$$\text{oder} \quad nx_0 = \sum x_k', \qquad ny_0 = \sum y_k', \qquad nz_0 = \sum z_k'.$$

Daraus folgt der Satz

III. *Ist $P_0=(x_0, y_0, z_0)$ eine (reelle) Nullstelle der Derivierten des Abstandspolynoms $F(x, y, z)$ n-ten Grades mit den Nullstellen Q_1, $Q_2, \cdots, Q_n$, für welche $F(x_0, y_0, z_0)\neq 0$ gilt, und bezeichnet Q_k' den Spiegelpunkt von Q_k in bezug auf eine Kugel vom Mittelpunkt P_0, so ist P_0 der Schwerpunkt der Punkte Q_k' $(k=1, 2, \cdots, n)$. Die Vektoren P_0Q_k' lassen sich in ein Raumvieleck zusammensetzen, weil ihre Summe verschwindet.*

4. Die Lage der Nullstellen der Derivierten eines Abstandspolynoms $F(x, y, z)$ in der Umgebung einer Nullstelle von $F(x, y, z)$. Bedeutet $|PQ|$ den Abstand der Punkte P, Q, so kann man aus dem Satz III den Satz ableiten:

IV. *Bezeichnen $Q_1, Q_2, \cdots, Q_m$ bzw. $p_1, p_2, \cdots, p_m$ die verschiedenen Nullstellen des Abstandspolynoms $F(x, y, z)$ n-ten Grades bzw. ihre Vielfachheiten $(p_k\geqq 1,\ p_1+p_2+\cdots+p_m=n)$ und bezeichnet K den Konvexkörper der gemeinsamen Punkte P der $m-1$ Kugeln*

$$|PQ_1| \leqq \frac{p_1}{n-p_1}|PQ_k| \qquad (k = 2, 3, \cdots, m),$$

so besitzt die Derivierte $F'(x, y, z)$ im Innern des Körpers K ausserhalb von Q_1 keine Nullstelle.

Ist $d=\text{Min}\ |Q_1Q_k|$ $(k=2, 3, \cdots, m)$, so ist die Kugel vom Mittelpunkt Q_1 und vom Halbmesser $p_1\cdot d/n$ ein Teilkörper von K. Sie enthält ausserhalb von Q_1 keine Nullstelle der Derivierten $F'(x, y, z)$ im Innern.

Für einen inneren Punkt $P_0(\neq Q_1)$ des Körpers K kann nämlich die Vektorgleichung

$$P_0N = \sum_{k=1}^{m} p_k\cdot P_0Q_k' = p_1\cdot P_0Q_1' + \sum_{k=2}^{m} p_k\cdot P_0Q_k' = 0$$

nicht bestehen, weil

$$|P_0N| \geqq p_1\cdot |P_0Q_1'| - \sum_{k=2}^{m} p_k\cdot |P_0Q_k'| > 0$$

ist.

Aus den Annahmen des Satzes IV folgen nämlich die Ungleichungen

$$\frac{|P_0Q_k'|}{|P_0Q_1'|} = \frac{|P_0Q_1|}{|P_0Q_k|} < \frac{p_1}{n-p_1} \qquad (k = 2, 3, \cdots, m),$$

$$\sum_{k=2}^{m} p_k \,|\, P_0Q_k' \,| < \frac{p_1}{n-p_1}(p_2 + p_3 + \cdots + p_m)\cdot |\, P_0Q_1' \,| = p_1 \,|\, P_0Q_1' \,|,$$

$$|\, P_0N \,| \geqq p_1 \,|\, P_0Q_1' \,| - \sum_{k=2}^{m} p_k \,|\, P_0Q_k' \,| > 0.$$

Damit ist der Satz IV bewiesen.

5. **Die Nullstellenverteilung eines Abstandspolynoms $F(x, y, z)$ in zwei Kugeln, die sich in einer Nullstelle der Derivierten von aussen berühren.** Es gilt die folgende Verallgemeinerung eines Satzes[1] von mir über die Polynome mit einer komplexen Veränderlichen:

V. *Bezeichnet $P_0 = (x_0, y_0, z_0)$ eine Nullstelle der Derivierten eines Abstandspolynoms $F(x, y, z)$ n-ten Grades (für welche $F(x_0, y_0, z_0) \neq 0$ ist), bezeichnen ferner K bzw. K′ die abgeschlossenen Inneren von zwei sich im Punkte P_0 von aussen berührenden beliebigen Kugeln mit dem Halbmesser R bzw. $R' = (n-1)R$ und besitzt endlich das Abstandspolynom $F(x, y, z)$ in K mindestens eine Nullstelle, so besitzt es auch in K′ mindestens eine. Enthält K′ keine Nullstelle von $F(x, y, z)$, so enthält auch K keine.*

Zum Beweis dieses Satzes kann man annehmen, das $x_0 = y_0 = z_0 = 0$ sind und dass die Berührungsebene der Kugeln K und K' im Punkte P_0 die yz-Ebene ist, weil man diese Lage durch eine Koordinatentransformation erreichen kann. Bei dieser Annahme hat die erste Gleichung von (12) die Form

$$\sum_{k=1}^{n} \frac{x_k}{x_k^2 + y_k^2 + z} = \sum_{k=1}^{n} X_k = 0, \qquad X_k = \frac{x_k}{x_k^2 + y_k + z_k^2}. \tag{14}$$

Ist $X_k = 0$, so liegt die Nullstelle Q_k in der yz-Ebene. Liegt ein Punkt Q_k ausserhalb der yz-Ebene, so besitzt die Summe von (14) mindestens je ein positives und negatives Glied.

Wir nehmen an, dass die Indizes der Punkte Q_k so gewählt sind, dass die ersten ν Glieder in der Summe von (14) positiv, die folgenden ν' Glieder negativ und die letzten $n-\nu-\nu'$ Glieder Null sind ($\nu \geqq 1$, $\nu' \geqq 1$, $n-\nu-\nu' \geqq 0$) und dass die Ungleichungen

$$X_1 \geqq X_2 \geqq \cdots \geqq X_\nu > 0, \qquad X_1' \geqq X_2' \geqq \cdots \geqq X_{\nu'}' > 0,$$
$$X_i' = -X_{\nu+i} \qquad (i = 1, 2, \cdots, \nu')$$

bestehen. Dann liegen die Punkte $Q_1, Q_2, \cdots, Q_\nu$ bzw. $Q_{\nu+1}$,

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[1] Gy. (F.) v. Sz. Nagy, *Über geometrische Relationen zwischen den Wurzeln einer algebraischen Gleichung und ihrer Derivierten*, Jber. Deutschen Math. Verein. bd. 27 (1918) pp. 44–88; *Über die Lage der Nullstellen der Derivierten eines Polynoms*, Tôhoku Math. J. bd. 35 (1932) pp. 126–135.

$Q_{\nu+2}, \cdots, Q_{\nu+\nu'}$ an der positiven bzw. negativen Seite der yz-Ebene, die übrigen Punkte $Q_{\nu+\nu'+1}, \cdots, Q_n$ liegen in der yz-Ebene.

Sind

$$X_h = 1/D_h, \quad X'_i = 1/D'_i \qquad (h = 1, 2, \cdots, \nu;\ i = 1, 2, \cdots, \nu')$$

so lässt sich die Gleichung (14) in der Form

$$\begin{gathered}\sum_{h=1}^{\nu} \frac{1}{D_h} = \sum_{i=1}^{\nu} \frac{1}{D'_i}, \qquad \frac{1}{D_1} \geqq \frac{1}{D_2} \geqq \cdots \geqq \frac{1}{D_\nu}, \\ \frac{1}{D'_1} \geqq \frac{1}{D'_2} \geqq \cdots \geqq \frac{1}{D_{\nu'}}\end{gathered} \tag{15}$$

schreiben. Hier bedeutet D_h bzw. D'_i den Durchmesser der durch den Punkt Q_h bzw. $Q_{\nu+i}$ gehenden und die yz-Ebene im Anfangspunkt berührenden Kugel.

Man erhält aus (15) die Ungleichungen

$$\frac{1}{D_1} \leqq \frac{\nu'}{D'_1} \leqq \frac{n-1}{D'_1}, \qquad \frac{p}{D_p} \leqq \frac{\nu'}{D'_1} \leqq \frac{n-1}{D'_1} \qquad (p \leqq \nu),$$

oder

$$D'_1 \leqq \nu' \cdot D_1 \leqq (n-1) \cdot D_1, \quad D'_1 \leqq \frac{\nu'}{p} D_p \leqq \frac{n-p}{p} D_p. \tag{16}$$

Aus der Ungleichung $D'_1 \leqq (n-1)D_1$ folgt der Satz V. In dieser Ungleichung besteht das Gleichheitszeichen nur dann, wenn $\nu = 1$, $\nu' = n-1$, $D'_1 = D'_2 = \cdots = D'_{\nu'}$ sind. Liegt der Punkt Q_1 auf der Kugel K $(D_1 = 2R)$ und liegt kein Punkt Q_k innerhalb der Kugel K' $(D'_1 = 2R')$, so fallen die $n-1$ Punkte $Q_2, Q_3, \cdots, Q_n$ auf die Kugel K'.

Aus der zweiten Ungleichung von (16) folgt die folgende Verallgemeinerung des Satzes V:

VI. *Bezeichnet $P_0 = (x_0, y_0, z_0)$ eine Nullstelle der Derivierten eines Abstandspolynoms $F(x, y, z)$ n-ten Grades, für welche $F(x_0, y_0, z_0) \neq 0$ ist, bezeichnen ferner K und K' sich im Punkte P_0 von aussen berührende Kugeln von Halbmessern R bzw. $R' = (n-p)/p \cdot R$ und besitzt das Abstandspolynom $F(x, y, z)$ in der abgeschlossenen Kugel K mindestens p Nullstellen, so besitzt es in der Kugel K' mindestens eine Nullstelle. Besitzt das Abstandspolynom $F(x, y, z)$ im Innern von K' keine Nullstelle, so besitzt es auf der Kugel K bzw. K' genau p bzw. $n-p$ Nullstellen.*

Ist $\nu = \nu' = 1$, so ist $D_1 = D'_1$. Es gilt dann der Satz V:

VII. *Besitzt ein Abstandspolynom n-ten Grades die Nullstellen* Q_1 *und* Q_2 *ausserhalb einer Ebene* E, *die übrigen* $n-2$ *Nullstellen* $Q_3, Q_4, \cdots, Q_n$ *aber in der Ebene* E *und enthält die Ebene* E *eine Nullstelle* P_0 *der Derivierten* $(P_0 \neq Q_k,\ k=1, 2, \cdots, n)$, *so haben die Kugeln, die durch den Punkt* Q_1 *bzw.* Q_2 *gehen und im Punkte* P_0 *die Ebene* E *berühren, gleiche Durchmesser.*

Aus der Gleichung (15) erhält man leicht den Satz

VIII. *Es sei* P_0 *eine Nullstelle der Derivierten eines Abstandspolynoms 4-ten Grades, deren Nullstellen* Q_1, Q_2, Q_3, Q_4 *nicht in einer Ebene liegen. Ist* K_{234} *die Umkugel des Tetraeders* $P_0Q_2Q_3Q_4$ *und ist* K_1 *die Kugel durch den Punkt* Q_1, *von der die Kugel* K_{234} *im Punkte* P_0 *berührt wird, so ist* $R_{234}=3\cdot R_1$, *wo* R_{234} *bzw.* R_1 *den Halbmesser von* K_{234} *bzw.* K_1 *bedeutet. Bezeichnet* E_{34} *die Ebene der Punkte* $P_0Q_3Q_4$ *und bezeichnet* K_1'' *bzw.* K_2' *die Kugel durch den Punkt* Q_1 *bzw.* Q_2, *von der die Ebene* E_{34} *im Punkte* P_0 *berührt wird, so ist* $R_1''=R_2'$ *(wo* R_1'' *und* R_2' *die Halbmesser von* K_1'' *und* K_2' *bedeuten). Bezeichnet* $E_{12,34}$ *die durch den Punkt* P_0 *gehende und zu den Geraden* Q_1Q_2 *und* Q_3Q_4 *parallele Ebene und bezeichnet* K_{12} *bzw.* K_{34} *die Kugel durch das Punktpaar* Q_1Q_2 *bzw.* Q_3Q_4, *von der die Ebene* $E_{12,34}$ *im Punkte* P_0 *berührt wird, so ist* $R_{12}=R_{34}$ *(wo* R_{12} *und* R_{34} *die Halbmesser von* K_{12} *und* K_{34} *sind).*

Im ersten, zweiten bzw. dritten Teil dieses Satzes sind nämlich

$$\nu = 3, \quad \nu' = 1, \quad D_1 = D_2 = D_3 = 2R, \quad D_1' = 2R', \quad D_1 = 3D_1';$$

$$\nu = \nu' = 1, \quad D_1 = D_1', \quad D_1 = 2R, \quad D_1' = 2R;$$

bzw.

$$\nu = \nu' = 2, \quad D_1 = D_2, \quad D_1' = D_2', \quad D_1 = D_1', \quad D_1 = 2R, \quad D_1' = 2R'.$$

Damit ist der Satz VIII bewiesen.

6. **Der Satz von Laguerre für ein Abstandspolynom.** Es gilt der Satz:

IX. *Ist* $P=(x, y, z)$ *keine Nullstelle des Abstandspolynoms* $F(x, y, z)$ *oder seiner Derivierten, so werden die Nullstellen von* $F(x, y, z)$ *von jeder Kugelfläche getrennt, die den Punkt* P *und den Punkt* $\Pi=(\xi, \eta, \zeta)$

$$(17) \qquad \xi = x - 2n\frac{F_x(x, y, z)}{H(x, y, z)}, \qquad \eta = y - 2n\frac{F_y(x, y, z)}{H(x, y, z)}, \qquad \zeta = z - 2n\frac{F_z(x, y, z)}{H(x, y, z)}$$

enthält, wo $H(x, y, z)=F_x^2(x, y, z)+F_y^2(x, y, z)+F_z^2(x, y, z)$ *ist.*

Der Punkt $\Pi=(\xi, \eta, \zeta)$ wird als *Polarpunkt* von P in bezug auf das Abstandspolynom $F(x, y, z)$ genannt. Der Punkt Π kann nach Laguerre als der derivierte Punkt[2] von P in bezug auf $F(x, y, z)$, oder nach Pólya-Szegö als der Schwerpunkt[3] der Nullstellen von $F(x, y, z)$ in bezug auf P genannt werden.

In diesem Satz sind die Ebenen durch das Punktpaar $P\Pi$ den Kugeln zu rechnen. Die Kugelfläche K trennt dann die Nullstellen von $F(x, y, z)$, wenn sie jede dieser Nullstellen enthält, oder wenn das Abstandspolynom $F(x, y, z)$ innerhalb und ausserhalb von K mindestens je eine Nullstelle besitzt.

Zum Beweis des Satzes IX können wir annehmen, dass $P=(0, 0, 0)$ ist. Haben die Nullstellen Q_k von $F(x, y, z)$ in diesem Koordinatensystem die Koordinaten (x_k, y_k, z_k) $(k=1, 2, \cdots, n)$, so hat der Spiegelpunkt Q_k' von Q_k in bezug auf die Einheitskugel (vom Mittelpunkt $P=(0, 0, 0)$ und vom Einheitshalbmesser) die Koordinaten

$$x_k' = \frac{x_k}{x_k^2+y_k^2+z_k^2}, \quad y_k' = \frac{y_k}{x_k^2+y_k^2+z_k^2}, \quad z_k' = \frac{z_k}{x_k^2+y_k^2+z_k^2}.$$

Der Schwerpunkt $S'=(X', Y', Z')$ der n Punkte Q_k' $(k=1, 2, \cdots, n)$ hat nach (10) die Koordinaten

$$X' = \frac{1}{n}\sum_{k=1}^{n} x_k' = \frac{-1}{2n}\cdot\frac{F_x(0, 0, 0)}{F(0, 0, 0)}, \qquad Y' = \frac{-1}{2n}\cdot\frac{F_y(0, 0, 0)}{F(0, 0, 0)},$$

$$Z' = \frac{-1}{2n}\cdot\frac{F_z(0, 0, 0)}{F(0, 0, 0)}.$$

Ist $S=(X, Y, Z)$ der Spielgelpunkt von $S'=(X', Y', Z')$ in bezug auf die Einheitskugel, so fällt S mit dem Punkt Π zusammen, weil

$$X = \frac{X'}{X'^2+Y'^2+Z'^2} = \frac{-2n\cdot F_x(0, 0, 0)}{H(0, 0, 0)}, \qquad Y = \frac{-2n\cdot F_y(0, 0, 0)}{H(0, 0, 0)},$$

$$Z = \frac{-2n\cdot F_z(0, 0, 0)}{H(0, 0, 0)}$$

sind.

Das Punktsystem $Q_1', Q_2', \cdots, Q_n'$ wird von jeder durch seinen Schwerpunkt S' gehenden Ebene E getrennt. Die Spiegelung in bezug auf die Einheitskugel führt die Ebene E in eine die Punkte P und $S\equiv\Pi$ enthaltende Kugelfläche K, die Punkte $Q_1', Q_2', \cdots, Q_n'$

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[2] E. Laguerre, *Oeuvres de Laguerre*, bd. II, pp. 56–63.

[3] G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, bd. II, pp. 55–66.

in die Punkte $Q_1, Q_2, \cdots, Q_n$ über, die also von der Kugelfläche K getrennt werden. Damit ist der Satz IX bewiesen.

7. **Abstandspolynom mit in bezug auf eine Ebene symmetrisch liegenden Nullstellen.** Der bekannte Jensensche Satz[4] lässt sich auf folgende Weise verallgemeinern:

X. *Ist das Nullstellenpaar* $Q_{2h-1}Q_{2h}$ $(h=1, 2, \cdots, p)$ *eines Abstandspolynoms* $F(x, y, z)$ *n-ten Grades symmetrisch in bezug auf eine Ebene* E *und liegen seine übrigen Nullstellen* Q_k $(k=2p+1, 2p+2, \cdots, n)$ *in der Ebene* E, *so fällt eine ausserhalb der Ebene* E *liegende und von den Punkten* Q_i $(i=1, 2, \cdots, 2p)$ *verschiedene Nullstelle* P *der Derivierten* $F'(x, y, z)$ *entweder in das Innere von mindestens einer der* p *Kugeln mit den Durchmessern* $Q_{2h-1}Q_{2h}$ $(h=1, 2, \cdots, p)$, *oder* P *ist ein gemeinsamer Punkt dieser* p *Kugelflächen.*

Zum Beweis dieses Satzes können wir annehmen, dass die Ebene E mit der xy-Ebene zusammenfält, dass also

$$Q_{2h-1} = (x_h, y_h, z_h), \quad Q_{2h} = (x_h, y_h, -z_h) \quad (z_h \neq 0,\ h = 1, 2, \cdots, p);$$
$$Q_k = (x_k, y_k, 0) \qquad (k = 2p+1, 2p+2, \cdots, n)$$

sind. Ist $P=(x, y, z)$ im Satz X, so sind $F'(x, y, z)=0$, $F(x, y, z)\neq 0$ und $z\neq 0$. Aus (10) ergibt sich die Gleichung

$$\begin{aligned}\frac{F_z(x, y, z)}{2F(x, y, z)} = {} & \sum_{h=1}^{p}\left[\frac{z-z_h}{(x-x_h)^2+(y-y_h)^2+(z-z_h)^2}\right. \\ & \left. + \frac{z+z_h}{(x-x_h)^2+(y-y_h)^2+(z+z_h)^2}\right] \\ & + \sum_{k=2p+1}^{n}\frac{z}{(x-x_k)^2+(y-y_k)^2+z^2} \\ = {} & 2z\left\{\sum_{h=1}^{p}\frac{(x-x_h)^2+(y-y_h)^2+z^2-z_h^2}{[(x-x_h)^2+(y-y_h)^2+(z-z_h)^2]\cdot[(x-x_h)^2+(y-y_h)^2+(z+z_h)^2]}\right. \\ & \left. + \frac{1}{2}\sum_{k=2p+1}^{n}\frac{1}{(x-x_k)^2+(y-y_k)^2+z^2}\right\} = 0.\end{aligned}$$

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[4] L. W. Jensen, *Recherches sur la théorie des équations*, Acta Math. bd. 36 (1913) pp. 181–195.

Gy. (F.) v. Sz. Nagy, *Zur Theorie der algebraischen Gleichungen*, Jber. Deutschen Math. Verein. bd. 31 (1922) pp. 258–251.

Der Faktor von $2z$ ($z \neq 0$) kann nur dann verschwinden, wenn die erste Summe mindestens ein Glied mit einem negativen Zähler besitzt, oder wenn jedes Glied der erste Summe verschwindet und die zweite Summe kein Glied enthält ($n=2p$).

Damit ist der Satz X bewiesen.

Der Faktor $2z$ verändert sich nicht, wenn man den Punkt $P=(x, y, z)$ durch den Punkt $P'=(x, y, -z)$ ersetzt. Ebenso bleiben die Summen in den ersten zwei Gleichungen von (10) bei dieser Ersetzung unverändert. Daraus folgt, dass auch die Lage der Nullstellen der Derivierten $F'(x, y, z)$ in bezug auf die Ebene E symmetrisch ist.

Aus dem Satz X folgt:

XI. *Liegen die Nullstellen* $Q_1, Q_2, \cdots, Q_n$ *eines Abstandspolynoms n-ten Grades in einer Ebene E, sind die Punktpaare* $Q_{2h-1}Q_{2h}$ $(h=1, 2, \cdots, p)$ *in bezug auf eine Gerade g der Ebene E symmetrisch und fallen die übrigen* $n-2p$ *Punkte* Q_k *auf die Gerade g, so ist jede ausserhalb von g liegende und von den Punkten* Q_k $(k \leqq 2p)$ *verschiedene Nullstelle der Derivierten ein Punkt von mindestens einer Kreisscheibe unter den p Kreisscheiben mit den Durchmessern* $Q_{2h-1}Q_{2h}$.

Die Nullstellen des Abstandspolynoms in diesem Satz sind nämlich auch in bezug auf die Ebene E' symmetrisch, die durch die Gerade g geht und zu E senkrecht steht.

8. Die Lage der Nullstellen eines Abstandspolynoms $F(x, y, z)$ in bezug auf zwei Nullstellen seiner Derivierten.

Es gilt der Satz:

XII. *Bezeichnen* $P_0=(x_0, y_0, z_0)$ *und* $P_0'=(x_0', y_0', z_0')$ *zwei verschiedene Nullstellen der Derivierten eines Abstandspolynoms* $F(x, y, z)$ *n-ten Grades, für welche* $F(x_0, y_0, z_0)$ *und* $F(x_0', y_0', z_0')$ *nicht verschwinden und bezeichnet E bzw.* E_0 *eine beliebige Ebene durch die Punkte* P_0 *und* P_0', *bzw. die Mittelebene der Punkte* P_0, P_0', *so trennt das Ebenenpaar* EE_0 *die Nullstellen von* $F(x, y, z)$. (*Enthält das Ebenenpaar nicht jede Nullstelle von* $F(x, y, z)$, *so besitzt* $F(x, y, z)$ *im Inneren beider von E und* E_0 *begrenzten Doppel-Raumwinkel mindestens je eine Nullstelle.*)

Dreht man eine gleichseitige Hyperbel mit der Hauptachse P_0P_0' *um die Gerade* P_0P_0' *herum, so trennt das so enstandene zweischalige Hyperboloid die Nullstellen des Abstandspolynoms* $F(x, y, z)$.

Zum Beweis dieses Satzes können wir annehmen, dass $P_0=(0, 0, z_0)$, $P_0'=(0, 0, -z_0)$, $z_0 \neq 0$ sind. Bezeichnen $Q_k=(x_k, y_k, z_k)$ die Nullstellen von $F(x, y, z)$ in diesem Koordinatensystem und ist

$$N_k = [x_k^2 + y_k^2 + (z_k - z_0)^2]\cdot[x_k^2 + y_k^2 + (z_k + z_0)^2] \qquad (k = 1, 2, \cdots, n),$$

so erhält man aus (10) die Gleichungen

$$\begin{aligned} G_1 &\equiv \frac{1}{2z_0}\left[\frac{F_x(0, 0, z_0)}{F(0, 0, z_0)} - \frac{F_x(0, 0, -z_0)}{F(0, 0, -z_0)}\right] \\ &= \sum_{k=1}^{n} \frac{-2x_k z_k}{N_k} = 0, \\ G_2 &\equiv \frac{1}{2z_0}\left[\frac{F_y(0, 0, z_0)}{F(0, 0, z_0)} - \frac{F_y(0, 0, -z_0)}{F(0, 0, -z_0)}\right] \\ &= \sum_{k=1}^{n} \frac{-2y_k z_k}{N_k} = 0, \\ G_3 &\equiv \frac{1}{2z_0}\left[\frac{F_z(0, 0, z_0)}{F(0, 0, z_0)} - \frac{F_z(0, 0, -z_0)}{F(0, 0, -z_0)}\right] \\ &= \sum_{k=1}^{n} \frac{x_k^2 + y_k^2 - z_k^2 + z_0^2}{N_k} = 0. \end{aligned}$$

Bezeichnen A und B beliebige reele Zahlen, so ist

$$AG_1 + BG_2 \equiv -\sum_{k=1}^{n} \frac{(Ax_k + By_k)z_k}{N_k} = 0.$$

Daraus folgt der erste Teil des Satzes XII. Verschwindet nämlich das Produkt $(Ax+By)z$ nicht für jede Nullstelle Q_k, so gibt es mindestens je einen Punkt Q_h bzw. Q_i, für den dieses Produkt positiv bzw. negativ ist. Das Punktpaar Q_hQ_i wird also von den Ebenen $Ax+By=0$ und $z=0$ getrennt. Ebenso folgt der zweite Teil von XII aus der Gleichung $G_3=0$.

Aus der Gleichung $A\cdot G_1+B\cdot G_2+G_3=0$ folgt die folgende Ergänzung des Satzes XII:

Die Nullstellen des Abstandspolynoms $F(x, y, z)$ werden im Falle $P_0=(0, 0, z_0)$, $P_0'=(0, 0, -z_0)$ von jeder Fläche zweiter Ordnung mit der Gleichung

$$(18) \qquad (x - Az)^2 + (y - Bz)^2 - (1 + A^2 + B^2)z^2 + z_0^2 = 0$$

getrennt, wo A und B beliebige reele Zahlen bedeuten.

Diese Flächen sind im allgemeinen zweischalige Hyperboloide. Im Grenzfall $A\rightarrow\infty$, $B\rightarrow\infty$, stellt die Gleichung (18) zwei senkrechte Ebenen dar.

Dieser Satz ist eine Verallgemeinerung eines Satzes von mir.[5]

9. Abstandspolynome in den mehrdimensionalen euklidischen Räumen. Ein Abstandspolynom n-ten Grades in dem m-dimensionalen euklidischen Raum ist eine reelle Funktion der m reellen Veränderlichen $x_1, x_2, \cdots, x_m$ von der Form

$$F(x_1, x_2, \cdots, x_m)$$
$$= C\prod_{k=1}^{n} [(x_1 - x_{1,k})^2 + (x_2 - x_{2,k})^2 + \cdots + (x_m - x_{m,k})^2], \tag{19}$$
$$C > 0.$$

Die Punkte $Q_k = (x_{1,k}, x_{2,k}, \cdots, x_{m,k})$ $(k=1, 2, \cdots, n)$ sind die Nullstellen des Abstandspolynoms.

Die Derivierte des Abstandspolynom (19) wird durch die Gleichung

$$F'(x_1, x_2, \cdots, x_m) = \frac{1}{4\cdot F}\sum_{k=1}^{m} F_{x_k}^2, \qquad F_{x_k} = \frac{\partial F}{\partial x_k} \tag{20}$$

definiert. Man sieht ebenso ein, wie im Falle $m=3$, dass F' ein Polynom $2(n-1)$-ten Grades der m Veränderlichen $x_1, x_2, \cdots, x_m$ ist.

Unsere Sätze lassen sich ohne grössere Schwierigkeiten auch für $m > 3$ verallgemeinern.

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[5] Vgl. meine in der Fussnote 4 angeführte Arbeit.

THE SOLUTION OF LINEAR INTEGRAL EQUATIONS BY MEANS OF WIENER INTEGRALS

THEODORE G. OSTROM

1. Introduction. In this paper, we obtain expressions for the solution, resolvent kernel, and Fredholm determinant of the integral equation

$$z(t) = x(t) + \int_0^1 K(t, s)x(s)ds$$

in terms of Wiener integrals. Besides appealing to the general interest which is always inherent in the relating of two apparently diverse fields, these results are of possible importance in two ways: (a) Though the Wiener integrals involved can be evaluated at present only when the kernel takes on certain relatively simple forms, it may be possible in the future to obtain at least approximate evaluations which will in turn offer approximate solutions of the integral equation which may converge faster than the Fredholm solution. Here we might mention specifically the case where the integral equation contains a parameter of large absolute value; (b) As a means of evaluating Wiener integrals in terms of the known (Fredholm) solution of the integral equation.

The Wiener integral is based on a measure defined by Wiener [1][1] on the space C of all real functions $x(t)$ continuous on the interval $0 \leqq t \leqq 1$ and vanishing at $t=0$. Cameron and Martin have investigated its properties and have, in particular, discovered how it transforms under translations [2] and linear transformations [3]. They have also been able to express the solution of a class of nonlinear integral equations in terms of limits of Wiener integrals [4]. We shall obtain our results by using their theorems on translations and linear transformations.

2. The basic solution. Given the integral equation

$$z(t) = F[x \mid t], \tag{2.1}$$

where

$$F[x \mid t] = x(t) + \int_0^1 K(t, s)x(s)ds, \tag{2.2}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

suppose that the following conditions are satisfied:

1. $z(t) \in C$ and $z'(t)$ exists and is of bounded variation on the interval (0, 1).

2. $$K(t, s) = \begin{cases} K_1(t, s), & 0 \leqq t < s, \\ K_2(t, s), & s < t \leqq 1, \\ (K_1 + K_2)/2, & t = s, \end{cases}$$

where $K_1(t, s)$ is continuous, $\{0 \leqq t \leqq s,\ 0 \leqq s \leqq 1\}$; $K(0, s) = 0,\ 0 \leqq s \leqq 1$, and $K_2(t, s)$ is continuous, $\{0 \leqq s \leqq t,\ 0 \leqq t \leqq 1\}$.

3. $K(t, s)$ is absolutely continuous in t for almost all s, $0 \leqq s \leqq 1$, after the jump at $t = s$ is removed by the addition of a step function.

4. For almost all s, $\partial K(t, s)/\partial t$ is essentially of bounded variation—that is, there exists a measurable function $K_t^*(t, s)$ which is of bounded variation in t for each s and which for almost all (t, s) $\{0 \leqq t \leqq 1,\ 0 \leqq s \leqq 1\}$ is equal to $\partial K(t, s)/\partial t$.

5. $K_t^*(t, s)$ can be so chosen that

$$\int_0^1 \sup_{0 \leqq t \leqq 1} | K_t^*(t, s) | \, ds < \infty,$$

$$\int_0^1 \operatorname*{var}_{0 \leqq t \leqq 1} [K_t^*(t, s)] ds < \infty.$$

6. $J(s) = K_2(s, s) - K_1(s, s)$ is of bounded variation.

7. $$D = 1 + \sum_{\mu=1}^{\infty} \frac{1}{\mu!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(s_1, s_1) & \cdots & K(s_1, s_\mu) \\ \cdot & \cdots & \cdot \\ K(s_\mu, s_1) & \cdots & K(s_\mu, s_\mu) \end{vmatrix} ds_1 \cdots ds_\mu \neq 0.$$

We shall show in this section that under the above conditions our integral equation can be solved in terms of Wiener integrals. The explicit form of the solution is given in the following theorem:

THEOREM I. *Let* $z(t)$ *and* $K(t, s)$ *satisfy conditions* 1–7 *above. Then the solution of the equation* (2.1) *with* $F[x|t]$ *defined by* (2.2) *is given by the formula*:

$$x(t) = | D | \exp\left(- \int_0^1 [z'(s)]^2 ds \right) \cdot \int_C^w u(t) \exp\left(2 \int_0^1 z'(s) dF[u | s] - \Phi(u) \right) d_w u; \tag{2.3}$$

where

$$\begin{aligned}\Phi(u) &= \int_0^1 \left[\frac{d}{d\xi}\int_0^1 K(\xi, s)u(s)ds\right]^2 d\xi \\ &\quad + 2\int_0^1 \left[\int_0^1 \frac{\partial K(\xi, s)}{\partial \xi} u(s)ds\right] du(\xi) + \int_0^1 J(\xi)d[u^2(\xi)].\end{aligned} \tag{2.4}$$

PROOF.[2] Under the conditions of Theorem I, the equation (2.1) determines a 1-1 mapping of C into itself. Hence for each $z(t)$ in C there is a unique solution $x(t)$ belonging to C which can be written as

$$x(t) = G[z \mid t]. \tag{2.5}$$

Thus

$$G[F(x \mid \cdot) \mid t] \,\underset{\cdot}{\equiv}\, x(t). \tag{2.6}$$

Moreover, it is apparent that, if, in (2.1) and (2.2), $G[y|t]$ is the corresponding solution when $z(t)$ is replaced by $y(t)$, then $aG[z|t]+bG[y|t]$ is the solution when $z(t)$ is replaced by $az(t)+by(t)$, where a and b are constants. That is, $G[az+by|t]=aG[z|t]+bG[y|t]$. Then, since G is odd,

$$\int_c^w G[y \mid t]d_wy = 0$$

and since $\int_c^w d_wy=1$, we have

$$\int_c^w G[z + y \mid t]d_wy = G[z \mid t]. \tag{2.7}$$

Next we quote a theorem of Cameron and Martin [2]:

Let $x_0(t)$ *be a given continuous function vanishing at* $t=0$ *and having a derivative* $x_0'(t)$ *of bounded variation,* $0 \leq t \leq 1$. *Then if* $H[y]$ *is any functional for which either member of* (2.8) (*below*) *exists, the other member also exists and the equation* (2.8) *holds*:

$$\begin{aligned}\int_c^w H[y]d_wy &= \exp\left(-\int_0^1 [x_0'(s)]^2ds\right) \\ &\quad \cdot \int_c^w H[x + x_0] \ \exp\left(-2\int_0^1 x_0'(s)dx(s)\right)d_wx.\end{aligned} \tag{2.8}$$

In (2.8), let $H[y]=G[z+y]$ and let $x_0=-z$, noting that, from (2.7), we know that $\int_c^w G[z+y]d_wy$ exists and by condition 1 of the hypotheses, $Z(t)$ satisfies the conditions placed on $x_0(t)$. (2.8) becomes:

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[2] At the suggestion of the referee, the form of the first part of this argument has been modified slightly, so that reference to the Fredholm resolvent kernel could be postponed to a later stage of the paper.

$$\int_C^w G[z+y]d_wy = \exp\left(-\int_0^1 [z'(s)]^2ds\right)\int_C^w G[x]\exp\left(2\int_0^1 z'(s)dx(s)\right)d_wx.$$

Applying (2.7) and changing the variable of integration on the right to y so that confusion with $x(t)$ in (2.2) may be avoided, we get

$$\begin{aligned} G[z\mid t] &= \int_C^w G[(z+y)\mid t]d_wy \\ &= \exp\left(-\int_0^1 [z'(s)]^2ds\right)\int_C^w G[y\mid t] \\ &\quad\cdot\exp\left(2\int_0^1 z'(s)dy(s)\right)d_wy. \end{aligned} \tag{2.9}$$

Next we shall consider the transformation

$$y(t) = u(t) + \int_0^1 K(t, s)u(s)ds \equiv F[u\mid t].$$

By another theorem of Cameron and Martin [3], if K satisfies hypotheses 2–7, and if $H[y]$ is any measurable functional for which either member of (2.10) (below) exists, then the other member exists and the equality holds:

$$\int_C^w H[y]d_wy = |D|\int_C^w H\left[u+\int_0^1 K(\cdot, s)u(s)ds\right] \cdot\exp(-\Phi(u))d_wu, \tag{2.10}$$

where $\Phi(u)$ is defined in (2.4).

Now in (2.10), let $H[y]=G[y]\exp(2\int_0^1 z'(s)dy(s))$ since from (2.9) we know that $\int_C^w G[y]\exp(2\int_0^1 z'(s)dy(s))d_wy$ exists and since Wiener integrability implies measurability, this value of $H[y]$ is a measurable functional. We obtain

$$\int_C^w G[y]\exp\left(2\int_0^1 z'(s)dy(s)\right)d_wy = |D|\int_C^w G[F(u\mid\cdot)]\exp\left(2\int_0^1 z'(s)dF[u\mid s]-\Phi(u)\right)d_wu,$$

which reduces, on applying (2.6), to

$$\begin{aligned}\int_C^w G[y \mid t] \exp\left(2\int_0^1 z'(s)dy(s)\right) d_w y \\ = |D| \int_C^w u(t) \exp\left(2\int_0^1 z'(s)dF[u \mid s] - \Phi(u)\right) d_w u.\end{aligned} \tag{2.11}$$

Putting together (2.9) and (2.11), we get

$$\begin{aligned}G[z \mid t] = |D| \exp\left(-\int_0^1 [z'(s)]^2 ds\right) \int_C^w u(t) \\ \cdot \exp\left(2\int_0^1 z'(s)dF[u \mid s] - \Phi(u)\right) d_w u.\end{aligned}$$

But, by (2.5), $x(t) = G[z \mid t]$ is the solution of (2.1) and our theorem is proved.

3. **Reduction to linear form.** This solution can be put into a form in which the Wiener integrals are simpler, although the solution is a more complicated function of the integrals. The process, particularly as regards justification of some of the steps, is somewhat laborious hence we shall only summarize the argument.

If, in (2.1), $z(t)$ be replaced by $\lambda z(t)$, the solution $x(t)$ will be replaced by $\lambda x(t)$. Hence

$$\begin{aligned}\lambda x(t) = |D| \exp\left(-\lambda^2 \int_0^1 [z'(s)]^2 ds\right) \int_C^w u(t) \\ \cdot \exp\left(2\lambda \int_0^1 z'(s)dF[u \mid s] - \Phi(u)\right) d_w u.\end{aligned} \tag{3.1}$$

It can be shown that differentiation under the Wiener integral sign is justifiable provided that $z''(s)$ is continuous. If we assume $z''(s)$ continuous, and differentiate both sides of (3.1) with respect to λ,

$$\begin{aligned}x(t) = |D| \exp\left(-\lambda^2 \int_0^1 [z'(s)]^2 ds\right)\left\{-2\lambda \int_0^1 [z'(s)]^2 ds\right\} \\ \cdot \int_C^w u(t) \exp\left(2\lambda \int_0^1 z'(s)dF[u \mid s] - \Phi(u)\right) d_w u \\ + |D| \exp\left(-\lambda^2 \int_0^1 [z'(s)]^2 ds\right) \int_C^w u(t) \\ \cdot \exp\left(2\lambda \int_0^1 z'(s)dF[u \mid s] - \Phi(u)\right)\left[2\int_0^1 z'(s)dF(u \mid s)\right] d_w u.\end{aligned} \qquad 2)$$

Now (3.2) is an identity in λ which holds in particular when $\lambda=0$, hence

$$(3.3)\quad x(t) = 2\,|D| \int_C^w u(t)\left[\int_0^1 z'(s)\,dF(u\,|\,s)\right] \exp\,(-\Phi(u))\,d_w u.$$

Now $F[u\,|\,s]=u(s)+\int_0^1 K(s,\,\xi)u(\xi)d\xi$, so that we obtain by substitution in (3.3)

$$(3.4)\quad \begin{aligned} x(t) &= 2\,|D| \int_C^w u(t)\left[\int_0^1 z'(s)\,du(s)\right] \exp\,(-\Phi(u))\,d_w u \\ &\quad + 2\,|D| \int_C^w u(t)\left\{\int_0^1 z'(s)\,\frac{d}{ds}\left[\int_0^1 K(s,\,\xi)u(\xi)d\xi\right] ds\right\} \\ &\qquad \cdot \exp\,(-\Phi(u))\,d_w u. \end{aligned}$$

Substituting $\int_0^1 z'(s)du(s)=z'(1)u(1)-\int_0^1 u(s)d[z'(s)]$ and

$$d\int_0^1 K(s,\,\xi)u(\xi)d\xi/ds = \int_0^1 \partial K(s,\,\xi)/\partial s u(\xi)d\xi + J(s)u(s),$$

we get

$$(3.5)\quad \begin{aligned} x(t) &= 2\,|D| \int_C^w u(t)\left\{u(1)z'(1) - \int_0^1 u(s)\,dz'(s)\right\} \\ &\qquad \cdot \exp\,(-\Phi(u))\,d_w u \\ &\quad + 2\,|D| \int_C^w u(t)\left\{\int_0^1 z'(s)\left[\int_0^1 \frac{\partial K(s,\,\xi)}{\partial s}\,u(\xi)d\xi\right] ds\right\} \\ &\qquad \cdot \exp\,(-\Phi(u))\,d_w u \\ &\quad + 2\,|D| \int_C^w u(t)\left\{\int_0^1 z'(s)J(s)u(s)\,ds\right\} \exp\,(-\Phi(u))\,d_w u. \end{aligned}$$

Interchanging the order of Wiener integration and ordinary integration (which can be justified by the mixed Fubini theorem), we obtain

$$(3.6)\quad \begin{aligned} x(t) = 2\,|D| \biggl\{ & z'(1)\int_C^w u(t)u(1) \exp\,(-\Phi)\,d_w u \\ & - \int_0^1 \int_C^w u(t)u(s) \exp\,(-\Phi)\,d_w u\,dz'(s) \\ & + \int_0^1 z'(s)\left[\int_0^1 \frac{\partial K(s,\,\xi)}{\partial s} \int_C^w u(t)u(\xi) \exp\,(-\Phi)\,d_w u\,d\xi\right] ds \\ & + \int_0^1 z'(s)J(s)\left[\int_C^w u(t)u(s) \exp\,(-\Phi)\,d_w u\right] ds \biggr\} \end{aligned}$$

$$\begin{aligned} &= 2\,|D|\,\left\{z'(1)W(t,1) - \int_0^1 W(t,s)d[z'(s)]\right. \\ &\quad + \int_0^1 z'(s)\left[\int_0^1 \frac{\partial K(s,\xi)}{\partial s}W(t,\xi)d\xi\right]ds \\ &\quad \left. + \int_0^1 z'(s)J(s)W(t,s)ds\right\}, \end{aligned} \tag{3.6}$$

where $W(t, s) = \int_c^w u(t)u(s)/\exp(-\Phi(u))d_w u$ and $z''(s)$ is continuous. It will be shown later that $W(t, s)$ is continuous in s, hence this may be written (since $W(t, 0) = 0$)

$$\begin{aligned} x(t) = 2\,|D|\,&\left\{\int_0^1 z'(s)d_s[W(t,s)]\right. \\ &\quad + \int_0^1 z'(s)\left[\int_0^1 \frac{\partial K(s,\xi)}{\partial s}W(t,\xi)d\xi\right]ds \\ &\quad \left. + \int_0^1 z'(s)J(s)W(t,s)ds\right\}. \end{aligned} \tag{3.7}$$

4. Expression of D and the Fredholm resolvent kernel in terms of Wiener integrals.

THEOREM II. *If, in addition to satisfying the conditions of Theorem* I, $K_1(t, s)$ *and* $K_2(t, s)$ *have continuous partial second derivatives with respect to* t, *then*

$$\begin{aligned} R(t,s) = -2&\left\{\int_c^w \exp(-\Phi(u))d_w u\right\}^{-1}\left\{\frac{\partial^2 W(t,s)}{\partial s^2}\right. \\ &\left. + \frac{\partial^2}{\partial s^2}\int_0^1 K(s,\xi)W(t,\xi)d\xi\right\} \end{aligned} \tag{4.1}$$

for all s, t *except* $s = t$, *where*

$$W(t,s) = \int_c^w u(t)u(s)\exp(-\Phi(u))d_w u$$

and $R(t, s)$ *is the resolvent kernel*

PROOF. Equation (3.6) holds if $z''(s)$ is continuous; let us restrict $z(s)$ still further, so that $z'(0) = 0$. Then

$$z'(1) = \int_0^1 z''(s)ds.$$

Moreover, we can write

$$\int_0^1 W(t, s)dz'(s) = \int_0^1 W(t, s)z''(s)ds,$$

$$\begin{aligned}\int_0^1 z'(s)\frac{\partial K(s, \xi)}{\partial s}ds &= \int_0^\xi z'(s)\frac{\partial K_1(s, \xi)}{\partial s}ds + \int_\xi^1 z'(s)\frac{\partial K_2(s, \xi)}{\partial s}ds \\ &= K(1, \xi)z'(1) - \int_0^1 K(s, \xi)z''(s)ds - J(\xi)z'(\xi)\end{aligned}$$

(integrating each integral by parts and then recombining into a single integral). Hence

$$\begin{aligned}&\int_0^1 z'(s)\left[\int_0^1 \frac{\partial K(s, \xi)}{\partial s} W(t, \xi)d\xi\right]ds \\ &= \left[\int_0^1 K(1, \xi)W(t, \xi)d\xi\right]\int_0^1 z''(s)ds \\ &\quad - \int_0^1 z''(s)\left[\int_0^1 K(s, \xi)W(t, \xi)d\xi\right]ds - \int_0^1 W'(t, \xi)J(\xi)z'(\xi)d\xi.\end{aligned}$$

Thus, (3.6) can be written

$$\begin{aligned}x(t) &= 2|D|\int_0^1 z''(s)\Big\{W(t, 1) - W(t, s) \\ &\quad + \int_0^1 [K(1, \xi) - K(s, \xi)]W(t, \xi)d\xi\Big\}ds.\end{aligned} \tag{4.2}$$

But also, under the conditions of Theorem I, we have, by ordinary Fredholm theory,

$$x(t) = z(t) + \int_0^1 R(t, s)z(s)ds.$$

Now $z(t) = \int_0^1 z'(s)\gamma(t, s)ds$, where

$$\gamma(t, s) = \begin{cases} 1, & 0 \leqq s \leqq t, \\ 0, & t < s \leqq 1. \end{cases}$$

Integrating by parts and noting that the indefinite integral of $\gamma(t, s)$ with respect to s is

$$\min(t, s) = \begin{cases} t, & t \leqq s, \\ s, & t \geqq s, \end{cases}$$

we obtain $z(t) = t\int_0^1 z''(s)ds - \int_0^1 z''(s)\min(t, s)ds$. Finally, let

$$R_1(t, s) = \int_0^s R(t, \xi)d\xi, \quad R_2(t, s) = \int_0^s R_1(t, \xi)d\xi;$$

then $\int_0^1 R(t, s)z(s)ds = R_1(t, 1)z(1) - \int_0^1 R_1(t, s)z'(s)ds$. Integrating by parts again, and replacing $z(1)$ by

$$\int_0^1 z''(s)ds - \int_0^1 sz''(s)ds,$$

we get

$$\begin{aligned}\int_0^1 R(t, s)z(s)ds = R_1(t, 1)\left[\int_0^1 z''(s)ds - \int_0^1 sz''(s)ds\right]& \\ - R_2(t, 1)\int_0^1 z''(s)ds + \int_0^1 R_2(t, s)z''(s)ds&\end{aligned}$$

and

$$\begin{aligned}x(t) = \int_0^1 z''(s)\{t - \min(t, s) + R_1(t, 1)(1 - s)& \\ - R_2(t, 1) + R_2(t, s)\}ds&\end{aligned} \tag{4.3}$$

which gives us, when combined with (4.2),

$$\begin{aligned}&2|D|\int_0^1 z''(s)\Big\{W(t, 1) - W(t, s) \\ &\qquad + \int_0^1 [K(1, \xi) - K(s, \xi)W(t, \xi)d\xi\Big\}ds \\ &\quad = \int_0^1 z''(s)\{t - \min(t, s) + R_1(t, 1)(1 - s) \\ &\qquad\qquad - R_2(t, 1) + R_2(t, s)\}ds.\end{aligned} \tag{4.4}$$

Transposing, we get

$$\begin{aligned}&\int_0^1 z''(s)\Big\{2|D|W(t, 1) - 2|D|W(t, s) \\ &\quad + 2|D|\int_0^1 [K(1, \xi) - K(s, \xi)]W(t, \xi)d\xi \\ &\quad - t + \min(t, s) - R_1(t, 1)(1 - s) + R_2(t, 1) - R_2(t, s)\Big\}ds = 0.\end{aligned} \tag{4.5}$$

Now (4.5) holds for $z(s)$ any function satisfying the conditions:

(a) $z(0)=z'(0)=0$,
(b) $z(t)$ continuous, $0\leqq t\leqq 1$,
(c) $z'(t)$ is of bounded variation on $(0, 1)$,
(d) $z''(t)$ continuous on $(0, 1)$.

Condition (d) implies (b) and (c); moreover, given any function $z''(t)$, the constants of integration can be so determined that (a) will be satisfied. Hence $z''(s)$ can be any continuous function. Next let us write (4.5) in the form

$$\int_0^1 z''(s)\phi(s)ds = 0, \tag{4.6}$$

that is, $\phi(s)$ represents the expression in brackets in (4.5). We see that $\phi(s)$ will be continuous if $W(t, s)$ is a continuous function of s. But, by (2.10), (2.2), (2.5) and (2.6),

$$\begin{aligned} W(t, s) &= \int_C^w u(t)u(s)\exp(-\Phi)d_w u = \frac{1}{|D|}\int_C^w G[v \mid t]G[v \mid s]d_w v \\ &= \frac{1}{|D|}\int_C^w \left[v(t) + \int_0^1 R(t, \xi)v(\xi)d\xi\right] \\ &\qquad \cdot\left[v(s) + \int_0^1 R(s, \xi)v(\xi)d\xi\right] d_w v.^{[3]} \end{aligned}$$

Multiplying out and interchanging order of integration, we get

$$\begin{aligned} W(t, s) &= \frac{1}{|D|}\left\{\int_C^w v(t)v(s)d_w v + \int_0^1 R(s, \xi)\int_C^w v(t)v(\xi)d_w v d\xi \right. \\ &\quad + \int_0^1 R(t, \xi)\int_C^w v(s)v(\xi)d_w v d\xi \\ &\quad \left. + \int_0^1\int_0^1 R(t, \xi)R(s, \eta)\int_C^w v(\xi)v(\eta)d_w v d\xi d\eta\right\} \\ &= \frac{1}{2|D|}\left\{\min(t, s) + \int_0^1 R(s, \xi)\min(t, \xi)d\xi \right. \\ &\quad + \int_0^1 R(t, \xi)\min(s, \xi)d\xi \\ &\quad \left. + \int_0^1\int_0^1 R(t, \xi)R(s, \eta)\min(\xi, \eta)d\xi d\eta\right\}. \end{aligned} \tag{4.7}$$

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[3] Using here the fact that the solution of (2.1) can be written alternately as $x(t)=G[z \mid t]$ or $x(t)=z(t)+\int_0^1 R(t, s)z(s)ds$, so that these two expressions are equal.

Since $R(t, s)$ is continuous at any point of continuity of $K(t, s)$, the right-hand side of (4.7) is continuous Thus $W(t, s)$ and, therefore, $\phi(s)$ are continuous on $(0, 1)$. Now $z''(s)$ may, in particular, be any of the complete orthonormal sets of polynomials on $(0, 1)$. From (4.6) and definition of completeness, $\phi(s)\equiv 0$. Therefore

$$\begin{aligned}(4.8)\qquad 2|D|\left\{W(t, 1) - W(t, s) + \int_0^1 [K(1, \xi) - K(s, \xi)]W(t, \xi)d\xi\right\} \\ - t + \min(t, s) - R_1(t, 1)(1 - s) \\ + R_2(t, 1) - R_2(t, s) \equiv 0.\end{aligned}$$

Now for every $t\neq s$ we note that each term in (4.8) has a derivative with respect to s except possibly $W(t, s)$; hence $\partial W(t, s)/\partial s$ exists. Differentiating, we get

$$\begin{aligned}2|D|\left\{\frac{\partial W(t, s)}{\partial s} + \frac{\partial}{\partial s}\int_0^1 K(s, \xi)W(t, \xi)d\xi\right\} + \gamma(t, s) \\ + R_1(t, 1) - R_1(t, s) \equiv 0.\end{aligned}$$

Differentiating again, we get

$$(4.9)\qquad -2|D|\left\{\frac{\partial^2 W(t, s)}{\partial s^2} + \frac{\partial^2}{\partial s^2}\int_0^1 K(s, \xi)W(t, \xi)d\xi\right\} - R(t, s) = 0.$$

As in the previous step, we infer the existence of $\partial^2 W(t, s)/\partial s^2$ from the existence of the derivatives of all other terms in the identity.

Finally, we can express D in terms of Wiener integrals. If, in (2.10), we set $H[y]\equiv 1$, then

$$1 = |D|\int_C^w \exp(-\Phi(u))d_w u.$$

Hence

$$|D| = \left\{\int_C^w \exp(-\Phi(u))d_w u\right\}^{-1}.$$

Theorem II then follows directly from (4.9).

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ON THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER INVARIANT UNDER CONTACT TRANSFORMATIONS

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In the classical Lie theory it is shown how to construct a differential equation invariant under a given group, and how to solve an equation when a group leaving the equation invariant is known. However, little is said about the problem of determining the group for a given differential equation, which is by far the most interesting problem.

In the present paper, necessary and sufficient conditions for the existence of an infinitesimal contact transformation leaving a given equation invariant are determined along with the general form of the characteristic function of the group. It will also be shown how to reduce, by a proper change of variables, the infinitesimal contact transformation to a point transformation. This enables one to solve the transformed differential equation by Lie's methods. Passing back to the original variables, a new differential equation is obtained which combined with the original equation gives its solution in parametric form.

Let

$$Bf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \pi \frac{\partial f}{\partial p}$$

be the symbol of the infinitesimal contact transformation leaving invariant the differential equation $u=F(v)$, with $u=u(x, y, p)$, $v=v(x, y, p)$, $p=dy/dx$, and F such that the equation $G(x, y, p) = u-F(v)=0$ satisfies the various conditions for the existence of solutions (but otherwise arbitrary). Throughout this paper we shall assume that:

(A) Both u and v have first derivatives with respect to x, y and p, at least in some region R of the (x, y, p)-space.

(B) The Jacobians

$$J_1 = \frac{\partial(u, v)}{\partial(y, p)}, \qquad J_2 = \frac{\partial(u, v)}{\partial(p, x)}, \qquad J_3 = \frac{\partial(u, v)}{\partial(x, y)}$$

have in R derivatives of the first and second orders, while J_1 and J_2 have also derivatives of the third order with respect to x, y and p,

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as are involved in the discussion.

(C) The functions u and v are not in involution, that is,

$$[uv] = \begin{vmatrix} u_p & u_x + pu_y \\ v_p & v_x + pv_y \end{vmatrix} = J_2 - pJ_1 \neq 0.$$

Since u and v are to be invariants under Bf they will satisfy the partial differential equations

$$\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \pi \frac{\partial u}{\partial p} = 0,$$

$$\xi \frac{\partial v}{\partial x} + \eta \frac{\partial v}{\partial y} + \pi \frac{\partial v}{\partial p} = 0,$$

from which we obtain

$$\frac{\xi}{\partial(u, v)/\partial(y, p)} = \frac{\eta}{\partial(u, v)/\partial(p, x)} = \frac{\pi}{\partial(u, v)/\partial(x, y)} = \sigma,$$

$\sigma = \sigma(x, y, p)$ being the common ratio. This can be written

$$(1) \qquad \xi = \sigma J_1, \qquad \eta = \sigma J_2, \qquad \pi = \sigma J_3.$$

If W is the so-called characteristic function of the infinitesimal contact transformation, we have also

$$(2) \qquad W = p\xi - \eta = \sigma(pJ_1 - J_2).$$

Now to find σ we recall that[1]

$$(3) \qquad \xi = \frac{\partial W}{\partial p}, \qquad \pi = -\frac{\partial W}{\partial x} - p\frac{\partial W}{\partial y}\cdot$$

As a consequence of (1), (2) and (3) we obtain the system of equations

$$(pJ_1 - J_2)\frac{\partial \sigma}{\partial p} + \left(p\frac{\partial J_1}{\partial p} - \frac{\partial J_2}{\partial p}\right)\sigma = 0,$$

$$(4) \quad (pJ_1 - J_2)\frac{\partial \sigma}{\partial x} + p(pJ_1 - J_2)\frac{\partial \sigma}{\partial y}$$
$$+ \left[\left(p\frac{\partial J_1}{\partial x} - \frac{\partial J_2}{\partial x}\right) + p\left(p\frac{\partial J_1}{\partial y} - \frac{\partial J_2}{\partial y}\right) + J_3\right]\sigma =$$

This system may be written in the homogeneous form

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[1] See Cohen, *An introduction to the Lie theory of one-parameter g*

$$(5)\qquad \begin{aligned} A_1f &= \frac{\partial f}{\partial p} + M_1\frac{\partial f}{\partial \sigma} = 0,\\ A_2f &= \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y} + M_2\frac{\partial f}{\partial \sigma} = 0, \end{aligned}$$

in which

$$(6)\quad M_1 = -\sigma\frac{p(\partial J_1/\partial p) - (\partial J_2/\partial p)}{pJ_1 - J_2},$$

$$(7)\quad M_2 = -\sigma\frac{(p(\partial J_1/\partial x) - \partial J_2/\partial x) + p(p(\partial J_1/\partial y) - \partial J_2/\partial y) + J_3}{pJ_1 - J_2}.$$

Adjoining to the system (5) the equations

$$(8)\quad A_3f = (A_1A_2)f = \frac{\partial f}{\partial y} + (A_1M_2 - A_2M_1)\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial y} + M_3\frac{\partial f}{\partial \sigma} = 0,$$

$$(9)\quad A_4f = (A_1A_3)f = (A_1M_3 - A_3M_1)\frac{\partial f}{\partial \sigma} = 0,$$

$$(10)\quad A_5f = (A_2A_3)f = (A_2M_3 - A_3M_2)\frac{\partial f}{\partial \sigma} = 0,$$

we see that the equations

$$(11)\qquad A_1M_3 - A_3M_1 = 0,\qquad A_2M_3 - A_3M_2 = 0,$$

are necessary and sufficient conditions in order that the system (4) have a solution. The system (5)–(8) implies the Jacobian complete system

$$(12)\qquad \begin{aligned} K_1f &= \frac{\partial f}{\partial x} + (M_2 - pM_3)\frac{\partial f}{\partial \sigma} = 0,\\ K_2f &= \frac{\partial f}{\partial y} + M_3\frac{\partial f}{\partial \sigma} = 0,\\ K_3f &= \frac{\partial f}{\partial p} + M_1\frac{\partial f}{\partial \sigma} = 0. \end{aligned}$$

Either we may solve (12) or the equivalent total differential equation

$$(13)\qquad (M_2 - pM_3)dx + M_3dy + M_1dp - d\sigma = 0.$$

If $f=\psi(x, y, p, \sigma)$ is the solution of (12), then

$$\psi(x, y, p, \sigma) = c \tag{14}$$

will be the solution of (13), and conversely. Equation (14) determines σ in terms of x, y, and p. Since σ enters as a factor in M_1 and M_2, it is also a factor of M_3.[2] Hence, equation (13) can be written

$$d\sigma/\sigma = d\omega(x, y, p),$$

and so σ has the form

$$\sigma = ke^{\omega(x,y,p)}. \tag{15}$$

Several special formulas for σ may be found. For instance, if

$$M_1 = \phi_1(p)\sigma, \qquad M_2 = \phi_2(x)\sigma,$$

then $M_3 = 0$, and equation (13) reduces to

$$\phi_2(x)\sigma dx + \phi_1(p)\sigma dp - d\sigma = 0,$$

from which we obtain

$$\sigma = k \exp\left(\int \phi_1(p)dp + \int \phi_2(x)dx\right).$$

Therefore, the characteristic function takes the form

$$W = k(pJ_1 - J_2) \exp\left(\int \phi_1(p)dp + \int \phi_2(x)dx\right) \tag{16}$$

by virtue of (2). This special case will be of use in some examples to be considered later.

We summarize our results in the following theorem.

THEOREM. *The characteristic function W of the infinitesimal contact transformation leaving invariant a given differential equation $u = F(v)$ can be found by the formula*

$$W = k(pJ_1 - J_2)e^{\omega(x,y,p)}$$

if, and only if, the equations

$$A_1M_3 - A_3M_1 = 0, \qquad A_2M_3 - A_3M_2 = 0$$

are both satisfied for all values of x, y and p.

Now, to solve the differential equation $u = F(v)$ invariant under the known group

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[2] If $M_1 = \sigma N_1$, $M_2 = \sigma N_2$, then $M_3 = A_1M_2 - A_2M_1 = \sigma(\partial N_2/\partial p - \partial N_1/\partial x - p\partial N_1/\partial y)$. This relation, together with (11), are the conditions in order that (13) be an exact differential when divided by σ.

$$(17)\qquad Bf = W_p\frac{\partial f}{\partial x} + (pW_p - W)\frac{\partial f}{\partial y} - (W_x + pW_y)\frac{\partial f}{\partial p},$$

we consider two cases:

(A) Both $\xi = W_p$ and $\eta = pW_p - W$ are free of p. This case occurs when W is linear in p. Then Bf represents an extended point transformation and the equation may be solved by introducing canonical variables.

(B) Either ξ or η, or both, contain p. Then Bf represents a general contact transformation.

In this case we may show that by a suitable change of variables the transformation reduces to a point transformation.[8] To this aim, let us define a finite contact transformation

$$(18)\qquad X = X(x, y, p),\qquad Y = Y(x, y, p),\qquad P = P(x, y, p)$$

in the following manner: $X = u$, $Y \neq X$ in involution with X, that is, such that $[XY] = 0$, or

$$(19)\qquad X_p\frac{\partial Y}{\partial x} + pX_p\frac{\partial Y}{\partial y} - (X_x + pX_y)\frac{\partial Y}{\partial p} = 0,$$

and P by the equation $P = Y_p/X_p$. The symbol for the transformed group will be

$$(20)\quad \overline{B}f = \bar{\xi}\frac{\partial f}{\partial X} + \bar{\eta}\frac{\partial f}{\partial Y} + \bar{\pi}\frac{\partial f}{\partial P} = BX\frac{\partial f}{\partial X} + BY\frac{\partial f}{\partial Y} + BP\frac{\partial f}{\partial P}.$$

But $\bar{\xi} = BX = Bu = 0$ since u is invariant under Bf. Since $\bar{\xi} = \overline{W}_P$ this implies that $\overline{W}$ is free of P. Also, we find that $\bar{\eta}$ does not contain P because $\bar{\eta} = P\overline{W}_P - \overline{W} = -\overline{W}$. Hence, $\overline{B}f$ is an extension of the point transformation group

$$(21)\qquad Uf = -\overline{W}(X, Y)\frac{\partial f}{\partial Y}.$$

This group can be reduced further by introducing the canonical variables

$$X^* = X,\qquad Y^* = -\int\frac{\partial Y}{\overline{W}(X, Y)}.$$

Then the symbol of the infinitesimal transformation assumes the simplest form

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[8] Cohen, loc. cit. p. 195, proves that the contact transformation reduces to a point transformation by assuming the corresponding differential equation solvable for p in the form $p = \omega(x, y)$.

$$U^*f = \frac{\partial f}{\partial Y^*}\,.$$

The equation $u=F(v)$ when written with the variables X, Y, P takes the form

$$\phi(X, Y, P) = 0. \tag{22}$$

This is also a differential equation, that is, $P=dY/dX$, since the relation $dY-PdX=\lambda(dy-pdx)$ which holds for any contact transformation implies $dY-PdX=0$ whenever $dy-pdx=0$. Since (22) will be invariant under (21) we are in position to solve (22), either directly or by introducing the canonical variables X^*, Y^* [this last step reduces the equation to the form $dY^*/dX^*=G(X^*)$]. Let

$$\psi(X, Y, c) = 0 \tag{23}$$

be the solution of (22). Passing back to the original variables we get a second differential equation

$$\Psi(x, y, p, c) = 0$$

which together with $u=F(v)$ determines the integral curves of the latter in terms of the parameter p.

Examples. I. Consider the differential equation

$$p + y/p = F(x + 2p). \tag{24}$$

Here $u=p+y/p$, $v=x+2p$. Hence, it follows that $J_1=2/p$, $J_2=1-y/p^2$, $J_3=-1/p$, $pJ_1-J_2=1+y/p^2$, $M_1=2\sigma/p$, $M_2=M_3=0$.

Formula (16) can be applied with $\phi_1(p)=2/p$, $\phi_2(x)=0$. Therefore, the characteristic function of the group is

$$W = k(1 + y/p^2)p^2 = k(p^2 + y).$$

Since a constant factor is irrelevant, we see that equation (24) is invariant under the infinitesimal contact transformation

$$Bf = 2p\frac{\partial f}{\partial x} + (p^2 - y)\frac{\partial f}{\partial y} - p\frac{\partial f}{\partial p}\,.$$

By taking $X=v=x+2p$ equation (19) reduces to

$$2\frac{\partial Y}{\partial x} + 2p\frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial p} = 0. \tag{25}$$

The corresponding system of ordinary differential equations is

$$\frac{dx}{2} = \frac{dy}{2p} = \frac{dp}{-1} = \frac{dY}{0},$$

from which we obtain $Y=p^2+y$ as a particular integral of (25). Finally we have $P=2p/2=p$. Introducing the new variables in (24) we get $dY/Y=dX/F(X)$. Hence, we have

$$Y = ce^{G(X)}, \qquad G(X) = \int \frac{dX}{F(X)}.$$

Passing back to the variables x, y, p we obtain

$$p^2 + y - ce^{G(x+2p)} = 0. \tag{26}$$

The system (24)–(26) furnishes the solution of the equation (24).

For instance, if $F(x+2p)=\tan(x+2p)$, equations (24) and (26) are respectively

$$p + y/p = \tan(x + 2p), \qquad p^2 + y = c \sin(x + 2p).$$

Solving for x and y we find

$$x = -2t + \operatorname{arc\,cos}(t/c),$$
$$y = -t^2 \pm (c^2 - t^2)^{1/2},$$

which are the parametric equations of the solution, where $t=p$ is the parameter.

II. To apply the method to find the group leaving invariant some familiar types of ordinary differential equations, let us consider first the homogeneous equation

$$p = F(y/x).$$

We have $u=p$, $v=y/x$, $J_1=-1/x$, $J_2=-y/x^2$, $J_3=0$, $pJ_1-J_2=(y-px)/x^2$, $M_1=0$, $M_2=2\sigma/x$, $M_3=0$. By using formula (16) with $\phi_1(p)=0$, $\phi_2(x)=2/x$, we get (taking $k=-1$)

$$W = px - y.$$

Since W is linear in p we obtain the point transformation with symbol

$$Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y},$$

which corresponds to the so-called homotetic transformation.

For the linear equation $p+P(x)y=F(x)$ we have $u=p+Py$, $v=x$, $J_1=0$, $J_2=1$, $J_3=-P$, $pJ_1-J_2=-1$, $M_1=0$, $M_2=-\sigma P$, $M_3=0$.

By putting $\phi_1(p)=0$, $\phi_2(x)=-P$, $ke^c=1$ in formula (16) we obtain

$$W = -\exp\left(-\int Pdx\right).$$

Hence the symbol for the group has the form

$$Uf = \exp\left(-\int Pdx\right)\frac{\partial f}{\partial y}\cdot$$

III. Finally, we shall give a short table of some general types with the corresponding characteristic functions.[4]

Differential Equations	Characteristic Functions
$y = px + F[x\phi(p)]$	$kx\phi(p)$
$y = px + pF[y\phi(p)]$	$kyp\phi(p)$
$y + \phi(p) = pF\left[x + \int \phi'(p)dp/p\right]$	$k[y + \phi(p)]$
$e^x\phi(x + y + p) = F[e^x(p + 1)]$	$k\phi(x + y + p)$
$\dfrac{y + x\phi(p)}{p + \phi(p)} = xF\left[\log x + \int \dfrac{\phi'(p)dp}{p + \phi(p)}\right]$	$k[y + x\phi(p)]$.

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[4] I am indebted to my former students Miss C. Santana and Dr. R. Peña for the fourth and fifth types shown in the list.

NOTE ON REGIONS OMITTED BY UNIVALENT FUNCTIONS

A. W. GOODMAN

1. Introduction.

Let $\mathfrak{F}$ denote the set of functions

$$w = f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad |a_1| = 1, \tag{1}$$

which are regular and univalent in E, the open unit circle. A classic result due to Koebe[1] states that if $f(z) \in \mathfrak{F}$ omits ν for z in E, then $|\nu| \geqq 1/4$, and that equality occurs (essentially)[2] only for the function

$$k(z) = \frac{z}{(1+z)^2}. \tag{2}$$

Hence the problem for a single omitted value is completely solved.

It is natural to inquire into the situation when the set of omitted values contains a region or a number of regions. This can be specialized in two ways. In §2 it is assumed that $f(z)$ omits all the values inside a circle of radius R, and a sharp upper bound for R is obtained. In §3 the area of the intersection of E and the set of omitted values is considered. The result obtained is not sharp.

2. The largest circle of omitted values.

We shall make use of the following theorem due to Pick[3] and Nevanlinna,[4] and since a short proof can be given, it is included here for completeness.

THEOREM 1. *Let $h(z) \in \mathfrak{F}$ omit γ in E. Further suppose that $|h(z)| < M$ for z in E. Then*

$$|\gamma| \geqq 2M^2 - M - 2M(M^2 - M)^{1/2}, \tag{3}$$

with equality (essentially) only for the function

$$h(z) = MK\left[\frac{1}{M} k(z)\right] \tag{4}$$

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[1] Titchmarsh, *The theory of functions*, Oxford University Press, p. 211.

[2] We use "(essentially)" to denote that all other functions for which the equality sign holds can be obtained from the one given by subjecting both planes to appropriate rotations.

[3] G. Pick, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Vienna, Abteilung IIa, vol. 126 (1917) pp. 247–263.

[4] R. Nevanlinna, Oversigt a Finska Vetenskaps Societetens Forhandlinger vol. 62 (1919).

where $K(z)$ denotes the inverse function of $k(z)$.

Proof. Without loss of generality take $\gamma > 0$. Clearly

$$f(z) = \frac{h(z)}{(1 + h(z)/M)^2} \tag{5}$$

is in $\mathfrak{F}$ and omits the positive value

$$\nu = \frac{\gamma}{(1 + \gamma/M)^2}. \tag{6}$$

Apply Koebe's Theorem. Since, for fixed M, (6) is an increasing function of γ in the interval $0 \leqq \gamma \leqq M$, the inequality $\nu \geqq 1/4$ implies $\gamma \geqq \gamma_c$, the least positive root obtained by solving

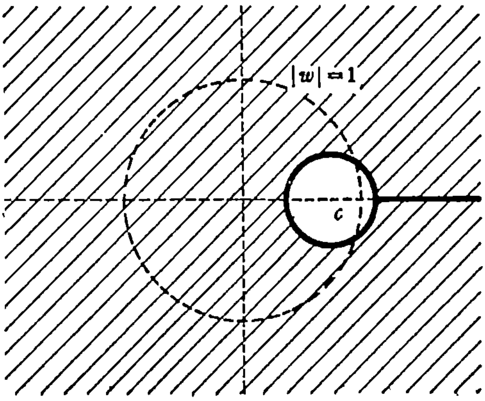

Fig. 1

$$\frac{\gamma}{(1 + \gamma/M)^2} = \frac{1}{4}. \tag{7}$$

This yields for γ_c exactly the right side of (3). Since $k(z)$ is (essentially) the only function for which $\nu = 1/4$, it is easy to see that (4) is (essentially) the only function for which the equality sign holds in (3). The function given by (4) maps E onto a circle of radius M with a slit along the positive real axis from M to γ.

Theorem 2. *Let $f(z) \in \mathfrak{F}$, Let c be fixed and suppose that for z in E, $f(z)$ omits all ξ for which*

$$|\xi - c| \leqq R. \tag{8}$$

Then

$$R \leqq |c| \frac{4|c|-1}{4|c|+1}, \tag{9}$$

with equality (essentially) only for the function which maps E onto the entire complex plane except for the closed circle of prescribed center and radius, and a radial slit running from the point $c+(c/|c|)R$ *to the point at infinity (see Fig.* 1).

PROOF. Without loss of generality take $f'(0)=1$ and $c>0$. Since $f(0)=0$, $c-R>0$. Let

$$g(z) = \frac{R}{f(-z)-c}. \tag{10}$$

Then for z in E, $g(z)$ is regular, univalent, $|g(z)|<1$, and $g(z)\neq 0$. Also $g(0)=-R/c$, and $g'(0)=R/c^2$. Next set

$$h(z) = \frac{(c^2-R^2)(g(z)-g(0))}{R(1-g(z)g(0))}. \tag{11}$$

Then $h(z)$ satisfies the conditions of Theorem 1 with

$$M = (c^2-R^2)/R, \qquad \gamma = (c^2-R^2)/c > 0. \tag{12}$$

We write (3) in the form

$$2M(M^2-M)^{1/2} \geqq 2M^2-(M+\gamma). \tag{13}$$

Now $M>1$ and $M>\gamma$ show that the right side of (13) is positive, and so we may square both sides and obtain

$$4M^2\gamma \geqq (M+\gamma)^2. \tag{14}$$

Using (12) in (14) and simplifying, one finds

$$4c(c-R) \geqq R+c, \tag{15}$$

and this is equivalent to (9).

To see that the inequality is sharp and to obtain the (essentially) unique function, reverse the steps of the proof. Take $h(z)$ as given by (4) with M determined by (12) and R determined by (9) with the equality sign. Then (11) together with the requirement that $g(0)=-R/c$ gives $g(z)$. Finally (10) defines $f(z)$, and it is not difficult to verify that the $f(z)$ so obtained has all of the required properties.

THEOREM 3. *Let* $h(z)\in\mathfrak{F}$ *omit* γ *in E. Further suppose that in E,* $\Re(h(z))>-c$, $c>0$. *Then*

$$|\gamma| \geqq c-(4c^2-2c)^{1/2}/2$$

with equality only in the case $h(z)$ *maps* E *onto the half-plane* $\Re(\omega) > -c$, *slit along the real axis from* $-c$ *to* $-c+(4c^2-2c)^{1/2}/2$.

The proof of this theorem is similar to that of Theorem 2, and will be omitted.

3. **The area question.** Instead of the point set mentioned in §1, it will be more convenient and completely equivalent to consider the set S_f, the intersection of E and the set of values assumed by $f(z)$ for z in E. Let A_f be the area of this set and let A denote the greatest lower bound of $\{A_f\}$ for $f(z)\in\mathfrak{F}$. Roughly speaking, A measures the amount of the unit circle which must always remain covered by the map of E under any normalized univalent function. We shall prove the following theorem.

THEOREM 4.

$$.5000\pi \leqq A < .7728\pi. \tag{16}$$

In establishing the lower bound use will be made of the rather elementary lemma:

LEMMA 1. *Let* $k>0$. *Let* $\phi(\theta)$ *be a positive continuous function for* $0\leqq\theta\leqq 2\pi$, *and*

$$\int_0^{2\pi} \phi(\theta)d\theta \leqq c. \tag{17}$$

Then

$$\int_0^{2\pi} \frac{d\theta}{\phi^k(\theta)} \geqq \frac{(2\pi)^{k+1}}{c^k} \tag{18}$$

with equality if and only if $\phi(\theta)\equiv c/2\pi$.

The Hölder inequality for integrals[5] states that if α, β, $f(x)$, $g(x)$ are positive and $\alpha+\beta=1$, then

$$\int_0^{2\pi} f^\alpha(\theta)g^\beta(\theta)d\theta \leqq \left[\int_0^{2\pi} f(\theta)d\theta\right]^\alpha \left[\int_0^{2\pi} g(\theta)d\theta\right]^\beta. \tag{19}$$

Take $\alpha=1/k+1$, $\beta=k/k+1$, $g(\theta)=\phi(\theta)$, $f(\theta)=(\phi(\theta))^{-\beta/\alpha}=\phi^{-k}(\theta)$, and obtain

$$2\pi \leqq \left[\int_0^{2\pi} \frac{d\theta}{\phi^k(\theta)}\right]^{1/(k+1)} c^{k/(k+1)}. \tag{20}$$

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[5] Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge University Press, p. 140.

This gives (18). Since equality occurs in (19) only when $f(\theta)$ and $g(\theta)$ are proportional, equality occurs in (20) only if $\phi(\theta)$ is constant.

To establish the lower bound for A consider, not $f(z)$, but $f(rz)/r = f_r(z)$, $0<r<1$. This insures the regularity of $f_r(z)$ on the boundary of E. To simplify matters, assume that the boundary of S_{f_r} is given

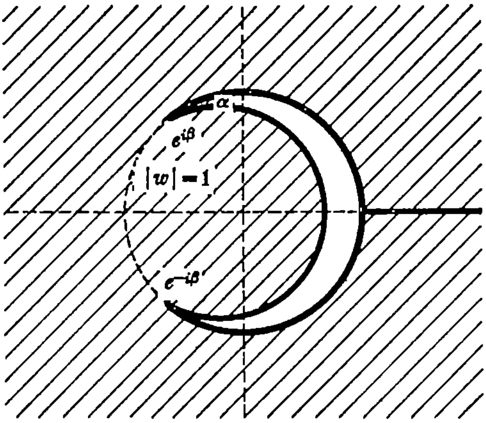

Fig. 2

by $R=\phi_r(\theta)$, a continuous single-valued function for $0\leqq\theta\leqq 2\pi$. Then

$$A_{f_r} = \frac{1}{2}\int_0^{2\pi} \phi_r^2(\theta)d\theta. \tag{21}$$

Hence the set of points omitted by $1/f_r(z)$, which lie outside the unit circle, has area

$$Q = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{\phi_r^2(\theta)} - \pi. \tag{22}$$

But by the Faber-Bieberbach[6] area theorem $Q\leqq\pi$ and so

$$\int_0^{2\pi} \frac{d\theta}{\phi_r^2(\theta)} \leqq 4\pi. \tag{23}$$

By Lemma 1, with $k=1$

$$2A_{f_r} = \int_0^{2\pi} \phi_r^2(\theta)d\theta \geqq \frac{(2\pi)^2}{4\pi} = \pi. \tag{24}$$

It is easy to remove the restriction that $\phi_r(\theta)$ be single-valued and continuous. The details will be omitted, but it should be noted that for the functions $f_r(z)$ each ray through the origin intersects the boundary of S_{f_r} in at most a finite number of points. Finally taking the limit as $r \to 1$ presents no difficulties.

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[6] Titchmarsh, loc. cit. page 209.

To establish the upper bound we construct, by combining the five simple functions (25), a function $w=f(z)$ which maps E onto the shaded region shown in Fig. 2.

$$
\begin{aligned}
&\eta = K[ck(z)], && 0 < c < 1,\\
&\zeta = \frac{\eta + \zeta_0}{\zeta_0\eta + 1}, && -1 < \zeta_0 < 0,\\
(25)\qquad &u = \frac{1 + i\zeta}{1 - i\zeta}, && \\
&v = u^{2-\alpha/\pi}, && 0 \leqq \alpha < \pi,\\
&w = \frac{e^{i\beta}v + e^{i(\pi-\alpha/2)}}{v + e^{i(\pi-\alpha/2+\beta)}}, && 0 < \beta < \pi.
\end{aligned}
$$

The first mapping takes E into E with a slit with length depending on the parameter c. The second preserves E and introduces a parameter ζ_0 which is later determined by the condition $f(0)=0$. The third mapping takes E into the right half-plane, and the fourth takes this half-plane into the entire plane except for a wedge of angular aperture α symmetrically located about the negative real axis with apex at the origin. The slit in E becomes a circular arc slit of unit radius and one end on the upper boundary of the wedge. Finally the fifth mapping takes this region onto the shaded region of Fig. 2.

The constant c is determined by requiring that the slit end go into the point at infinity. Using subscripts to denote the images of the points in the w-plane it is easy to verify the following.

$$
\begin{aligned}
&u_0 = e^{i\psi_0}, && \psi_0 = \frac{-\pi}{2\pi - \alpha}\left(\beta + \frac{\alpha}{2}\right),\\
&u_\infty = e^{i\psi_\infty}, && \psi_\infty = \frac{\pi}{2\pi - \alpha}\left(\beta - \frac{\alpha}{2}\right),\\
(26)\qquad &\zeta_0 = \frac{\sin \psi_0}{1 + \cos \psi_0}, && \zeta_\infty = \frac{\sin \psi_\infty}{1 + \cos \psi_\infty},\\
&\eta_\infty = \frac{\sin((\psi_\infty - \psi_0)/2)}{\cos((\psi_\infty + \psi_0)/2)}, && c = 4k(\eta_\infty).
\end{aligned}
$$

This gives the two parameters c and ζ_0 as functions of α and β. Finally

$$|f'(0)| = \frac{2\pi - \alpha}{\pi \sin \beta} 4k(\eta_\infty) \cos \psi_0. \tag{27}$$

Setting $|f'(0)| = 1$ gives a transcendental equation which determines α as a function of β. Fortunately, however, it is not necessary to carry the precise analysis any further to secure a valid upper bound for A. The particular selection $\alpha = 28°$, $\beta = 90°$ gives $|f'(0)| > 1.0015$ and $A_f < .7728\pi$.

To place our conclusion on a completely sound basis it must be remarked that by a well known lemma[7] $|f'(0)|$, for fixed β, is a strictly decreasing function of α. Further $|f'(0)|$ is a continuous function of α. So with $\beta = 90°$ there is a unique α' for which $|f'(0)| = 1$ and $\alpha' > 28°$.

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[7] If $f(z)$ and $g(z)$ map E onto F and G respectively with $f(0) = g(0)$ and $F \subset G$, then $|f'(0)| < |g'(0)|$.

ON SONINE'S INTEGRAL FORMULA AND ITS GENERALIZATION

HSIEN-YÜ HSÜ

1. **Introduction.** Let $J_\nu(x)$ denote Bessel function of order ν and let $\nu > -1/2$. The value of the integral

$$S(\nu;\, a,\, b,\, c) = \int_0^\infty J_\nu(ax)J_\nu(bx)J_\nu(cx)x^{1-\nu}dx, \tag{1.1}$$

where a, b, c are positive parameters, was evaluated by Sonine [8, p. 411][1] and given by his formula

$$S(\nu;\, a,\, b,\, c) = \frac{2^{\nu-1}\Delta^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu + 1/2)(abc)^\nu} \quad \text{or} \quad 0, \tag{1.2}$$

according as a triangle can, or cannot, be constructed with sides a, b, c. The area of the triangle is denoted by Δ. Derivation of (1.2) may be established by means of a limiting process from either Dougall's integral formula or Dougall's expansion formula, both of which involve the product of three ultra-spherical polynomials [5, pp. 379–382]. It has further been remarked [5, p. 375] that a proof of Sonine's theorem (1.2) can also be given by a procedure similar to that employed for the proof of the latter of Dougall's results by making use of Hankel's inversion formula.

In this paper, (a) we shall show the content of the previous remark with detailed explanations. (b) By use of the result (2.7) and Sonine's formula (1.2) a generalization of Sonine's integral (3.6) can be obtained. (c) Similarly, a generalization of Dougall's expansion formula may also be derived, which will contain (3.6) as a limiting case. And (d) we shall further give an explicit expression of the generalization of Dougall's integral formula, whence by means of a limiting process we also get the same generalized Sonine's integral. For $\nu = 0$, the well known Nicholson result [8, p. 414] can be displayed as a special case of (3.6).

2. **Derivation of Sonine's integral (1.2) by means of Hankel's inversion formula.** We shall assume that the area of the triangle as constructed by the given lengths a, b, c is greater than zero. For the special case when the area Δ is zero, then we may replace the expres-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

sion $\Delta^{2\nu-1}$ in the right member of (1.2) by 0, 1/2 and $+\infty$ for $\nu > 1/2$, $\nu = 1/2$ and $\nu < 1/2$ respectively [7, p. 685]. When the construction of the triangle becomes impossible it is easy to show that the value of the integral (1.1) vanishes [8, p. 413].

Let us consider a plane triangle with two fixed sides β and γ, and that the angle ϕ between β and γ should vary from 0 to π. Then the opposite side α to the angle ϕ of the triangle can be evaluated from the cosine law, namely:

$$\alpha^2 = \beta^2 + \gamma^2 - 2\beta\gamma \cos \phi. \tag{2.1}$$

The addition theorem for Bessel function of order ν [8, p. 363] gives

$$\frac{J_\nu(\alpha)}{\alpha^\nu} = 2^\nu\Gamma(\nu) \sum_{m=0}^{\infty} (\nu + m) \frac{J_{\nu+m}(\beta)J_{\nu+m}(\gamma)}{(\beta\gamma)^\nu} P_m^{(\nu)}(\cos \phi), \tag{2.2}$$

which is valid for all values of β, γ and ϕ, and for all values of ν with the exception of[2] 0, -1, -2, $\cdots$. Hence we have by integration

$$\begin{aligned} \int_0^\pi \frac{J_\nu(\alpha)}{\alpha^\nu} \sin^{2\nu} \phi d\phi &= 2^\nu\Gamma(\nu + 1) \frac{J_\nu(\beta)J_\nu(\gamma)}{(\beta\gamma)^\nu} \int_0^\pi \sin^{2\nu} \phi d\phi \\ &= 2^\nu\Gamma(\nu + 1/2)\pi^{1/2} \frac{J_\nu(\beta)J_\nu(\gamma)}{(\beta\gamma)^\nu}. \end{aligned} \tag{2.3}$$

The above formula (2.3) holds true even when $\nu = 0$ (see footnote 2).

Now we introduce α instead of ϕ as the variable of the integration. Since

$$\beta\gamma \sin \phi = 2\Delta_1, \tag{2.4}$$

where Δ_1 is the area of the triangle with sides α, β, γ; and since $\beta\gamma \sin \phi d\phi = \alpha d\alpha$, we find for the left member of (2.3)

$$\int_0^\pi \frac{J_\nu(\alpha)}{\alpha^\nu} \sin^{2\nu-1} \phi \sin \phi d\phi = \frac{2^{2\nu-1}}{(\beta\gamma)^{2\nu}} \int_{|\beta-\gamma|}^{\beta+\gamma} \frac{J_\nu(\alpha)}{\alpha^\nu} \Delta_1^{2\nu-1} \alpha d\alpha. \tag{2.5}$$

After this transformation, (2.3) can be written in the form

$$\int_{|\beta-\gamma|}^{\beta+\gamma} \frac{2^{\nu-1} J_\nu(\alpha)\Delta_1^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu + 1/2)(\alpha\beta\gamma)^\nu} \alpha d\alpha = J_\nu(\beta)J_\nu(\gamma). \tag{2.6}$$

Put $\alpha = Rx$, $\beta = bx$, $\gamma = cx$, we obtain

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[2] In case $\nu = 0$, the addition theorem is given by (2.2') $J_0(\alpha) = \sum_{m=0}^{\infty} \epsilon_m J_m(\beta) J_m(\gamma) \cdot \cos m\phi$, where $\epsilon_0 = 1$; $\epsilon_m = 2$, $m \geqq 1$. By integration, we have (2.3') $(1/\pi)\int^\pi J_0(\alpha) d\phi = J_0(\beta) J_0(\gamma)$, which is indeed the particular form of (2.3) when $\nu = 0$.

$$(2.7)\qquad \begin{aligned}\int_{|b-c|}^{b+c} \frac{2^{\nu-1}J_\nu(Rx)\Delta_2^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)(Rbc)^\nu}RdR &= \int_0^\infty f(R)J_\nu(Rx)RdR\\ &= x^{-\nu}J_\nu(bx)J_\nu(cx),\end{aligned}$$

where Δ_2 is the area of the triangle with sides R, b, c and where we set

$$(2.8)\ f(R) = \begin{cases} \dfrac{2^{\nu-1}\Delta_2^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)(Rbc)^\nu}, & |b-c| \leqq R \leqq b+c,\\ 0, & R < |b-c|, \text{ or } R > b+c.\end{cases}$$

Now the function $f(R)$ as defined above in (2.8) satisfies the condition that

$$\int_0^\infty f(R)R^{1/2}dR$$

exists and is absolutely convergent for $\nu > -1/2$. Therefore we can apply Hankel's inversion formula [8, p. 456] and have

$$(2.9)\qquad \begin{aligned} S(\nu;\, a,\, b,\, c) &= \int_0^\infty xJ_\nu(ax)\left\{\int_0^\infty f(R)J_\nu(Rx)RdR\right\}dx\\ &= \frac{1}{2}\left\{f(a-0)+f(a+0)\right\} = f(a),\end{aligned}$$

provided that the positive number "a" lies inside the interval in which $f(R)$ is of bounded variation. This is actually satisfied because of the fact that $f(R)$ has a bounded derivative for the whole interval except the end points $R=|b-c|$ and $R=b+c$, at which points the area of the corresponding triangle vanishes.

3. **Generalization of Sonine's integral formula.** We are given four positive parameters a, b, c, d, say $a \geqq b \geqq c \geqq d$ and let $2\sigma = a+b+c+d$. When a quadrilateral can not be constructed with lengths a, b, c, d as sides, the value of the integral

$$(3.1)\qquad S(\nu;\, a,\, b,\, c,\, d) = \int_0^\infty J_\nu(ax)J_\nu(bx)J_\nu(cx)J_\nu(dx)x^{1-2\nu}dx$$

may easily be shown to vanish [8, p. 413]. Moreover, from the well known asymptotic formula for Bessel functions [8, p. 199]

$$(3.2)\qquad J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{x^{3/2}}\right),\quad x\to\infty,$$

the principal part of (3.1) will become

$$
\begin{aligned}
&\left(\frac{2}{\pi}\right)^2 \frac{1}{(abcd)^{1/2}} \int_0^1 t^{-1-2\nu} \left\{ \cos\left(at - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \right. \\
&\qquad \cos\left(bt - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cos\left(ct - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
&\qquad \left. \cos\left(dt - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \right\} dt.
\end{aligned} \tag{3.3}
$$

We can point out without difficulty the conditions under which the integral (3.1) diverges. Recalling the identity

$$
\begin{aligned}
&8 \cos\left(m\phi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cos\left(n\phi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
&\qquad \cdot \cos\left(p\phi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \cos\left(q\phi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
&= \cos\left\{(m + n + p + q)\phi - (2\nu + 1)\pi\right\} \\
&\quad + \cos\left\{(n + p + q - m)\phi - \nu\pi - \frac{\pi}{2}\right\} \\
&\quad + \cos\left\{(p + q + m - n)\phi - \nu\pi - \frac{\pi}{2}\right\} \\
&\quad + \cos\left\{(q + m + n - p)\phi - \nu\pi - \frac{\pi}{2}\right\} \\
&\quad + \cos\left\{(m + n + p - q)\phi - \nu\pi - \frac{\pi}{2}\right\} \\
&\quad + \cos\,(m + n - p - q)\phi \\
&\quad + \cos\,(m + p - n - q)\phi + \cos\,(m + q - n - p)\phi,
\end{aligned} \tag{3.4}
$$

we readily see that, for the special case when $\sigma = a$, namely, a rigid quadrilateral with area zero, the integral (3.1) is properly divergent for $\nu < 0$; however, for $\nu \geqq 0$, it converges (and $=0$). We further note that for the case when $\sigma = a + d = b + c$, (3.1) diverges for $\nu \leqq 0$.

For the subsequent considerations we shall assume that a quadrilateral is constructible, and notice that the construction may be arranged in several ways. Let us consider one of the arrangements, say, the quadrilateral is formed by sides a, b, c, d which are connected with one another in the related order. One of the diagonals, which is denoted by R, divides the quadrilateral into two triangles

ΔabR and ΔcdR. Since the diagonal R is of variable length, we let $R=c+d-2t$ (or $R=a+b-2u$) where t (or u) is a variable. We further denote the area of the two triangles by Δ_{abR} and Δ_{cdR} respectively.

For sake of definiteness, suppose $c-d\leqq a-b$; when the equality sign is used we remember that $\nu>0$. Replacing b by c, and c by d in (2.7), multiplying both sides of the modified equation by $x^{1-\nu}J_\nu(ax)J_\nu(bx)$ and then integrating with respect to x from 0 to ∞, we obtain

$$\begin{aligned} S(\nu;\, a,\, b,\, c,\, d) &= \int_0^\infty x^{1-\nu}J_\nu(ax)J_\nu(bx) \\ &\quad\cdot\left\{\int_{c-d}^{c+d} \frac{2^{\nu-1}J_\nu(Rx)\Delta_{cdR}^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)(Rcd)^\nu}\, RdR\right\} dx \\ &= \frac{2^{\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)(cd)^\nu}\int_{c-d}^{c+d}\frac{\Delta_{cdR}^{2\nu-1}}{R^{\nu-1}} \\ &\quad\cdot\left\{\int_0^\infty J_\nu(ax)J_\nu(bx)J_\nu(Rx)x^{1-\nu}dx\right\} dR. \end{aligned} \tag{3.5}$$

Here the interchange of the above integrations is permissible for $\nu>-1/2$, since the improper integral $S(\nu;\, a,\, b,\, R)$ is uniformly convergent ***in general*** with respect to R in the interval $c-d$ to $c+d$; and

$$\int_{c-d}^{R} \frac{2^{\nu-1}J_\nu(Rx)\Delta_{cdR}^{2\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)(Rcd)^\nu}\, RdR$$

is a continuous function of R for the range $c-d\leqq R\leqq c+d$ [4, p. 473]. Write $R=c+d-2t$ and by Sonine's formula (1.2) we may rewrite (3.5) as follows:

$$\begin{aligned} &S(\nu;\, a,\, b,\, c,\, d) \\ &\qquad = \frac{2^{2\nu-1}}{\pi\{\Gamma(\nu+1/2)\}^2(abcd)^\nu}\int_0^{c-a}\left(\frac{c+d-2t}{\Delta_{abR}\Delta_{cdR}}\right)^{1-2\nu} dt, \end{aligned} \tag{3.6}$$

which we shall call the generalized Sonine's integral formula.

4. Generalized Dougall's expansion formula. Similar to the related line of reasoning, we use instead of (2.7) the known result [5, p. 378]

$$\begin{aligned} &\int_{|\beta-\gamma|}^{B} P_n^{(\lambda)}(\cos\alpha)(\sin\alpha\sin\beta\sin\gamma)^{1-2\lambda}E^{\lambda-1}\sin^{2\lambda}\alpha d\alpha \\ &\qquad = 2\{\Gamma(\lambda)\}^2\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}P_n^{(\lambda)}(\cos\beta)P_n^{(\lambda)}(\cos\gamma), \end{aligned} \tag{4.1}$$

where $P_n^{(\lambda)}$ $(\cos\theta)$ denotes ultra-spherical polynomial. In (4.1) we note that β and γ are two fixed sides of a spherical triangle: $0<\beta<\pi$, $0<\gamma<\pi$; α is the third side; that

$$E=\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\gamma+\alpha-\beta}{2}\sin\frac{\alpha+\beta-\gamma}{2}, \tag{4.2}$$

and that $B=\beta+\gamma$ if $\beta+\gamma\leqq\pi$; or $B=2\pi-\beta-\gamma$, if $\beta+\gamma>\pi$.

Suppose $\alpha\geqq\beta\geqq\gamma\geqq\delta$, and for sake of definiteness let $\gamma-\delta\leqq\alpha-\beta$. For the latter relation when the equality sign is used, we consider only $\lambda>1/2$. Now replacing β, γ, α, E, B in (4.1) by γ, δ, ρ, $E_{\gamma\delta\rho}$, B' respectively; multiplying both sides of the modified equation by

$$(n+\lambda)\left\{\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\right\}^2 P_n^{(\lambda)}(\cos\alpha)P_n^{(\lambda)}(\cos\beta),$$

and then summing with respect to n from 0 to ∞, we finally obtain

$$\begin{aligned}
&D_2(\lambda;\alpha,\beta,\gamma,\delta)\\
&=\sum_{n=0}^{\infty}(n+\lambda)\left\{\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\right\}^2\\
&\qquad\cdot P_n^{(\lambda)}(\cos\alpha)P_n^{(\lambda)}(\cos\beta)P_n^{(\lambda)}(\cos\gamma)P_n^{(\lambda)}(\cos\delta)\\
&=\frac{1}{2}\{\Gamma(\lambda)\}^{-2}\sum_{n=0}^{\infty}(n+\lambda)\left\{\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\right\}^2 P_n^{(\lambda)}(\cos\alpha)P_n^{(\lambda)}(\cos\beta)\\
&\qquad\cdot\int_{\gamma-\delta}^{B'}(\sin\rho\sin\gamma\sin\delta)^{1-2\lambda}E_{\gamma\delta\rho}^{\lambda-1}P_n^{(\lambda)}(\cos\rho)\sin^{2\lambda}\rho d\rho\\
&=\frac{1}{2}\{\Gamma(\lambda)\}^{-2}\int_{\gamma-\delta}^{B'}(\sin\rho\sin\gamma\sin\delta)^{1-2\lambda}E_{\gamma\delta\rho}^{\lambda-1}\sin^{2\lambda}\rho\\
&\qquad\cdot\left\{\sum_{n=0}^{\infty}(n+\lambda)\left\{\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\right\}^2\right.\\
&\qquad\qquad\left.\cdot P_n^{(\lambda)}(\cos\alpha)P_n^{(\lambda)}(\cos\beta)P_n^{(\lambda)}(\cos\rho)\right\}d\rho.
\end{aligned}\tag{4.3}$$

Here the interchange of the infinite series and the integration is permissible for $\lambda>0$, since the infinite series is uniformly convergent with respect to ρ in the interval between $\gamma-\delta$ and B'; and the integral

$$\int_{\gamma-\delta}^{B'}E_{\gamma\delta\rho}^{\lambda-1}\sin\rho d\rho$$

is absolutely convergent [2, p. 495]. By the result of Dougall's expansion formula we may rewrite (4.3) as follows:

$$D_2(\lambda;\alpha,\beta,\gamma,\delta) = 2^{-1-2\lambda}\pi\{\Gamma(\lambda)\}^{-6}\{\sin\alpha\sin\beta\sin\gamma\sin\delta\}^{1-2\lambda} \tag{4.4}$$
$$\cdot\int_{\gamma-\delta}^{B'}\left\{\frac{E_{\alpha\beta\rho}E_{\gamma\delta\rho}}{\sin^2\rho}\right\}^{\lambda-1}d\rho,$$

which we shall call the generalized Dougall's expansion formula. We shall remark that by means of a limiting process [5, p. 380] the generalized Sonine's integral formula (3.6) may also be derived from (4.4).

5. **Generalized Dougall's integral formula.** Bailey [1, p. 284] has attempted to generalize Busbridge's formula [3, p. 95], which involves products of Hermite polynomials, by evaluating the integral

$$D_1(\lambda;m,n,p,q) \tag{5.1}$$
$$= \int_{-1}^{1}(1-x^2)^{\lambda-1/2}P_m^{(\lambda)}(x)P_n^{(\lambda)}(x)P_p^{(\lambda)}(x)P_q^{(\lambda)}(x)dx.$$

He did not give an explicit, elementary expression for this generalized integral, but remarked that the result is complicated and involves a well-poised series of the type ${}_{11}F_{10}$ with unit argument.

By the known result of Dougall's integral formula [5, p. 374] we immediately have

$$P_p^{(\lambda)}(x)P_q^{(\lambda)}(x) = \sum_{r=0}^{q}\pi_r^{(\lambda;p,q)}P_{p+q-2r}^{(\lambda)}(x), \tag{5.2}$$

where

$$\pi_r^{(\lambda;p,q)} = \frac{p+q-2r+\lambda}{p+q-r+\lambda}\,\frac{\Gamma(p+q-2r+1)}{\Gamma(p+q-r+1)} \tag{5.3}$$
$$\cdot\frac{\Gamma(p+q-r+2\lambda)}{\Gamma(p+q-2r+2\lambda)}$$
$$\cdot\frac{\binom{p-r+\lambda-1}{p-r}\binom{q-r+\lambda-1}{q-r}\binom{r+\lambda-1}{r}}{\binom{p+q-r+\lambda-1}{p+q-r}}.$$

Now by an analogous method, as employed by Bailey in dealing with Hermite polynomials, we may obtain an explicit expression for the value of the integral (5.1), namely:

$$
\begin{aligned}
&D_1(\lambda; m, n, p, q)\\
&= \frac{2^{1-2\lambda}}{\{\Gamma(\lambda)\}^2} \sum_{r=0}^{\min(q,k-m)} \frac{\pi}{k-r+\lambda}\,\frac{\Gamma(k-r+2\lambda)}{\Gamma(k-r+1)}\,\frac{p+q-2r+\lambda}{p+q-r+\lambda}\\
&\quad\cdot\frac{\Gamma(p+q-2r+1)}{\Gamma(p+q-r+1)}\,\frac{\Gamma(p+q-r+2\lambda)}{\Gamma(p+q-2r+2\lambda)}\\
(5.4)\qquad &\quad\cdot\frac{\binom{p-r+\lambda-1}{p-r}\binom{q-r+\lambda-1}{q-r}\binom{r+\lambda-1}{r}}{\binom{p+q-r+\lambda-1}{p+q-r}}\\
&\quad\cdot\frac{\binom{k-m-r+\lambda-1}{k-m-r}\binom{k-n-r+\lambda-1}{k-n-r}\binom{k-p-q+r+\lambda-1}{k-p-q+r}}{\binom{k-r+\lambda-1}{k-r}}\\
&= \frac{2^{1-2\lambda}}{\{\Gamma(\lambda)\}^2} \sum_{r=0}^{\min(q,k-m)} F_r(\lambda; m, n, p, q),
\end{aligned}
$$

where we let $2k=m+n+p+q$ and $m\geqq n\geqq p\geqq q$, k being an integer not less than m; otherwise the integral is zero. For $\lambda=1/2$, the explicit expression for the integral involving product of Legendre polynomials was given by Shabde [6, pp. 31–32]. We may further remark that by means of a limiting process [5, p. 379], (5.4) will contain the same generalized Sonine's integral formula (3.6) as a special case.

We finally note that Bailey [1, p. 282] also remarked that for any four positive numbers, say $m\geqq n\geqq p\geqq q$, there are several ways of obtaining the value of the integral (5.1). It is easy to show, for example, with n and q interchanged in the related order and following the analogous argument for the evaluation, that we have

$$
\begin{aligned}
&D_1(\lambda; m, n, p, q) = D_1(\lambda; m, q, p, n)\\
(5.5)\qquad &\qquad = \frac{2^{1-2\lambda}}{\{\Gamma(\lambda)\}^2} \sum_{r=\max(0,k-m-q)}^{k-m} F_r(\lambda; m, q, p, n).
\end{aligned}
$$

Making use of a limiting process and particularly for the case when $\nu=0$, we have, if $a_1\geqq a_2\geqq a_3\geqq a_4>0$, and $\omega=a_2+a_3-2x$,

$$
\begin{aligned}
&S(0, a_1, a_2, a_3, a_4) = \int_0^\infty \prod_{n=1}^{4} J_0(a_n t)\,t\,dt\\
(5.6)\qquad &\qquad = \frac{1}{2\pi^2}\int_{\max(0,\sigma-a_1-a_4)}^{\sigma-a_1} \frac{a_2+a_3-2x}{\Delta_{a_1a_4\omega}\Delta_{a_2a_3\omega}}\,dx.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
S(0;\, a_1,\, a_2,\, a_3,\, a_4) &= \frac{4}{\pi^2}\int_{a_1-a_4}^{\min(a_2+a_3,\, a_1+a_4)} \left[\left\{(a_1+a_4)^2-\omega^2\right\}\right. \\
&\quad \cdot\left.\left\{(a_1-a_4)^2-\omega^2\right\}\left\{(a_2+a_3)^2-\omega^2\right\}\left\{(a_2-a_3)^2-\omega^2\right\}\right]^{-1/2}\omega\, d\omega, \qquad (5.7)
\end{aligned}
$$

which is the well known Nicholson's result [8, p. 414].

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A NOTE ON THE ERGODIC THEOREMS

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Introduction, definitions and remarks. The purpose of this note is to give an example of a measurable transformation of a measure space onto itself for which the individual ergodic theorem holds while the mean ergodic theorem does not hold.

Let S be a measure space of finite measure, m the measure defined on the measurable subsets of S, and T a 1-1 point transformation of S onto itself which is measurable (both T and T^{-1} transform measurable sets into measurable sets). Let the points of S be denoted by y and let $f(y)$ be any real valued function defined on S. We denote by $F_h(y)$ the average $(1/h)\sum_{i=0}^{h-1} f(T^iy)$.

We shall say that the individual ergodic theorem holds for $f(y)$ if the sequence of averages $\{F_h(y)\}$ converges to a finite limit almost everywhere. If the individual ergodic theorem holds for every integrable function $f\in L_1(m)$ we shall say that the individual ergodic theorem holds (with respect to m).[1]

We shall say that the mean ergodic theorem holds in $L_p(m)$ $(p\geqq 1)$ for a function $f\in L_p(m)$ if $F_h(y)\in L_p(m)$ for $h=1, 2, \cdots$ and the sequence $\{F_h(y)\}$ converges in the norm of $L_p(m)$. If the mean ergodic theorem holds in $L_p(m)$ for every function $f(y)\in L_p(m)$ then we shall say that the mean ergodic theorem holds in $L_p(m)$.

The following relations between the two ergodic theorems are known: If T is measure preserving, both the individual [1] and the mean [4][2] ergodic theorems hold. Without assuming that T is measure preserving, the mean ergodic theorem in $L_p(m)$ for any $p\geqq 1$ implies the individual ergodic theorem for all functions in $L_p(m)$ ([2, p. 1061], see also [3, p. 539] for the case $p=1$).

The question arises whether, conversely, the individual ergodic theorem implies the mean ergodic theorem in $b_p(m)$ for some $p\geqq 1$. This question has significance only when $L_p(m)$ is transformed into itself by the transformation induced on it by T. For in this case and only in this case is it true that for any $f\in L_p(m)$ the averages $\{F_h\}$ also belong to $L_p(m)$ for $h=1, 2, 3, \cdots$.[3] We answer this question in

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[1] The words in the parenthesis will be omitted if there is no reason for ambiguity.

[2] Only the case $p=2$ is proved in [4]; see [2, p. 1053] for all $p\geqq 1$. The numbers in brackets refer to the bibliography at the end of the paper.

[3] It is easy to give examples for which the individual ergodic theorem does hold while $L_p(m)$ is not transformed into itself. Such an example for instance is given if T is periodic while m is non-atomic and T^{-1} is singular.

the negative by constructing for each given $p \geqq 1$ an example of a transformation of a measure space (S, m) onto itself for which (1) the individual ergodic theorem holds, (2) the mean ergodic theorem does not hold in $L_p(m)$, and (3) every function in $L_p(m)$ is transformed into a function belonging to $L_p(m)$.

REMARK. Even though the individual ergodic theorem does not imply the mean ergodic theorem with respect to the original measure m it is known [2, p. 1059] that in case the individual ergodic holds, it is possible to introduce a new measure μ defined on the measurable sets of S such that μ has no more null sets than m and μ is also invariant under T. It follows then from the statements made above that the mean ergodic theorem holds in $L_p(\mu)$ for every $p \geqq 1$.

The example for $p=1$. Let S be the totality of all points on the circumferences $c_1, c_2, \cdots$ of a sequence of circles. Let the length of c_n be $1/2^n$. Let the measure m and the family of measurable sets in S be the obvious ones determined by the Lebesgue measure on each of the circumferences. On each circumference c_n we fix the polar coordinates $\rho = (1/2^n \cdot 2\pi) \cdot e^{i\theta}$. Let $x = \theta/2\pi$. Let us divide c_n into $2n+2$ arcs, the end points of the arcs being $x=0$, $x=1/2$, and $x = \pm 1/2^{k+1}$, $k=1, 2, \cdots, n$. The arcs are

$$A_{nk}: \frac{1}{2^k} \geqq x \geqq \frac{1}{2^{k+1}},$$

$$A_{n,n+1}: \frac{1}{2^{k+1}} \geqq x \geqq 0,$$

while if $n+2 \leqq k \leqq 2n+2$, A_{nk} is the reflection in $\theta=0$ of $A_{n,2n+3-k}$.

We define T as follows: For the points of A_{nk} let T be the unique transformation given by $x' = ax+b$, $a>0$, which transforms A_{nk} onto $A_{n,k+1}$ for $k=1, \cdots, 2n-1$, and A_{nk} onto $A_{n,1}$ for $k=2n+2$. T is clearly a 1:1 point transformation of S onto itself. Moreover it is easily seen that T is measurable and pointwise periodic with the period of $2n+2$ for the points of c_n. T also satisfies the following two conditions:

(1) $m(T^{-1}A) \leqq 2m(A)$ for every measurable set A in S.

(2) The set of ratios

$$R_A^n = \frac{1}{2n+2} \cdot \sum_{i=0}^{2n-1} \frac{m(T^{-i}A)}{m(A)}$$

where $n=1, 2, \cdots$ and A varies over all measurable sets $A \subseteq S$ is not a bounded set of numbers. (1) follows from the fact that T^{-1} is determined by a linear transformation with a stretching factor of at most 2. (In fact, it is either 1/2, 1, or 2.)

To prove (2) consider the sequence of sets $A_{n,n+1}$. Since $\bigcup_{i=0}^{2n-1} m(T^{-i}A_{n,n+1}) = c_n$ we have that

$$\frac{1}{2n+2}\cdot\sum_{i=0}^{2n-1} m(T^{-i}A_{n,n+1}) = \frac{1}{2n+2}\cdot\frac{1}{2^n}.$$

On the other hand $m(A_{n,n+1}) = (1/2^{n+1})\cdot(1/2^n)$ and hence

$$R^n_{A_{n,n+1}} = \frac{2^{n+1}}{2n+2},$$

which is an unbounded sequence.

We can now show that the above example satisfies the required conditions specified in the introduction.

(a) The individual ergodic theorem holds. In fact let $f(y)$ be any real-valued function defined on S, then since T is pointwise periodic the sequence $\{F_h(y)\}$ converges to a finite limit for every y, that is, the individual ergodic theorem holds for every real-valued function defined on S. A fortiori it holds (with respect to m).

(b) $L_1(m)$ is transformed into itself by T: In fact, it can be easily seen that $\bar{m}(A) = m(T^{-1}A)$ is a completely additive non-negative set function (that is, a measure) defined on the measurable sets of S. It can also be shown by considering approximating sums to the integrals that if $f \in L_1(m)$ then

$$\int_S |f(Ty)|\, dm = \int_S |f(y)|\, d\bar{m}.$$

By (1), $\bar{m}(A) \leqq 2m(A)$ for every measurable set A and hence

$$\int_S |f(y)|\, d\bar{m} \leqq 2\int_S |f(y)|\, dm < \infty$$

from which follows that $f(Ty) \in L_1(m)$, that is, $L_1(m)$ is transformed into itself by T.

(c) The mean ergodic theorem does not hold in $L_1(m)$. To prove this statement we use the following result due to Miller and Dunford [3, p. 539]: Suppose that the mean ergodic theorem did hold in $L_1(m)$; then there would exist a positive constant c independent of A and h such that

$$\text{(i)} \qquad \frac{1}{h}\cdot\sum_{i=0}^{h-1} m(T^{-i}A) < c\cdot m(A)$$

for all measurable sets A and $h=1, 2, \cdots$. But (i) is in contradiction with (2) above. Hence the mean ergodic theorem does not hold in $L_1(m)$.

It is possible to prove the last statement also directly by exhibiting functions in $L_1(m)$ for which the mean ergodic theorem does not hold in $L_1(m)$. In fact let $\bar{f}(y)$ be defined as follows: $\bar{f}(y)=2^n$ on $A_{n,n+1}$, $n=1, 2, \cdots$, $\bar{f}(y)=0$ everywhere else on S, then $\bar{f}(y)\in L_1(m)$ but $\{\bar{F}_k(y)\}$ is not convergent in $L_p(m)$, for if it were then the limit function $\bar{f}^*(y)$ of $\{\bar{F}_k(y)\}$ would have to belong to $L_1(m)$. But $\bar{f}^*(y)$ is seen to be equal to $(1/2n+2)\cdot 2^n$ for $y\in c_n$. We have

$$\int_S |\bar{f}^*(y)|\, dm = \sum_{n=1}^{\infty} (1/2n+2)$$

which is a divergent series, that is, $\bar{f}^*\notin L_1(m)$ and hence $\{F_k(y)\}$ is not convergent in $L_1(m)$.

The example for $p\geqq 1$. Let p be a fixed integer $\geqq 1$. Let S be the same sequence of circumferences $c_1, c_2, \cdots$ as before. We divide each c_n into $2(n\cdot 2^p-n+1)$ arcs, the end points being $x=0$, $x=1/2$ and $x=r/2^{kp+1}$, $k=1, 2, \cdots, n$, $r=1, 2, \cdots, 2^p-1$. Again we define T by the transformation given by $x'=ax+b$, $a>0$, which transforms each arc into the next adjacent one.

T is again seen to be a 1-1 measurable pointwise periodic transformation of S onto itself with the period $2(n\cdot 2^p-n+1)$ for the points of c_n. As before it follows that (a) the individual ergodic theorem holds. (b) $L_p(m)$ is transformed into itself since there is a bound (the bound being 2^p) on the stretching factor of T^{-1}. (c) The mean ergodic theorem does not hold in $L_p(m)$. To prove this last statement we use the following generalization of Miller and Dunford's result stated above: Let t be any real number $\geqq 1$, then if the mean ergodic theorem holds in $L_t(m)$ there exists a constant C independent of A and h such that

$$\text{(ii)} \qquad \left[\frac{1}{h}\cdot\sum_{i=0}^{h-1} m(T^{-i}A)\right]^t < C\cdot m(A).$$

The proof is almost the same as for the special case $t=1$. If, however, we consider the sequence of sets A_{nq}, where $q=n(2^p-1)+1$ and where the enumeration of the arcs on each c_n is analogous to that used in the case $p=1$, we can easily see that the sequence of ratios

$$R_n = \left[\frac{1}{2q}\sum_{i=0}^{2q-1} m(T^{-i}A_{nq})\right]^p \Big/ m(A_{nq})$$

is an unbounded sequence. This is in contradiction with (ii) for $t=p$. Hence the mean ergodic theorem does not hold in $L_p(m)$. Again the statement made in (c) may be proved directly by exhibiting functions in $L_p(m)$ for which the mean ergodic theorem does not hold in $L_p(m)$.

Let p_1 be any fixed number not less than 1. Let p be the first integer not less than p_1. Then the example constructed above for p is also a valid example for p_1, for the individual ergodic theorem clearly holds and $L_{p_1}(m)$ is transformed into itself for the same reasons as before, while it follows from the fact that $m(A) \leqq 1$ for every measurable set A and the fact that $p \geqq p_1$ that the same sequence of sets which violates (ii) for the case $t=p$ also violates (ii) for $t=p_1$. Hence the mean ergodic theorem does not hold in $L_{p_1}(m)$.

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RADCLIFFE COLLEGE

ORTHOGONALITY PROPERTIES OF *C*-FRACTIONS

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1. **Introduction.** It has been indicated in the work of Tchebichef and Stieltjes that the denominators $D_p(z)$ of the successive approximants of a J-fraction

$$\frac{b_0}{d_1+z}-\frac{b_1}{d_2+z}-\frac{b_2}{d_3+z}-\cdots \tag{1.1}$$

constitute a sequence of orthogonal polynomials. The orthogonality relations which exist between the $D_p(z)$ may be expressed in the following way (cf. [4, 7]).[1] Let S' be defined as the operator which replaces every z^p by c_p in any polynomial upon which it operates, where the $\{c_p\}$ are a given sequence of constants. Then the orthogonality relations

$$S'(D_p(z)D_q(z))\begin{cases}=0 & \text{for} \quad p\neq q,\\ \neq 0 & \text{for} \quad p=q,\end{cases} \tag{1.2}$$

hold relative to the operator S' and the sequence $\{c_p\}$. The polynomials $D_p(z)$ are given recurrently by the formulas $D_0(z)=1$, $D_p(z)=(d_p+z)D_{p-1}(z)-b_{p-1}D_{p-2}(z)$, $p=1, 2, \cdots$ $(D_{-1}(z)=0)$.

In this paper orthogonality relations similar to (1.2) are developed for the polynomials $B_p^*(z)$ which are derived from the denominators $B_p(z)$ of the successive approximants of a C-fraction

$$1+\frac{a_1z^{\alpha_1}}{1}+\frac{a_2z^{\alpha_2}}{1}+\frac{a_3z^{\alpha_3}}{1}+\cdots, \tag{1.3}$$

where the a_p denote complex numbers and the α_p positive integers (cf. [3]). In fact, conditions (1.2) for a certain J-fraction are shown to be a specialization of the orthogonality relations for a C-fraction. Furthermore, necessary and sufficient conditions are obtained for the unique existence of the polynomials $B_p^*(z)$ in terms of the sequence $\{c_p\}$ (Theorem 2.2).

2. **Orthogonal polynomials constructed from the denominators of the approximants of a *C*-fraction.** Let $A_p(z)$ and $B_p(z)$ denote the numerator and denominator, respectively, of the pth approximant

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

of a C-fraction (1.3). The recurrence formulas

$$
\begin{aligned}
&A_0(z) = 1, \qquad A_1(z) = 1 + a_1 z^{\alpha_1}, \\
(2.1)\quad &A_{p+1}(z) = A_p(z) + a_{p+1} z^{\alpha_{p+1}} A_{p-1}(z), \\
&B_0(z) = 1, \qquad B_1(z) = 1, \\
&B_{p+1}(z) = B_p(z) + a_{p+1} z^{\alpha_{p+1}} B_{p-1}(z), \qquad p = 1, 2, \cdots,
\end{aligned}
$$

show that $A_p(z)$ and $B_p(z)$ are polynomials of the form

$$
(2.2)\qquad
\begin{aligned}
A_p(z) &= \gamma_0^{(p)} + \gamma_1^{(p)} z + \cdots + \gamma_{s_p}^{(p)} z^{s_p}, \\
B_p(z) &= 1 + \beta_1^{(p)} z + \cdots + \beta_{t_p}^{(p)} z^{t_p}.
\end{aligned}
$$

From the determinant formula

$$A_p(z)B_{p-1}(z) - A_{p-1}(z)B_p(z) = (-1)^p a_1 a_2 \cdots a_{p+1} z^{\alpha_1+\alpha_2+\cdots+\alpha_{p+1}},$$

it follows that there is determined uniquely a power series

$$(2.3)\qquad P(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

such that the power series expansion for $A_p(z)/B_p(z)$ agrees term by term with the power series $P(z)$ up to the term involving $z^{\alpha_1+\alpha_2+\cdots+\alpha_p+1}$, that is,

$$P(z)B_p(z) - A_p(z) = (-1)^p a_1 a_2 \cdots a_{p+1} z^{\alpha_1+\alpha_2+\cdots+\alpha_{p+1}}.$$

This uniquely determined power series is called the corresponding power series.

In [3] it was shown that the algorithm

$$
(2.4)\quad (c_n, c_{n-1}, c_{n-2}, \cdots) \begin{pmatrix} 1 \\ \beta_1^{(p)} \\ \beta_2^{(p)} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}
= \begin{cases} 0 \text{ if } \alpha_0 + \alpha_1 + \cdots + \alpha_p < n < \alpha_0 + \alpha_1 + \cdots + \alpha_{p+1}, \\ (-1)^p a_1 a_2 \cdots a_{p+1} \text{ if } n = \alpha_0 + \alpha_1 + \cdots + \alpha_{p+1}, \\ p = 0, 1, \cdots \ (\alpha_0 = 0), \end{cases}
$$

combined with formulas (2.1), gives the a_p and α_p of (1.3) corresponding to the power series $P(z)$ (2.3). Conversely, the coefficients c_p of the power series expansion $P(z)$ are determined by (2.1) and (2.4) when the corresponding C-fraction (1.3) is given.

Let the polynomials $B_p^*(z)$ be defined as follows:

$$(2.5)\qquad \begin{aligned} &B_p^*(z) = z^{n_p}B_p(1/z) = z^{n_p} + \beta_1^{(p)} z^{n_p-1} + \beta_2^{(p)} z^{n_p-2} + \cdots, \\ &\qquad\qquad \text{where } n_p = \begin{cases} s_p & \text{if } \ s_p \geqq t_p, \\ t_p & \text{if } \ s_p < t_p, \ p = 0, 1, \cdots . \end{cases} \end{aligned}$$

Let S be defined as the operator which replaces every z^p by c_{p+1} in any polynomial upon which it operates, and let the sequence

$$(2.6)\qquad \{c_p\}: c_1, c_2, c_3, \cdots, \qquad\qquad (c_0 = 1, c_i = 0, i < 0),$$

be the coefficients of the power series (2.3) corresponding to the C-fraction (1.3). From the algorithm (2.4)

$$(2.7)\qquad \begin{aligned} &\text{(i)}\ \ S(z^k B_p^*(z)) \begin{cases} = 0 & \text{for} \quad k = 0, 1, \cdots, \sum_{i=1}^{p+1} \alpha_i - n_p - 2, \\ \neq 0 & \text{for} \quad k = \sum_{i=1}^{p+1} \alpha_i - n_p - 1; \end{cases} \\ &\text{(ii)}\ \ S(z^k B_p^*(z)) = (-1)^p a_1 a_2 \cdots a_{p+1} \quad \text{for} \quad k = \sum_{i=1}^{p+1} \alpha_i - n_p - 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 0, 1, \cdots . \end{aligned}$$

Let $B_q^*(z) = z^{n_q} + \beta_1^{(q)} z^{n_q - 1} + \beta_2^{(q)} z^{n_q - 2} + \cdots$ be one of the polynomials defined by (2.5). From (2.7) the following orthogonality relations hold between $B_p^*(z)$ and $B_q^*(z)$ relative to the operator S and the sequence $\{c_p\}$:

$$(2.8)\qquad S(B_p^*(z) B_q^*(z)) = \begin{cases} 0 & \text{for} \quad n_q = 0, 1, \cdots, \sum_{i=1}^{p+1} \alpha_i - n_p - 2, \\ (-1)^p a_1 a_2 \cdots a_{p+1} \neq 0 & \\ \text{for} \quad n_q = \sum_{i=1}^{p+1} \alpha_i - n_p - 1, & p = 0, 1, \cdots . \end{cases}$$

Consequently the following theorem holds.

THEOREM 2.1. *Let the C-fraction* (1.3) *be given. By the algorithm* (2.4) *the sequence* $\{c_p\}$ (2.6) *of coefficients of the corresponding power series* (2.3) *can be computed. Then the polynomials* $B_p^*(z)$, *which are found by* (2.5) *from the denominators* $B_p(z)$ *of the successive approximants of* (1.3), *satisfy the orthogonality relations* (2.8) *relative to the operator S and the sequence* $\{c_p\}$.

From formulas (2.4) or conditions (2.7) there follows directly a condition necessary for the existence of the polynomials $B_p(z)$ and consequently the polynomials $B_p^*(z)$, namely, that

$$\begin{aligned}
&\Delta(0, 0) \neq 0, \Delta(0, i) = 0, i = 1, 2, \cdots, s_1 - 1, \Delta(0, s_1) \neq 0;\\
&\Delta(t_p - 1, s_p) \neq 0, \Delta(t_p - 1 + i, s_p + i) = 0,\\
&\qquad\qquad\qquad i = 1, 2, \cdots, \sum_{i=1}^{p+1} \alpha_i - t_p - s_p - 1,\\
&\Delta\left(\sum_{i=1}^{p+1} \alpha_i - s_p - 1, \sum_{i=1}^{p+1} \alpha_i - t_p \right) \neq 0, \qquad p = 1, 2, \cdots,
\end{aligned} \tag{2.9}$$

where

$$\Delta(j, k) = \begin{vmatrix} c_{k-j}, & c_{k-j+1}, & \cdots, & c_k \\ c_{k-j+1}, & c_{k-j+2}, & \cdots, & c_{k+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_k, & c_{k+1}, & \cdots, & c_{k+j} \end{vmatrix}, \qquad j, k = 0, 1, \cdots$$

(cf. [3, (3.2)]). In fact, the coefficients $a_p \neq 0$ of (1.3) may be found in terms of the determinants Δ provided conditions (2.9) hold.

Conversely, given a sequence $\{c_p\}$ for which conditions (2.9) hold, one can construct a unique system of polynomials $B_p^*(z)$ such that (2.7) hold, and consequently (2.8), and the $B_p^*(z)$ are given by formulas (2.1), (2.5), and (2.7) (ii). For, by conditions (2.9), $\Delta(0, 0) = c_0 = 1$, $\Delta(0, i) = 0$, $i = 1, \cdots, s_1 - 1$, or $c_1 = \cdots = c_{s_1-1} = 0$, and $\Delta(0, s_1) = c_{s_1} \neq 0$, it follows that $s_1 = \alpha_1$, since $c_{\alpha_1} \neq 0$. Then equations (2.7) (i) hold, that is, $S(z^k B_0^*(z)) = 0$, $k = 0, \cdots, \alpha_1 - 2$; if $B_0^*(z) = B_0(z) = 1$, and $S(z^{\alpha_1 - 1} B_0^*(z) = a_1$ if $c_{\alpha_1} = a_1$. Thus α_1 and a_1 are determined. Suppose now $B_1^*(z) = z^{n_1} + \beta_1^{(1)} z^{n_1 - 1} + \beta_2^{(1)} z^{n_1 - 2} + \cdots = z^{\alpha_1}$ is computed by (2.1) and (2.5). By the algorithm (2.4) .

$$\sum_{i=0}^{\alpha_1} c_{\alpha_1 - i + j} \beta_i^{(1)} = 0, \qquad j = 1, \cdots, \alpha_2 - 1, \tag{2.10}$$

$$\sum_{i=0}^{\alpha_1} c_{\alpha_1 + \alpha_2 - i} \beta_i^{(1)} = -a_1 a_2. \tag{2.11}$$

These equations are exactly conditions (2.7) for $n_1 = \alpha_1$. But since equations (2.7)(i) may be solved for the coefficients $\beta^{(1)}$ uniquely because (2.9) hold, the polynomial $B_1^*(z)$ as determined by (2.1) and (2.5) is the unique polynomial which satisfies (2.7). The value of a_2 may be determined from (2.11) and the value of α_2 from (2.9). In exactly the same way one may show for $p = 2, 3, \cdots$ that, from a given sequence $\{c_p\}$ for which conditions (2.9) hold, unique polynomials $B_p^*(z)$ may be found by (2.1), (2.5), and (2.7)(ii) such that (2.7) hold and consequently conditions (2.8). This completes the proof of the following theorem.

THEOREM 2.2. *Let* $\{c_p\}$ *be a given sequence of constants. There exists a unique sequence of polynomials* $B_p^*(z)$ (2.5) *which satisfy conditions* (2.7) *and consequently relations* (2.8) *relative to the operator* S *and the sequence* $\{c_p\}$ *if and only if conditions* (2.9) *hold. These polynomials are determined by the recurrence formulas* (2.1), (2.5), *and* (2.7) (ii).

The polynomials $B_p(z)$ (2.1) *are the denominators of the successive approximants of the C-fraction* (1.3) *corresponding to the power series* (2.3) *with coefficients* c_p.

If conditions (2.9) hold for $p=1, \cdots, m$, then a finite sequence of orthogonal polynomials $B_p^*(z)$ may be constructed. On the other hand, if (2.9) hold for all values of m, there exists an infinite sequence of orthogonal polynomials $B_p^*(z)$.

3. **Special** ***C*****-fractions.** The C-fraction (1.3) (and its corresponding power series) is called regular if all of its approximants are Padé approximants (cf. [5]). In [3] necessary and sufficient conditions for regularity are found in terms of the s_p and t_p (cf. (2.2)). There is a special class of regular C-fractions called α-regular, in which the condition of regularity depends only upon the exponents α_p, that is, the α_p must satisfy the relations

$$
\begin{aligned}
&\alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1} = s_{2p+1} \geqq \omega + 1 + t_{2p+1}, \\
&\alpha_0 + \alpha_2 + \cdots + \alpha_{2p} \quad = t_{2p} \geqq s_{2p} - \omega, \qquad p = 0, 1, \cdots,
\end{aligned}
\tag{3.1}
$$

where ω is an integer. In [3] it is shown that necessary and sufficient for α-regularity are the conditions

$$
\begin{aligned}
&\Delta(i, \omega + 1 + i) = 0, \; i = 0, 1, \cdots, h_1 - \omega - 2, \\
&\Delta(h_1 - \omega - 1, h_1) \neq 0; \\
&\Delta(h_p - \omega - 2, h_p) \neq 0, \; \Delta(h_p - \omega - 1 + i, h_p + 1 + i) = 0 \\
&\qquad\qquad i = 0, 1, \cdots, g_p - h_p + \omega - 1, \\
&\Delta(g_p - 1, g_p + \omega + 1) \neq 0; \\
&\Delta(g_p - 1, g_p + \omega) \neq 0, \; \Delta(g_p + i, g_p + \omega + 1 + i) = 0, \\
&\qquad\qquad i = 0, 1, \cdots, h_{p+1} - g_p - \omega - 2, \\
&\Delta(h_{p+1} - \omega - 1, h_{p+1}) \neq 0, \qquad p = 1, 2, \cdots,
\end{aligned}
\tag{3.2}
$$

where $g_0=0$, $g_p=\alpha_2+\alpha_4+ \cdots +\alpha_{2p}$, $h_p=\alpha_1+\alpha_3+ \cdots +\alpha_{2p-1}$. Conditions (3.2) reduce to (2.9) when s_p and t_p satisfy (3.1).

For α-regular continued fractions, let the polynomials $B_p^*(z)$ be defined as follows:

$$(3.3)\qquad B^*_{2p+1}(z) = z^{h_{p+1}} B_{2p+1}(1/z), \quad B^*_{2p}(z) = z^{g_p+\omega} B_{2p}(1/z), \qquad p = 0, 1, \cdots .$$

Then the following theorem is a corollary to Theorem 2.2.

THEOREM 3.1. *Let* $\{c_p\}$ *be a given sequence of constants which are the coefficients of an* α*-regular power series. There exists a set of unique polynomials* $B^*_p(z)$ (3.3) *which satisfy the orthogonality relations*

$$(3.4)\qquad \begin{aligned} S(B^*_{2p+1}(z)B^*_q(z)) &= \begin{cases} 0 & \text{for } n_q < g_{p+1} - 1, \\ (-1)^{2p+1} a_1 a_2 \cdots a_{2p+2} \neq 0 & \\ & \text{for } n_q = g_{p+1} - 1; \end{cases} \\ S(B^*_{2p}(z)B^*_q(z)) &= \begin{cases} 0 & \text{for } n_q < h_{p+1} - \omega - 1, \\ (-1)^{2p} a_1 a_2 \cdots a_{2p+1} \neq 0 & \\ \text{for } n_q = h_{p+1} - \omega - 1, \ p = 0, 1, \cdots , \end{cases} \end{aligned}$$

relative to the operator S *and the sequence* $\{c_p\}$. *These polynomials are determined by the recurrence formulas* (2.1), (3.3), *and the relations*

$$(3.5)\qquad \begin{aligned} S(z^i B^*_{2p+1}(z)) &= (-1)^{2p+1} a_1 a_2 \cdots a_{2p+2} \quad \text{for} \quad i = g_{p+1} - 1, \\ S(z^i B^*_{2p}(z)) &= (-1)^{2p} a_1 a_2 \cdots a_{2p+1} \\ &\qquad \text{for} \quad i = h_{p+1} - \omega - 1, \ p = 0, 1, \cdots . \end{aligned}$$

The polynomials $B_p(z)$ (2.1) *are the denominators of the successive approximants of an* α*-regular C-fraction.*

It may be remarked that for an α-regular C-fraction the orthogonality relations (3.4) depend entirely on the exponents α_p of the C-fraction and the integer ω, and are independent of the coefficients a_p.

A second special case of the orthogonality relations (2.8) is obtained from the denominators of the approximants of a certain continued fraction (1.1). If one specializes the C-fraction (1.3) by putting $\alpha_i = 1$, $i = 1, 2, \cdots$, and then replacing z by $1/z$, the even part (cf. [5, p. 201]) of the resulting continued fraction after certain equivalence transformations is

$$1 + \frac{a_1}{a_2 + z} - \frac{a_2 a_3}{(a_3 + a_4) + z} - \frac{a_4 a_5}{(a_5 + a_6) + z} - \cdots .$$

This is a J-fraction for which the orthogonality relations (2.8) reduce to the known conditions (1.2).

4. Analogue to the Christoffel-Darboux formula. The denominators D_p of the successive approximants of a J-fraction (1.1) are connected by the identity (cf. [1], also [2])

$$D_0(z)D_0(w) + D_1(z)D_1(w) + \cdots + D_p(z)D_p(w) = \frac{D_{p+1}(z)D_p(w) - D_p(z)D_{p+1}(w)}{z - w}, \qquad p = 0, 1, \cdots .$$

A similar relation for the denominators of the approximants of a C-fraction is shown in the following theorem.

THEOREM 4.1. *The denominators B_p of the approximants of a C-fraction* (1.3) *are connected by the identity*

$$\begin{aligned} &B_{p+1}(z)B_p(w) - B_p(z)B_{p+1}(w) \\ &\quad = a_{p+1}B_{p-1}(z)B_{p-1}(w)\left[z^{\alpha_{p+1}} - w^{\alpha_{p+1}}\right] \\ &\qquad + a_{p+1}a_pB_{p-2}(z)B_{p-2}(w)\left[z^{\alpha_{p+1}}w^{\alpha_p} - w^{\alpha_{p+1}}z^{\alpha_p}\right] \\ &\qquad + a_{p+1}a_pa_{p-1}B_{p-3}(z)B_{p-3}(w)\left[z^{\alpha_{p+1}}w^{\alpha_p}z^{\alpha_{p-1}} - w^{\alpha_{p+1}}z^{\alpha_p}w^{\alpha_{p-1}}\right] \\ &\qquad + \cdots + a_{p+1}a_p \cdots a_2B_0(z)B_0(w)\left[z^{\alpha_{p+1}}w^{\alpha_p} \cdots \right. \\ &\qquad\qquad \left. - w^{\alpha_{p+1}}z^{\alpha_p} \cdots \right]. \end{aligned} \tag{4.1}$$

PROOF. (4.1) is derived from the recurrence formula (2.1) for B_p. For $B_{p+1}(z)B_p(w) - B_p(z)B_{p+1}(w) = B_p(w)\left[B_p(z) + a_{p+1}z^{\alpha_{p+1}}B_{p-1}(z)\right] - B_p(z)\left[B_p(w) + a_{p+1}w^{\alpha_{p+1}}B_{p-1}(w)\right] = a_{p+1}\left[z^{\alpha_{p+1}}B_p(w)B_{p-1}(z) - w^{\alpha_{p+1}}B_p(z)B_{p-1}(w)\right] = a_{p+1}B_{p-1}(z)B_{p-1}(w)\left[z^{\alpha_{p+1}} - w^{\alpha_{p+1}}\right] + a_{p+1}a_p \cdot \left[z^{\alpha_{p+1}}w^{\alpha_p}B_{p-1}(z)B_{p-2}(w) - w^{\alpha_{p+1}}z^{\alpha_p}B_{p-1}(w)B_{p-2}(z)\right]$. By a repetition of this process, formula (4.1) is ultimately obtained.

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CONVERGENCE OF CONTINUED FRACTIONS IN PARABOLIC DOMAINS

H. S. WALL

1. **Introduction.** The principal object of this paper is to establish the following theorem.

THEOREM A. *Let* $c_1, c_2, c_3, \cdots$ *be a sequence of complex numbers such that, for* $p=1, 2, 3, \cdots,$

$$|c_p| - R(c_p e^{i(\phi_p+\phi_{p+1})}) \leqq 2r\cos\phi_p\cos\phi_{p+1}(1-g_{p-1})g_p, \tag{1.1}$$

where $r, \phi_1, \phi_2, \phi_3, \cdots, g_0, g_1, g_2, \cdots$ *are real numbers satisfying the inequalities*

$$\begin{aligned} 0<r<1, \quad -\pi/2+c \leqq \phi_p &\leqq +\pi/2-c \qquad (0<c<\pi/2), \\ 0 \leqq g_{p-1} &\leqq 1, \qquad p=1, 2, 3, \cdots, \end{aligned} \tag{1.2}$$

c and r being independent of p. The continued fraction

$$\cfrac{1}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\cdots}}} = \mathop{K}_{p=1}^{\infty}\frac{c_{p-1}}{1} \qquad (c_0=1) \tag{1.3}$$

converges if, and only if, (a) *some* c_p *vanishes, or* (b) $c_p \neq 0,$ $p=1, 2, 3, \cdots,$ *and the series* $\sum|d_p|$ *diverges, where*

$$d_1 = 1, \quad d_{p+1} = \frac{1}{c_p d_p}, \qquad p=1, 2, 3, \cdots. \tag{1.4}$$

We note the following particular cases of Theorem A.

(a) The continued fraction

$$\mathop{K}_{p=1}^{\infty}\frac{1}{k_p e^{i\phi_p}} = \frac{1}{k_1 e^{i\phi_1}}\mathop{K}_{p=1}^{\infty}\frac{c_{p-1}}{1}, \quad c_0=1, \quad c_p = \frac{e^{-i(\phi_p+\phi_{p+1})}}{k_p k_{p+1}},$$

in which $k_p>0$, $-\pi/2+c \leqq \phi_p \leqq +\pi/2-c$, $0<c<\pi/2$, converges if, and only if, the series $\sum k_p$ diverges (Stieltjes [6] ($\phi_p=\phi$); E. B. Van Vleck [8]).[1] For an extension of this theorem in a direction different from Theorem A, see Scott and Wall [5].

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

(b) If

$$|c_p| - R(c_p e^{2i\phi}) \leqq 2^{-1}r \cos^2 \phi, \qquad p = 1, 2, 3, \cdots,$$

where $0<r<1$, $-\pi/2<\phi<+\pi/2$, then the continued fraction (1.3) converges if, and only if, (a) some c_p vanishes, or (b) $c_p \neq 0$, $p=1, 2, 3, \cdots$, and the series $\sum |d_p|$, defined by (1.4), diverges (Paydon and Wall [3]). The case $\phi=0$ of this theorem holds with $r=1$ (Scott and Wall [4]).

(c) Inasmuch as

$$\frac{1}{(|c_p|)^{1/2}} \leqq \frac{|d_p| + |d_{p+1}|}{2},$$

it follows from Theorem A that a sufficient condition for convergence of the continued fraction (1.3), satisfying (1.1) and (1.2), is the divergence of the series $\sum(1/(|c_p|)^{1/2})$ (Wall and Wetzel [7]). This sufficient condition is not necessary, as is shown by the example $d_{2p-1}=1$, $d_{2p}=s^p$, $0<s<1$.

2. **Preliminary theorem.** Let $x_p = X_p(z)$ and $x_p = Y_p(z)$ be the solutions of the system of equations

$$(2.1) \qquad -a_{p-1}x_{p-1} + (b_p + z_p)x_p - a_p x_{p+1} = 0, \qquad p = 1, 2, 3, \cdots,$$

under the initial conditions $x_0 = -1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. We suppose that $a_0 = 1$, a_1, a_2, a_3, $\cdots$ are constants not zero, b_1, b_2, b_3, $\cdots$ are constants, and z_1, z_2, z_3, $\cdots$ are parameters. The *theorem of invariability* [2, 5] states that if the series

$$(2.2) \qquad \sum |X_p(z)|^2, \qquad \sum |Y_p(z)|^2$$

converge for $z_p = h_p$, $p=1, 2, 3, \cdots$, then they converge uniformly for $|z_p - h_p| \leqq M$, for every finite constant M independent of p. The *determinate case* is said to hold for the continued fraction

$$(2.3) \qquad - \mathop{K}_{p=1}^{\infty} \frac{-a_{p-1}}{b_p + z_p}$$

if at least one of the series (2.2) diverges for $z_p=0$, $p=1, 2, 3, \cdots$. In the contrary event, the *indeterminate case* is said to hold.

THEOREM 2.1. *If* $|b_p| \leqq M$, $p=1, 2, 3, \cdots$, *where* M *is a finite constant independent of* p, *then the determinate case holds for the continued fraction* (2.3) *if, and only if, the series* $\sum |d_p'|$ *diverges, where*

$$(2.4) \qquad d_1' = 1, \qquad d_{p+1}' = \frac{1}{a_p^2 d_p'}, \qquad p = 1, 2, 3, \cdots.$$

PROOF. From the condition imposed upon b_p, and the theorem of invariability, it follows immediately that the determinate case holds if, and only if, at least one of the series (2.2) diverges for $z_p = -b_p$, $p = 1, 2, 3, \cdots$. On putting these values of the z_p in (2.1) we find that $|X_{2p}(z)|^2$, $|Y_{2p}(z)|^2$, $|X_{2p+1}(z)|^2$ and $|Y_{2p+1}(z)|^2$ take on the values $|d'_{2p}|$, 0, 0, and $|d'_{2p+1}|$, respectively. Therefore, the determinate case holds if, and only if, the series $\sum |d'_p|$ is divergent.

It is easy to see that when we drop the condition that the $|b_p|$ be bounded, then the determinate case may hold when the series $\sum |d'_p|$ converges. It seems likely, however, that the divergence of the series $\sum |d'_p|$ implies the determinate case whether or not the $|b_p|$ are bounded.

3. **Proof of Theorem A.** Let $\delta > 0$ be chosen sufficiently small in order that

$$r\left[1 + \delta \sec\left(\frac{\pi}{2} - c\right)\right]^2 \leqq 1.$$

Determine numbers a_p^2 by means of the equations

$$c_p = \frac{a_p e^{-i(\phi_p + \phi_{p+1})}}{(1 + \delta \sec \phi_p)(1 + \delta \sec \phi_{p+1})}, \qquad p = 1, 2, 3, \cdots.$$

Let the partial numerators a_p^2 in (2.3) have these values, and there take

$$z_p = i\delta, \qquad b_p + z_p = ie^{i\phi_p}(1 + \delta \sec \phi_p).$$

Then that continued fraction and (1.3) are equivalent, except for an unessential factor. Moreover, by (1.1),

$$|a_p^2| - R(a_p^2) \leqq 2\beta_p\beta_{p+1}(1 - g_{p-1})g_p, \qquad p = 1, 2, 3, \cdots,$$

where $\beta_p = I(b_p) = \cos \phi_p > 0$. Thus, the continued fraction (2.3) is *positive definite* [7, 1]. Since $I(z_p) = \delta > 0$, it follows that the continued fraction (2.3) converges if (a) some a_p vanishes, that is, some c_p vanishes, or (b) $a_p \neq 0$, $p = 1, 2, 3, \cdots$, and the determinate case holds. Since the $|b_p|$ are bounded, it follows from Theorem 2.1 that the determinate case holds if the series $\sum |d'_p|$ defined by (2.4) diverges. We note that this series diverges if, and only if, the series $\sum |d_p|$ defined by (1.4) diverges. Therefore, the continued fraction (1.3) converges if (a) some c_p vanishes, or (b) $c_p \neq 0$, $p = 1, 2, 3, \cdots$, and the series $\sum |d_p|$, defined by (1.4), diverges, If, on the other hand, this series converges, then the continued fraction diverges by virtue of a theorem of von Koch.

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THE UNIVERSITY OF TEXAS

REMARKS ON THE NOTION OF RECURRENCE

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We give in several lines a simple proof of Poincaré's recurrence theorem.

THEOREM. *Let* Ω *be a point set of finite Lebesgue measure, and* T *a one-to-one measure-preserving transformation of* Ω *into itself.*[1] *Let* $B \subset A \subset \Omega$ *be measurable sets such that, if* $b \in B$, $T^n b \notin A$ *for all positive integral* n. *Then the measure* $m(B)$ *of* B *is* 0.

PROOF. First we show that, if $i < j$, $(T^i B)(T^j B) = 0$. Suppose $c \in T^j B$; then from the hypothesis on B it follows that j is the smallest integer such that $T^{-j} c \in A$. Hence $c \notin T^i B$. Now if $m(B) = \delta > 0$, Ω would contain infinitely many disjunct sets $T^n B$, each of measure δ. This contradiction proves the theorem.

The following generalization of the above theorem is trivially obvious: The result holds if we replace the hypothesis that T is measure-preserving by the following: If $m(D) > 0$, $\limsup_i m\{T^i(D)\} > 0$.

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[1] For a discussion in probability language see M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1002–1010.

Another obvious generalization is this: Let C be the set of all points c of A such that $T^nc\in A$ for only finitely many n. Then $m(C)=0$ (for $C\subset\sum_{l=0}^{\infty} T^{-l}B$).

The following is a simple derivation of Kac's theorem on the mean recurrence time.[2]

THEOREM. *Let T above be metrically transitive. Let $a\in A-B$, and $n(a)$ be the smallest positive integer such that $T^na\in A$. Let $m(A)>0$. Then*

$$\int_{A-B} n(a)dm = m(\Omega).$$

PROOF. Define $A_k=\{n(a)=k\}$. Let $i<j$, $i'<j'$, $j\neq j'$. We notice:

(a) $(T^iA_j)(T^{i'}A_{j'})=0$. For T has a single-valued inverse and $A_jA_{j'}=0$. If T^iA_j and $T^{i'}A_{j'}$ had a point s in common, then $T^{-i}s\in A_j$, $T^{-i'}s\in A_{j'}$, in violation of the definition of j and j'.

(b)
$$\int_{A-B} n(a)dm = m\left(\sum_{k=1}^{\infty}\sum_{l=0}^{k-1} T^lA_k\right).$$

(c) Metric transitivity implies that almost every point in Ω lies in some T^lA_k, that is, $m(\sum\sum T^lA_k)=m(\Omega)$.

This proves the desired result.

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[2] Kac, loc. cit. Theorem 2.

THE NORMAL APPROXIMATION TO THE POISSON DISTRIBUTION AND A PROOF OF A CONJECTURE OF RAMANUJAN[1]

TSENG TUNG CHENG

1. **Summary.** The Poisson distribution with parameter λ is given by

$$F(x) = \sum_{r=0}^{n} p_r \quad \text{where} \quad p_r = \frac{\lambda^r}{r!} e^{-\lambda}, \qquad n = [x], \tag{1.1}$$

for $x \geqq 0$. It is well known that $F(x)$ converges to the normal distribution as $\lambda \to \infty$. We shall prove the following theorem.

THEOREM I. *Let* $x = \lambda^{-1/2}(n - \lambda + 1/2)$. *Then*

$$\begin{aligned}\sum_{r=0}^{n} \frac{\lambda^r}{r!} e^{-\lambda} = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-(1/2)t^2} dt \\ + (1/6)\,(2\pi\lambda)^{-1/2}(1 - x^2)e^{-(1/2)x^2} + \delta,\end{aligned} \tag{1.2}$$

where δ *satisfies the inequality*

$$| \delta | < .076\lambda^{-1} + .043\lambda^{-3/2} + .13\lambda^{-2}. \tag{1.3}$$

This formula is analogous to Uspensky's [1][2] estimate of the error term in the normal approximation to the binormal distribution and our proof consists in an adaptation of Uspensky's method.

At the same time we verify the following conjecture of Ramanujan: *for every positive integer* n *the value of* θ *which satisfies the equation*

$$\left\{1 + \frac{n}{1!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \theta \frac{n^n}{n!}\right\} e^{-n} = 1/2 \tag{1.4}$$

lies between 1/2 *and* 1/3, *and tends to* 1/3 *as* $n \to \infty$. A proof of the above statement was given by Szegö [3]. We give a more elementary proof, using only standard tools, for the following theorem.

THEOREM II. *For* $n \geqq 7$ *the root of the equation* (1.4) *in* θ *lies between* .37 *and* 1/3.

Finally, we shall obtain the following asymptotic expansion of θ:

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[1] This paper contains the results of a master's thesis done under the direction of Professor W. Feller and accepted by Cornell University in September, 1947.

[2] Numbers in brackets refer to the references cited at the end of the paper.

$$\theta = 1/3 + (4/135)n^{-1} + (8/2835)n^{-2} + O(n^{-3}). \tag{1.5}$$

2. **Proof of Theorem I.** From the characteristic function

$$\sum_{r=0}^{\infty} p_r e^{irt} = e^{\lambda(e^{it}-1)},$$

we get

$$\int_{-\pi}^{\pi} e^{-irt}\cdot e^{\lambda(e^{it}-1)}dt = 2\pi p_r. \tag{2.1}$$

From (2.1) it follows that

$$\begin{aligned}\sum_{r=0}^{n} p_r = (2\pi)^{-1}\int_0^{\pi} &(\sin t/2)^{-1} \\ &\cdot \exp\{\lambda(\cos t - 1)\} \sin[(n + 1/2)t - \lambda \sin t]dt \\ &+ (2\pi)^{-1}\int_0^{\pi}(\sin t/2)^{-1}\exp\{\lambda(\cos t - 1)\}\sin(\lambda \sin t + t/2)dt.\end{aligned}$$

The second integral is independent of λ, since its derivative with respect to λ vanishes, as may easily be verified. Putting $\lambda = 0$ we see the value of the integral is 1/2. Hence we have

$$\begin{aligned}\sum_{r=0}^{n}\frac{\lambda^r}{r!}e^{-\lambda} - \frac{1}{2} = (2\pi)^{-1}\int_0^{\pi}&(\sin t/2)^{-1} \\ &\cdot\exp\{\lambda(\cos t - 1)\}\sin[(n + 1/2)t - \lambda \sin t]dt \\ = I.&\end{aligned} \tag{2.2}$$

To estimate the integral I we introduce

$$I_1 = (2\pi)^{-1}\int_0^{\infty} e^{-\lambda t^2/2}(t/2)^{-1}\sin[\lambda^{1/2}xt + \lambda(t - \sin t)]dt.$$

Then

$$\begin{aligned}|I - I_1| \leqq (2\pi)^{-1}&\int_{\pi}^{\infty} e^{-\lambda t^2/2}(t/2)^{-1}dt \\ &+ (2\pi)^{-1}\int_0^{\pi}\left\{e^{-2\lambda \sin^2 t/2}(\sin t/2)^{-1} - e^{-\lambda t^2/2}(t/2)^{-1}\right\}dt \\ = J_1 + J_2 + J_3,&\end{aligned} \tag{2.3}$$

where the integrals J_k are defined and estimated as follows:

$$(2.4)\quad \begin{aligned} J_1 &= (2\pi)^{-1}\int_0^\pi \left\{e^{-2\lambda \sin^2 t/2}(\sin t/2)^{-1}\cos t/2 - e^{-\lambda t^2/2}(t/2)^{-1}\right\}dt \\ &= -(2\pi)^{-1}\int_2^\pi e^{-\lambda t^2/2}(t/2)^{-1}dt < 0; \end{aligned}$$

$$(2.5)\quad J_2 = (2\pi)^{-1}\int_\pi^\infty e^{-\lambda t^2/2}(t/2)^{-1}dt < (32\pi)^{-1}\int_\pi^\infty t^3e^{-2\lambda t^2/\pi^2}dt;$$

$$(2.6)\quad \begin{aligned} J_3 &= (2\pi)^{-1}\int_0^\pi e^{-2\lambda \sin^2 t/2}(\sin t/2)^{-1}(1-\cos t/2)dt \\ &< (2\pi)^{-1}\int_0^\pi e^{-2\lambda \sin^2 t/2}\cdot(\sin t/2)^{-1} \\ &\quad \cdot\left\{\frac{\sin^2 t/2\cos t/2}{2} + (1-\cos t/2)\sin^2 t/2\right\}dt \\ &< (8\pi)^{-1}\int_0^2 te^{-\lambda t^2/2}dt + (32\pi)^{-1}\int_0^\pi t^3e^{-2\lambda t^2/\pi^2}dt. \end{aligned}$$

It follows from (2.3)–(2.6) that

$$(2.7)\quad \begin{aligned} |I - I_1| &< (8\pi)^{-1}\int_0^\infty te^{-\lambda t^2/2}dt + (32\pi)^{-1}\int_0^\infty t^3e^{-2\lambda t^2/\pi^2}dt \\ &= (8\pi\lambda)^{-1} + (\pi^3/256)\lambda^{-2} < (8\pi\lambda)^{-1} + .13\lambda^{-2}. \end{aligned}$$

Now we write

$$(2.8)\quad \begin{aligned} I_1 &= \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\cdot\sin\lambda^{1/2}xt\cdot\cos\lambda(t-\sin t)dt \\ &\quad + \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\cdot\cos\lambda^{1/2}xt\cdot\sin\lambda(t-\sin t)dt \\ &= \left\{\pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\sin\lambda^{1/2}xt\,dt + J_4\right\} \\ &\quad + \left\{\pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\cdot\cos\lambda^{1/2}xt\cdot\frac{\lambda t^3}{6}dt + J_5\right\} \\ &= (2\pi)^{-1/2}\int_0^x e^{-t^2/2}dt + \frac{1}{6}(2\pi\lambda)^{-1/2}(1-x^2)e^{-x^2/2} + J_4 + J_5, \end{aligned}$$

where

$$J_4 = \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}[\cos\lambda(t-\sin t) - 1]dt,$$

$$J_5 = \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\cos\lambda^{1/2}xt\cdot\left[\sin\lambda(t-\sin t)-\frac{\lambda t^3}{6}\right]dt.$$

Upper limits for $|J_4|$ and $|J_5|$ are obtained as follows:

$$\begin{aligned}|J_4| &< \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}.\frac{\lambda^2(t-\sin t)^2}{2}\,dt\\ &< \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}.\frac{\lambda^2 t^6}{72}\,dt = (9\pi\lambda)^{-1};\end{aligned}\tag{2.9}$$

$$\begin{aligned}|J_5| &< \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\left\{\left[\frac{\lambda t^3}{6}-\lambda(t-\sin t)\right]\right.\\ &\qquad \left.+\left[\lambda(t-\sin t)-\sin\lambda(t-\sin t)\right]\right\}dt\\ &< \pi^{-1}\int_0^\infty t^{-1}e^{-\lambda t^2/2}\left[\frac{\lambda t^5}{120}+\frac{\lambda^3 t^9}{6^4}\right]dt\\ &= (2\pi)^{-1/2}\lambda^{-3/2}[1/40+35/432] < .043\lambda^{-3/2}.\end{aligned}\tag{2.10}$$

From (2.7)–(2.10) and the inequality $(8\pi)^{-1}+(9\pi)^{-1}<.076$ the required result follows immediately.

3. **Proof of Theorem II.** Solving (1.4) for θ and using the formula (2.2), we get

$$2\theta - 1 = -I'/(n^n e^{-n}/n!),\tag{3.1}$$

where

$$\begin{aligned}I' &= (2\pi)^{-1}\int_0^\pi\left(\sin\frac{t}{2}\right)^{-1}\cdot\exp\{n(\cos t-1)\}\\ &\quad\cdot\left\{\sin\left[\left(n+\frac{1}{2}\right)t-n\sin t\right]\right.\\ &\quad\left.+\sin\left[\left(n-\frac{1}{2}\right)t-n\sin t\right]\right\}dt\\ &= \pi^{-1}\int_0^\pi\left(\sin\frac{t}{2}\right)^{-1}\cdot\exp\{n(\cos t-1)\}\cdot\cos\frac{t}{2}\cdot\sin n(t-\sin t)dt\\ &= 2\pi^{-1}\int_0^2 t^{-1}e^{-nt^2/2}\cdot\sin n\left[2\arcsin\frac{t}{2}-t\left(1-\frac{t^2}{4}\right)^{1/2}\right]dt.\end{aligned}$$

By Maclaurin expansion,

$$2\arcsin t/2 - t(1-t^2/4)^{1/2} = t^3/6 + t^5/80 + 3t^7/1792 + t^9 f(t),$$

where $f(t)$ is an increasing function in the interval $0 \leqq t \leqq 2$. Hence

$$\begin{aligned}
\sin n[2 \operatorname{arc} \sin t/2 - t(1 - t^2/4)^{1/2}] & \\
&\leqq \sin (nt^3/6 + nt^5/80) + 3nt^7/1792 + nt^9 f(2) \\
&\leqq (nt^3/6 + nt^5/80) - (nt^3/6 + nt^5/80)^3/6 \\
&\quad + (nt^3/6 + nt^5/80)^5/120 + 3nt^7/1792 + .0024nt^9.
\end{aligned}$$

Assume $n \geqq 7$. Then the last expression is positive when $t > 2$, and therefore

$$\begin{aligned}
I' &< 2\pi^{-1} \int_0^\infty t^{-1} e^{-nt^2/2} \cdot \left\{ \frac{nt^3}{6} + \frac{nt^5}{80} - \frac{1}{6}\left(\frac{nt^3}{6} + \frac{nt^5}{80}\right)^3 \right. \\
&\qquad \left. + \frac{1}{120}\left(\frac{nt^3}{6} + \frac{nt^5}{80}\right)^5 + \frac{3nt^7}{1792} + .0024nt^9 \right\} dt \qquad (3.2) \\
&< (2\pi n)^{-1/2} \{ 1/3 - .087n^{-1} + .012n^{-2} + 1.87n^{-3} \\
&\qquad + 4\cdot 1n^{-4} + 6n^{-5} + 5n^{-6} + 2n^{-7} \}.
\end{aligned}$$

In obtaining the last inequality we have, as before, used the familiar relation:

$$2(2\pi)^{-1/2} \int_0^\infty t^{2k} e^{-(1/2)t^2} dt = 1 \cdot 3 \cdot \cdots \cdot (2k - 1).$$

Taking in (3.2) $n \geqq 7$ we get

$$I' < (2\pi n)^{-1/2}(1/3 - .03n^{-1}). \qquad (3.3)$$

On the other hand we can write

$$\begin{aligned}
2 \operatorname{arc} \sin t/2 - t(1 - t^2/4)^{1/2} &= t^3/6 + t^5 g(t) \\
&\geqq t^3/6 + .057t^5,
\end{aligned}$$

because $g(t)$ increases and $g(2) < .057$. Hence

$$\begin{aligned}
I' &> 2\pi^{-1} \int_0^2 t^{-1} e^{-(1/2)nt^2} \left\{ \sin \frac{nt^3}{6} - .057nt^5 \right\} dt \\
&> 2\pi^{-1} \int_0^\infty t^{-1} e^{-(1/2)nt^2} \left\{ \frac{nt^3}{6} - \frac{1}{6}\left(\frac{nt^3}{6}\right)^3 - .057nt^5 \right\} dt.
\end{aligned}$$

Therefore

$$I' > (2\pi n)^{-1/2}(1/3) - .505n^{-1}) > .26(2\pi n)^{-1/2}, \text{ when } n \geqq 7. \qquad (3.4)$$

From (3.1), (3.3), (3.4), and Stirling's formula,

$$(2\pi n)^{1/2} n^n e^{-n} < n! < (2\pi n)^{1/2} n^n e^{-n+1/12n},$$

it follows that $2\theta - 1 < -.26$, or $\theta < .37$; and

$$\begin{aligned} 2\theta - 1 &> -\, e^{1/12n}\cdot(2\pi n)^{1/2}\cdot(2\pi n)^{-1/2}(1/3 - .03/n) \\ &> -\,(1 + .09n^{-1})(1/3 - .03n^{-1}) \\ &= -\,1/3 + .0027n^{-2} > -\,1/3, \end{aligned}$$

and therefore $\theta > 1/3$.

4. Proof of the equation (1.5).

It can be verified without difficulty that

$$I' = (2\pi n)^{-1/2}\{1/3 - 47/540n^{-1} + 71/6048n^{-2}\} + O(n^{-7/2}).$$

From a modified form of Stirling's formula due to Feller [2], namely, for $n \geqq 4$,

$$n! = (2\pi n)^{1/2} n^n \exp\left\{-\, n + \frac{1}{12n} - \frac{1+\delta}{360n^3}\right\}, \qquad |\delta| < 1/6,$$

it follows that

$$\begin{aligned} 2\theta - 1 &= -\, [1/3 - 47/540n^{-1} + 71/6048n^{-2} + \cdots] \\ &\quad \cdot [1 + 1/12n^{-1} + 1/(2.12^2)n^{-2} + \cdots] \\ &= -\, 1/3 + 8/135n^{-1} + 16/2835n^{-2} + O(n^{-3}). \end{aligned}$$

An obvious reduction then gives the equation (1.5).

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SPACES OF CONTINUOUS FUNCTIONS

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Let X be a completely regular topological space, $B(X)$ the Banach space of real-valued bounded continuous functions on X, with the usual norm $\|b\|=\sup_{x\in X}|b(x)|$. A subset $G\subset B(X)$ is called completely regular (c.r.) over X if given any closed subset $K\subset X$ and point $x_0\in X-K$, there exists a $b\in G$ such that $b(x_0)=\|b\|$ and $\sup_{x\in K}|b(x)|<\|b\|$. A topological space X is completely regular in the usual sense if and only if $B(X)$ is c.r. over X.

A Banach space B is said to act completely regularly (c.r.) on X if B is equivalent to a closed linear subspace of $B(X)$ which is c.r. over X. It is known [6][1] that if X is compact,[2] a closed linear subspace of $B(X)$ c.r. over X determines the topology of X. By this is meant that if X_1 and X_2 are compact, and a Banach space B acts c.r. on both X_1 and X_2, then X_1 is homeomorphic to X_2. If B acts c.r. on X (compact or not), X is homeomorphically imbeddable in the surface of the unit sphere in B^*_w, the conjugate space to B under the weak-* topology, and for each $b\in B$ and $x\in X$ we have the formula $b(x)=\inf_{t\in T}[\|b+t\|-\|t\|]$, where $T=\{t\in B\,|\,t(x)=\|t\|\}$.

If we weaken the definition of complete regularity so that G is c.r. over X means that for every closed set $K\subset X$ and point $x_0\in X-K$ there is a $b\in G$ such that $b(x_0)=\|b\|$, $\sup_{x\in K} b(x)<\|b\|$, then a closed linear subspace of $B(X)$ c.r. over X does not necessarily determine the topology of X. For example, if X consists of just two points, x_1 and x_2, then the subspace G of $B(X)$ consisting of all $b\in B(X)$ such that $b(x_1)=-b(x_2)$ is c.r. over X according to the weakened definition, yet it is equivalent to the space $B(Y)$, where Y consists of a single point.

Proper closed linear subspaces of $B(X)$ which are c.r. over X exist in general for both compact and non-compact X, and may contain the constant functions. This is in contrast to the situation when $B(X)$ is made into a normed ring (Banach algebra) $R(X)$ or into a Banach lattice $L(X)$; if X is compact, no proper closed subring of $R(X)$ containing the constant functions can be c.r. over X [8], and no proper closed sublattice of $L(X)$ containing the constant functions can be c.r. over X [4].

Since topological properties of X must be reflected in metric and

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] "Compact" means bicompact and Hausdorff.

algebraic properties of $B(X)$, and in such properties of every closed linear subspace G of $B(X)$ which is c.r. over X, it would appear to be fruitful to investigate the possibilities of existence of a G of a particularly simple sort, for example, separable, or finite-dimensional, or reflexive. Such a study is made here.

In the next few paragraphs necessary and sufficient conditions are proved that for a given Banach space B there exists a compact X such that B acts c.r. on X. Later, we point out that $B(X)$ is itself rarely separable, finite-dimensional, or reflexive, and the existence of a c.r. subspace of $B(X)$ with such properties is investigated.

Let B be any Banach space; by a T-set, we mean a maximal subset T of B with the property that for any finite subset $b_1, \cdots, b_n$ of T, $\|\sum b_i\| = \sum \|b_i\|$. Every $b \in B$ is contained in such a T-set.

Let E_w^* be the solid unit sphere in B_w^*, and let S_w^* be the surface of E_w^*. It is known that E_w^* is compact [1].

THEOREM 1. *Given a Banach space B, the following conditions are necessary and sufficient that there exist a compact X such that B acts* c.r. *on X:*

(1) *For each T-set in B, there is a unique point $x_T \in S_w^*$ such that $x_T(t) = \|t\|$ for all $t \in T$.*

(2) *The set $M \subset S_w^*$ of all such x_T is closed in B_w^*.*

(3) *M is the union of two disjoint closed antipodal subsets Ω, $-\Omega$.*

PROOF. The necessity of the conditions follows directly from [6, Theorem 4.1 and Lemma 2.3]. To prove the sufficiency, we show that hypotheses (1), (3) imply that B acts c.r. on Ω. Consider the linear mapping $C(B) \subset B(\Omega)$, which assigns to each $b \in B$ the function $b(x)$ defined by the formula $b(x) = x(b)$ for all $x \in \Omega$. It is clear that $C(0)$ is the function $\equiv 0$ over Ω. Also, if $C(b) = C(0)$, then $b(x) \equiv 0$ over Ω, hence over M, and in particular $b(x_T) = 0$ where x_T is the point in M which corresponds by hypothesis (1) to the T-set containing b; hence $\|b\| = 0$. It follows that C is one-to-one. Since $M \subset S_w^*$, $-\|b\| \leq b(x) \leq \|b\|$ for all $b \in B$ and all $x \in M$, and since $b(x_T) = \|b\|$ and $b(-x_T) = -\|b\|$, we see that $\|b\| = \sup_{x \in \Omega} |b(x)|$. Thus C is norm-preserving, hence an equivalence. Since B is complete, $C(B)$ is complete and hence closed in $B(\Omega)$. It remains to show that $C(B)$ is c.r. over Ω. Let x_0 be any point in Ω, let D be any closed subset of Ω not containing x_0. Let $K = D \cup -D$; K is closed in M. Let T be the T-set in B such that $x_0(t) = \|t\|$ for all $t \in T$, and let $\bar{K}$ be the closure of K in E_w^*; $\bar{K}$ does not contain x_0. For each $x \in \bar{K}$, there is a $t \in T$ such that $t(x)\,[= x(t)] < \|t\|$; for if x is in S_w^*, this follows from (1), and if $\|x\| < 1$ then $t(x) < \|t\|$. Since $\bar{K}$ is compact, and $t(x)$ is a continuous function of x for every $t \in B$, there is a finite set $t_1, \cdots, t_n \in T$ such that for each $x \in \overline{K}$, at least one t_i has the property $t_i(x) < \|t_i\|$. Hence if we let $\bar{t} = \sum t_i$, $\sup_{x \in \overline{K}} \bar{t}(x) < \|\bar{t}\|$, in particular $\sup_{x \in K} \bar{t}(x) < \|\bar{t}\|$. Since $K = -K$, we have $\sup_{x \in K} |\bar{t}(x)| < \|\bar{t}\|$, hence $\sup_{x \in D} |\bar{t}(x)| < \|\bar{t}\|$. Thus B acts c.r. on Ω. The compactness of Ω follows from (2), (3) and the compactness of E_w^*.

Condition (1) of Theorem 1 could be replaced by the equivalent condition that for each T-set in B the functional $F_T(b) = \inf_{t \in T} [\|b+t\| - \|t\|]$ be linear over B (see [6]). Furthermore, when (1) is satisfied, $x_T = F_T$.

The following three results, the first two of which are stated without proof since they are known, indicate properties which $B(X)$ in general fails to have.

THEOREM 2. *If X is completely regular, $B(X)$ is separable if and only if X is compact and metrizable* [5].

THEOREM 3. *If X is completely regular, $B(X)$ is finite-dimensional if and only if X consists of a finite number of points.*

THEOREM 4. *If X is completely regular, $B(X)$ is reflexive if and only if X consists of a finite number of points.*

PROOF. If X has only a finite number of points, $B(X)$ is reflexive because it is finite-dimensional. Now suppose X completely regular and infinite, and $B(X)$ reflexive. Let $\overline{X}$ be the Čech compactification of X. Then $B(\overline{X})$ is equivalent to $B(X)$, hence reflexive. Therefore the unit sphere E in $B(\overline{X})$ is weakly compact. We obtain a contradiction by constructing a sequence in E with no weak limit point. Let $\{x\}$ be an infinite sequence of distinct points of $\overline{X}$, and let x be a limit point of $\{x\}$ not a member of $\{x\}$; let b_i be an element of $B(\overline{X})$ with the property that $\|b_i\| = 1$, $b_i(x) = 1$, $b_i(x_j) = 0$ for $j \leqq i$; b_i exists due to the complete regularity of X. If b is a weak limit point of $\{b\}$, it follows that $b(x) = 1$, $b(x_j) = 0$ for $j = 1, 2, \cdots$, contradicting the fact that b is continuous and x is a limit point of $\{x\}$.

THEOREM 5. *If E is the unit sphere in any Hilbert (inner product) space H, there exists a closed linear reflexive subspace L of $B(E)$ which is c.r. over E and is isomorphic to H if H is of infinite dimension and $n+2$-dimensional if H is n-dimensional.*

PROOF. Let c, d be real numbers, let $h \in H$, let $P \in E$. For fixed c, d, h the function $f_{c,d,h}(P) = c\|P\|^2 + P \cdot h + d$ is a bounded continuous function over E. The set of all such functions as c, d range over the real numbers and h ranges over H, with $\|f_{c,d,h}\| = \sup_{P \in E} |f_{c,d,h}(P)|$, forms a linear subspace L of $B(E)$.

Consider the mapping $C(H_1\times H_1\times H)=L$, where $\times$ denotes direct sum, $C(c, d, h)=f_{c,d,h}$, and H_1 is the euclidean line. C is clearly linear and continuous. C is also one-to-one, for if $C(c, d, h)=C(\bar{c}, \bar{d}, \bar{h})$ then $(c-\bar{c})\ \|P\|^2+P\cdot(h-\bar{h})+(d-\bar{d})=0$ for all $P\in E$; by taking $P=0$ we see that $d=\bar{d}$, and by taking $P=k(h-\bar{h})$, where $k>0$ is so small that $k(h-\bar{h})$ is in E and $k\neq 1/(\bar{c}-c)$, we see that $h=\bar{h}$, hence $c=\bar{c}$. To show that C^{-1} is continuous, we must show that given $\epsilon>0$ there exists a δ such that if $\sup_{P\in E}|f_{c,d,h}(P)|<\delta$ then $(c^2+d^2+\|h\|^2)^{1/2}<\epsilon$. Using $\delta=\epsilon/11^{1/2}$, by taking $P=0$ we get $|d|<\epsilon/11^{1/2}$. By taking $P=h/\|h\|$ we get $|c+\|h\|+d|<\epsilon/11^{1/2}$, and by taking $P=-h/\|h\|$ we get $|c-\|h\|+d|<\epsilon/11^{1/2}$, hence $\|h\|<\epsilon/11^{1/2}$. Hence $c<3\epsilon/11^{1/2}$, and $(c^2+d^2+\|h\|^2)^{1/2}<\epsilon$.

Thus C is an isomorphism between $H_1\times H_1\times H$ and L. But $H_1\times H_1\times H$ is complete and reflexive, hence L is complete (and so closed in $B(E)$) and reflexive.[3]

If H is infinite-dimensional, H is isomorphic to $H_1\times H_1\times H$, hence to L. If H is n-dimensional, $H_1\times H_1\times H$ is $n+2$-dimensional, hence so is L.

To show L is c.r. over E, let $P_0\in E$, and let K be any closed set in $E-P_0$. Then by taking $c=-1/2$, $h=P_0$, $d=2$, we obtain $f_{c,d,h}(P)=-(\|P-P_0\|^2)/2+2+\|P_0\|^2/2$, which is non-negative over E, and has the properties

$$f_{c,d,h}(P_0)=\|f_{c,d,h}\|,\ \sup_{P\in K}|f_{c,d,h}(P)|<f_{c,d,h}(P_0).$$

COROLLARY. *If X is imbeddable in a Hilbert (inner product) space H,[4] there exists a linear subspace of $B(X)$ which is c.r. over X, and which is the linear continuous image of $H_1\times H_1\times H$ (direct sum).*

If X is imbeddable in H it is imbeddable in the solid unit sphere E in H. Let $p(X)\subset E$ be this imbedding. The mapping $F[B(E)]\subset B[p(X)]$, obtained by cutting down to $p(X)$ each continuous function on E, is clearly continuous and linear. If L is the linear subspace of $B(E)$ furnished by Theorem 5, then $F(L)$ is a linear subspace of

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[3] A Banach space $\bar{B}$ isomorphic to a reflexive Banach space B is reflexive. For let I be such an isomorphism of $\bar{B}$ onto B, and let I^* be the induced isomorphism of B^* onto $\bar{B}^*$ defined by $[I^*(x)](\bar{b})=x[I(\bar{b})]$, where $x\in B^*$. Then if $\bar{F}$ is a continuous linear functional on $\bar{B}^*$, we see that $\bar{F}(\bar{x})=\bar{x}[I^{-1}(b)]$ for all $\bar{x}\in\bar{B}^*$, where b is the point in B such that $\bar{F}[I^*(x)]=x(b)$ for all $x\in B^*$ (b exists because B is reflexive). Hence $\bar{B}$ is reflexive.

[4] X is imbeddable in a Hilbert space if and only if it is metrizable; this follows by combining recent results of C. H. Dowker (Duke Math. J. vol. 14 (1947) pp. 639–645) and A. H. Stone (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 977–982).

$B[p(X)]$ and clearly acts c.r. on X and is the linear continuous image of L, hence of $H_1 \times H_1 \times H$.

THEOREM 6. *X is a finite-dimensional separable metric space if and only if there exists a finite-dimensional linear subspace of $B(X)$ which is c.r. over X.*

Assume X is finite-dimensional separable metric, and let n be the smallest dimension of any euclidean space E^n in which X is homeomorphically imbeddable. Then X is imbeddable in the solid unit sphere E in E^n; let $p(X) \subset E$ be such an imbedding. Either $p(X)$ lies on no spherical surface in E, or else $p(X)$ (and hence X) is an $(n-1)$-sphere; for if $p(X)$ is a proper subset of an $(n-1)$-sphere it is contained in the manifold obtained by removing a point from the $(n-1)$-sphere, which is homeomorphic to E^{n-1}, contradicting the definition of n.

Suppose $p(X)$ lies on no spherical surface in E. Let $F(L)$ be the linear continuous image of the linear subspace L of $B(E)$ (furnished by Theorem 5) under the mapping $F[B(E)] \subset B[p(X)]$ obtained by cutting down to $p(X)$ each continuous function on E. $F(L)$ is equivalent to a linear subspace of $B(X)$, and acts c.r. on X. Now L is the space consisting of all functions on E of the form

$$f_{c,d,\bar{y}}(y) = c\sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} y_i\bar{y}_i + d \tag{1}$$

where $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n)$ ranges over E^n and $(y) = (y_1, \cdots, y_n)$ ranges over E. L is closed in $B(E)$, and is isomorphic to E^{n+2}. The mapping F is 1-1 over L; for if $f_{c,d,\bar{y}} = f_{c',d',\bar{y}'}$ over $p(X)$, then $f_{c-c',d-d',\bar{y}-\bar{y}'}(y) = 0$ for $y \in p(X)$, which by (1) means that $p(X)$ lies on an $(n-1)$-sphere or an $(n-1)$-plane in E^n, contradicting hypotheses. Hence $F(L)$ is the linear 1-1 continuous image of E^{n+2}, hence isomorphic to E^{n+2}.

If $p(X)$ is an $(n-1)$-sphere in E, it is clear that (unless X consists of one or two points, in which case it is clear from the start that $B(X)$ is finite-dimensional) by deformation of $p(X)$ there is an imbedding $q(X)$ of X into E such that $q(X)$ lies on no sphere, which reduces the problem to the case just treated.

Now assume there is an n-dimensional linear subspace M of $B(X)$ which is c.r. over X. A consequence of the complete regularity of M over X is that given $x, \bar{x} \in X$ there is a continuous function $b \in M$ such that $b(x) \neq b(\bar{x})$. Since M is n-dimensional, we can write $b(x) = \sum a_i b_i(x)$ where $b_1(x), \cdots, b_n(x)$ is a base in M and, for at least one value of i, $b_i(x) \neq b_i(\bar{x})$. Then the mapping $f(X) \subset E^n$ which assigns to each $x \in X$ the point in E^n with coordinates $b_1(x), \cdots, b_n(x)$

is one-to-one. It is clearly continuous. Also f^{-1} is continuous. For let $x_0 \in X$, and let $O(x_0)$ be any open set containing x_0. Since M is c.r. over X, there is a $b \in M$ and a $\delta > 0$ such that $b(x_0) - b(x) > \delta$ for all $x \in X - O(x_0)$. Suppose $b = \sum a_i b_i$, and let $a = \max |a_i|$. Then for all x such that $|b_i(x) - b_i(x_0)| < \delta/an$ we have $|b(x_0) - b(x)| < \delta$, hence $x \in O(x_0)$. Thus X is imbeddable in E^n, so X is finite-dimensional separable metric.

THEOREM 7. *If X is completely regular, $B(X)$ contains a separable linear subspace c.r. over X if and only if X is separable metric.*

First assume X is separable metric. Then X is imbeddable in ordinary Hilbert space H. By the corollary to Theorem 5, there is a linear subspace L of $B(X)$ which is c.r. over X and is the linear continuous image of H. Since the continuous image of a separable space is separable, L is separable.

Now assume X is completely regular, and $B(X)$ contains a subspace M both separable and c.r. over X. The unit sphere E^*_w in M^*_w contains a subset X_1 homeomorphic to X. The points of E^*_w form an equicontinuous set of linear functionals on M. According to known results (see [7, Lemma 3.2 and Theorem 4.1]), the weak-$*$ topology on E^* and the compact-open topology on E^* as a set of mappings of M into the real line are identical, and the closure of X_1 in E^*_w is compact metric. Hence X_1 is separable metric, and so is X.

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ON A THEOREM OF REPRESENTATION

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Introduction. The main result of this paper is the characterization of the linear ring (BC) of all bounded continuous real functions defined in a locally compact Hausdorff space which vanish at infinity (see Theorem 5). This is done by making use of some of the properties of the functional $\phi(x)$ = greatest lower bound of x, defined for the elements of (BC) (see §. 1 and §. 8).

In this study we noticed that ϕ could give to the linear ring (BC) both its structures of natural partial order and topology. We had then the idea of seeing when this could be done in general, that is to say, when is it true that for a partially ordered normed linear ring or a normed linear lattice there is a functional which could give both the structures of partial order and topology of the space. The rest of our paper deals with such a question.

For the demonstration of our theorem 5 we made very much use of Professor Stone's paper, *A general theory of spectra*, I, the reading of which we recommend for having a clear understanding of this one.

We are deeply indebted to Professors I. Kaplansky and M. H. Stone for many valuable suggestions and to Dr. L. Nachbin for kind discussions.

1. **Definition of S-space.** A linear space with real scalars V is said to be *an S-space with respect to a real function* $\phi(x)$, defined in V, if and only if

(S1) $\phi(x) \leqq 0$ for any $x \in V$;

(S2) $\phi(x+y) \geqq \phi(x)+\phi(y)$ for any $x \in V$ and any $y \in V$;

(S3) $\phi(\lambda x) = \lambda\phi(x)$ for any $x \in V$ and any real number $\lambda \geqq 0$;

(S4) $\phi(0)=0$;

(S5) $\phi(x)+\phi(-x)=0$ implies $x=0$ $(x \in V)$;

(S6) the equations in the unknown ω

$$\text{(E)} \qquad \phi(\omega) = 0, \qquad \phi(\omega - x) = 0 \qquad (x \in V)$$

have a solution $x^+ \in V$ which is such that any other solution $x' \in V$ of (E) satisfies the condition $\phi(x'-x^+)=0$;

(S7) $\lim_{n,m\to\infty} \phi(x_n - x_m) = 0$ implies the existence of an element $x \in V$ such that $\lim_{n\to\infty} \phi(x_n - x) = \lim_{n\to\infty} \phi(x - x_n) = 0$ ($\{x_n\}$ is a sequence of elements of V).

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2. Examples of S-spaces.

EXAMPLE 1. The set B of all bounded real functions defined in a topological space is an S-space with respect to the function

$$I(x) = \text{g. l. b. } (x_-) \qquad (x \in B),$$

where g.l.b. means "greatest lower bound of" and x_- is the negative part of x.

PROOF. (1) $I(x) \leqq 0$ because $x_- \leqq 0$; (2) we have $x_- + y_- \leqq (x+y)_-$ and consequently $I(x)+I(y) \leqq \text{g.l.b. } (x_- + y_-) \leqq I(x+y)$; (3) we have $(\lambda x)_- = \lambda x_-$ for any real number $\lambda \geqq 0$ and so $I(\lambda x) = \lambda I(x)$ for $\lambda \geqq 0$; (4) $I(0)=0$ evidently; (5) we have $I(x)+I(-x) \leqq \text{g.l.b. } (x_- + (-x)_-) = \text{g.l.b. } (-|x|)$ and consequently $I(x)+I(-x)=0$ implies $x=0$; (6) we have $(x_+)_- = (x_+ - x)_- = 0$ where x_+ is the positive part of x and consequently $I(x_+)=0$ and $I(x_+ - x)=0$. If x' is such that $I(x')=0$ and $I(x'-x)=0$ then $x' \geqq 0$ and $x' \geqq x$. But now $x' \geqq x_+$ and consequently $I(x'-x_+)=0$; (7) $I(x_n - x_m)+I(x_m - x_n) \leqq \text{g.l.b. } (-|x_n - x_m|)$ and consequently $I(x_n - x_m) \rightarrow 0$ implies $(x_n - x_m) \rightarrow 0$ uniformly. But now there is an $x \in B$ such that $(x_n - x) \rightarrow 0$ uniformly. Consequently $(x_n - x)_- \rightarrow 0$ uniformly, that is to say, to each positive ϵ there is a suitable integer N such that $-\epsilon < (x_n - x)_- < \epsilon$ for $n \geqq N$. Consequently

$$-\epsilon \leqq I(x_n - x) < \epsilon \qquad \text{for } n \geqq N,$$

which means $\lim I(x_n - x) = 0$. Similarly we get $\lim I(x - x_n) = 0$.

EXAMPLE 2. The set C of all continuous real functions defined in a compact Hausdorff space is an S-space with respect to the function $I(x)$ considered above (the proof of this is easily deducible from that of Example 1).

EXAMPLE 3. The set L^p of all real functions defined in a compact measure space which have integrable pth potence ($p \geqq 1$) is an S-space with respect to the function

$$J(x) = -\left(\int (-x_-)^p\right)^{1/p} \qquad (x \in L^p).$$

PROOF. (1) $J(x) \leqq 0$ evidently; (2) we have $-x_- - y_- \geqq -(x+y)_- \geqq 0$ and consequently $(\int(-x_- - y_-)^p)^{1/p} \geqq (\int(-(x+y)_-)^p)^{1/p}$. But we know from the theory of the integral that $(\int(-x_-)^p)^{1/p} + (\int(-y_-)^p)^{1/p} \geqq (\int(-x_- - y_-)^p)^{1/p}$ and consequently we have $-J(x)-J(y) \geqq -J(x+y)$, that is to say $J(x+y) \geqq J(x)+J(y)$; (3) we have for $\lambda \geqq 0$, $(\lambda x)_- = \lambda x_-$ and consequently $J(\lambda x) = \lambda J(x)$ for $\lambda \geqq 0$; (4) $J(0)=0$ obviously; (5) $J(x)+J(-x)=0$ means in our case $J(x)=0$ and $J(-x)=0$ and consequently $x_-=0$ and $(-x)_-=0$. But then we have $x \geqq 0$ and $-x \geqq 0$ and therefore $x=0$; (6) see (6) of the proof of example 1; (7) we have $J(x_n-x_m)+J(x_m-x_n) \leqq -(\int(-(x_n-x_m)_- -(x_m-x_n)_-)^p)^{1/p} = -(\int|x_n-x_m|^p)^{1/p} \leqq 0$. Therefore, $J(x_n-x_m) \to 0$ implies $\int|x_n-x_m|^p \to 0$. Consequently there is an $x \in L^p$ such that $\int|x_n-x|^p \to 0$. But $-(x_n-x)_- \leqq |x_n-x|$ and $-(x-x_n)_- \leqq |x_n-x|$. Consequently $J(x_n-x) \to 0$ and $J(x-x_n) \to 0$.

Example 4. A Hilbert space is an S-space with respect to the function $H(x) = -\|x_-\|$, where $\|x\|$ is the norm of x.

Example 5. The set of all bounded real functions defined in a topological space which vanish at some point of the topological space is an S-space with respect to the function

$$I'(x) = \text{g.l.b. } (x),$$

where x is a generic member of the set of functions.

3. Structure of linear lattice for an S-space.

Proposition 1. *An S-space with respect to $\phi(x)$ is a linear lattice if we put: $x \geqq 0$ if and only if $\phi(x)=0$ and positive part of $x=x^+$ (see axiom* (S6)).

Proof. (1) $x \geqq 0$ and $y \geqq 0$ imply $\phi(x+y)=0$ and consequently $x+y \geqq 0$; (2) $x \geqq 0$ and $\lambda \geqq 0$ imply $\phi(\lambda x)=0$ and hence $\lambda x \geqq 0$; (3) $x \geqq 0$ and $-x \geqq 0$ imply $\phi(x)=\phi(-x)=0$ and then $x=0$; (4) by (S6) we have $x^+ \geqq 0$ and $x^+ \geqq x$ and if $x' \geqq 0$ and $x' \geqq x$ then $x' \geqq x^+$.

Proposition 2. *In an S-space with respect to ϕ we have:*

$$x \geqq y \textit{ implies } \phi(x) \geqq \phi(y).$$

Indeed, we have $x=y+(x-y)$ and therefore $\phi(x) \geqq \phi(y)+\phi(x-y)$. Then, from the hypothesis $x \geqq y$ we get $\phi(x) \geqq \phi(y)$.

Corollary. *It is true that* $\phi(x_-) \leqq \phi(x)$ and $\phi(x_- + y_-) \leqq \phi((x+y)_-)$.

4. Norm for an S-space.

Proposition 3. *Given an S-space L (with respect to ϕ), if $g(r, s$ is any symmetric real function defined for $-\infty < r \leqq 0$, $-\infty < s \leqq 0$ such that*

(D1) $g(r, s) >$ *for* $(r, s) \neq (0, 0)$; $g(0, 0)=0$;

(D2) $g(\lambda r, \lambda s) = \lambda g(r, s)$ *for any real number* $\lambda \geqq 0$;

(D3) $g(r+r', s+s') \leqq g(r, s)+g(r', s')$;

(D4) $g(r, s)$ *is monotone nonincreasing in each variable,*

then the real number $g(\phi(x), \phi(-x))$ is a norm for L and we have

$$\text{(N)} \qquad 0 \leqq x \leqq y \quad \text{imply} \quad \|x\| \leqq \|y\|.$$

PROOF. (1) For any $x\neq 0$ we cannot have simultaneously $\phi(x)=\phi(-x)=0$ (by (S5)) and so, by (D1), $\|x\|>0$ for $x\neq 0$; for $x=0$ we have obviously $\|x\|=0$; (2) we have by (S3) and (D2): $g(\phi(\lambda x),\ \phi(-\lambda x))=\lambda g(\phi(x),\ \phi(-x))$ for $\lambda\geqq 0$ and consequently $g(\phi(-\lambda x),\ \phi(\lambda x))=-\lambda g(\phi(x),\ \phi(-x))$ for $\lambda<0$. But g is symmetric, then $\|\lambda x\|=|\lambda|\,\|x\|$; (3) we have by (D4): $g(\phi(x+y),\ \phi(-x-y))\leqq g(\phi(x)+\phi(y),\ \phi(-x)+\phi(-y))$. Then, by (D3), $\|x+y\|\leqq\|x\|+\|y\|$.

Let us prove now property (N): We have $\|x\|=g(\phi(x),\ \phi(-x))=g(0,\ \phi(-x))$ for $x\geqq 0$ and similarly $\|y\|=g(0,\ \phi(-y))$ for $y\geqq 0$. If $x\leqq y$, then $\phi(-x)\geqq\phi(-y)$ and consequently $g(0,\ \phi(-x))\leqq g(0,\ \phi(-y))$ (by (D4)), which means $\|x\|\leqq\|y\|$.

5. **Indices for an S-space.** A real function $\phi(x)$ defined in a linear space L is said to be an *index* for L if L is an S-space with respect to ϕ. Two indices of an S-space are said to be *equivalent* when they give rise to the same ordering and equivalent topologies.

An S-space L with respect to an index ϕ satisfying the condition

(S8) $\phi(x_n)\rightarrow 0$ and $\phi(-x_n)\rightarrow 0$ imply $\phi(-|x_n|)\rightarrow 0$

is said to be a *regular S-space.*

PROPOSITION 4. *A regular S-space with respect to an index ϕ can be normalized in such a way that it becomes an S-space with respect to an index $W(x)$ equivalent to $\phi(x)$ and such that $W(x)=W(x_-)$.*

PROOF. The properties D1, 2, 4 of the function $g(r,\ s)$ considered above (see §4) imply that $g(r,\ s)\rightarrow 0$ implies $(r,\ s)\rightarrow(0,\ 0)$ and reciprocally $(r,\ s)\rightarrow(0,\ 0)$ implies $g(r,\ s)\rightarrow 0$. We shall prove that the real function $W(x)=\phi(x_-)$ is an index for our space and that W is equivalent to ϕ. In fact: (1) $W(x)\leqq 0$ obviously; (2) we have $W(x+y)=\phi((x+y)_-)\geqq\phi(x_-+y_-)$ because $(x+y)_-\geqq x_-+y_-$ and consequently $W(x+y)\geqq W(x)+W(y)$; (3) $W(\lambda x)=\lambda W(x)$ for $\lambda\geqq 0$ obviously; (4) $W(0)=0$ evidently; (5) We have $W(x)\leqq\phi(x)$ and so $W(x)+W(-x)=0$ implies $\phi(x)+\phi(-x)=0$, hence $x=0$; (6) we have $W(x_+)=W(x_+-x)=0$ obviously. If x' is such that $W(x')=W(x'-x)=0$, then $x'\geqq 0$ and $x'\geqq x$ obviously. Then $x'\geqq x_+$ and consequently $W(x'-x_+)=0$; (7) $W(x_n-x_m)\rightarrow 0$ means $\phi((x_n-x_m)_-)\rightarrow 0$ and consequently $\phi(x_n-x_m)\rightarrow 0$. Then, by (S7), there is an x such that $\phi(x_n-x)\rightarrow 0$ and $\phi(x-x_n)\rightarrow 0$. By (S8) we have then $\phi(-|x_n-x|)\rightarrow 0$. But $(x_n-x)_-\geqq -|x_n-x|$, then $W(x_n-x)\rightarrow 0$. Similarly we get $W(x-x_n)\rightarrow 0$.

Let us prove now that W is equivalent to ϕ. Indeed: (1) $W(x)\leqq\phi(x)$ and so $W(x)=0$ implies $\phi(x)=0$; reciprocally, $\phi(x)=0$ means $x\geqq 0$ and then $W(x)=0$; (2) we have by (D4)

$$0 \leqq g(\phi(x), \phi(-x)) \leqq g(W(x), W(-x)) = g(\phi(x_-), \phi((-x)_-)).$$

Then, when $g(W(x), W(-x)) \to 0$ we have that $g(\phi(x), \phi(-x)) \to 0$. Reciprocally, if $g(\phi(x), \phi(-x)) \to 0$, then, by our hypothesis, $\phi(x)$ and $\phi(-x)$ tend to zero. By (S8), $\phi(-|x|) \to 0$. But $\phi(x_-) \geqq \phi(-|x|)$ and $\phi((-x)_-) \geqq \phi(-|x|)$. Then $\phi(x_-) \to 0$, $\phi((-x)_-) \to 0$ and so $g(W(x), W(-x))$ tends to zero.

Finally, $W(x_-) = \phi(x_-) = W(x)$.

PROPOSITION 5. *For a regular S-space we have:*

(N1) $$\|x_-\| \leqq \|x\|,$$

(N2) $$\|(x+y)_-\| \leqq \|x_- + y_-\|.$$

PROOF. (1) we have $\|x_-\| = g(W(x_-), 0) = g(W(x), 0) \leqq g(W(x), W(-x)) = \|x\|$; (2) we have $-(x+y)_- \leqq -(x_- + y_-)$, then, by (N), $\|-(x+y)_-\| \leqq \|-(x_- + y_-)\|$ and so $\|(x+y)_-\| \leqq \|x_- + y_-\|$.

6. Regular S-spaces and quasi Banach lattices. A *quasi Banach lattice* is a linear lattice which is also a Banach space with respect to a norm $\|\cdot\|$ satisfying conditions (N1) and (N) (see §4).

THEOREM 1. *A linear space L is a quasi Banach lattice if and only if L is a regular S-space.*

PROOF. We shall prove first that a regular S-space is a quasi Banach lattice. Indeed: (1) by Proposition 1, a regular S-space L is a linear lattice; (2) by Proposition 3, L is a normed space; (3) by Propositions 3 and 5, the norm satisfies conditions (N) and (N1). Then, all we have to do to prove that L is a quasi Banach lattice is to show that L is complete in the norm. Let $\{x_n\}$ $(n=1, 2, \cdots)$ be a sequence of L and suppose $\|x_n - x_m\| \to 0$. This means (see §4) that $g(\phi(x_n - x_m), \phi(x_m - x_n)) \to 0$. But now if we take g such that $g(r, s)$ tends to zero implies $(r, s) \to (0, 0)$ and reciprocally, we get: $\phi(x_n - x_m)$ tends to zero. By (S7) there is an $x \in L$ such that $\phi(x_n - x) \to 0$ and $\phi(x - x_n) \to 0$ and so $g(\phi(x_n - x), \phi(x - x_n)) = \|x_n - x\| \to 0$.

Let us prove now that a quasi Banach lattice is a regular S-space; precisely we shall prove that the function $\phi(x) = -\|x_-\|$ is an index for the space and that this space is regular: (1) $\phi(x) \leqq 0$ obviously; (2) $\phi(x+y) \geqq \phi(x) + \phi(y)$ is easily deducible from (N2); (3) For $\lambda \geqq 0$, $\phi(\lambda x) = \lambda \phi(x)$ obviously; (4) $\phi(0) = 0$ evidently; (5) $\phi(x) + \phi(-x) = 0$ means in our case $x \geqq 0$ and $-x \geqq 0$ and consequently $x = 0$; (6) see (6) of the proof of example 1, §1; (7) we have $\phi(x_n - x_m) + \phi(x_m - x_n) = -(\|(x_n - x_m)_-\| + \|(x_m - x_n)_-\|) \leqq -\| |x_n - x_m| \|$. Consequently $\phi(x_n - x_m) \to 0$ implies $\| |x_n - x_m| \| \to 0$. But $(x_n - x_m)_+$

$\leqq |x_n - x_m|$ and $-(x_n - x_m)_- \leqq |x_n - x_m|$, hence, by (N), $\| x_n - x_m \| \to 0$. Now there is an x such that $\| x_n - x \| \to 0$. Then, by (N1), $\| (x_n - x)_- \| \to 0$, that is to say, $\phi(x_n - x) \to 0$. Similarly, we get $\phi(x - x_n) \to 0$.

Let us prove now that our S-space is regular. Let $\phi(x_n) \to 0$ and $\phi(-x_n) \to 0$. Then, $\| (x_n)_- \| \to 0$ and $\| (-x_n)_- \| \to 0$. Consequently, $\| |x_n| \| \to 0$. But $-\| |x_n| \| = \phi(-|x_n|)$; then $\phi(-|x_n|) \to 0$.

Corollary. *A Banach lattice, that is to say, a linear lattice which is also a Banach space with respect to a norm* $\| \cdot \|$ *such that*

$$\text{(N4)} \qquad |x| \leqq |y| \quad \textit{implies} \quad \|x\| \leqq \|y\|,$$

is a regular S-space.

Indeed, we have $|x_-| \leqq |x|$ and so $\|x_-\| \leqq \|x\|$, which is (N1), and $0 \leqq x \leqq y$ imply $\|x\| \leqq \|y\|$, which is (N).

7. Continuity of the operations in a regular S-space.

Proposition 6. (a) $x \to x_0$ *and* $y \to y_0$ *imply* $(x+y) \to (x_0 + y_0)$; (b) $x \to x_0$ *implies* $|x| \to |x_0|$.

Proof. (a) we have $\| (x+y) - (x_0 + y_0) \| \leqq \| x - x_0 \| + \| y - y_0 \|$, which proves part (a) of our proposition; (b) we have by (N1): $\| x_- \| \leqq \| x \|$ and $\| (-x)_- \| \leqq \| x \|$; consequently, $\| |x| \| \leqq 2 \| x \|$, which proves part (b) of our proposition.

Corollary 1. *The operations* $x \cup y$ *and* $x \cap y$ *are continuous.*

Indeed, we have $x \cup y = (x+y)/2 + |x-y|/2$ and $x \cap y = (x+y)/2 - |x-y|/2$.

Corollary 2. $x \leqq y$, $x \to x_0$, $y \to y_0$ *imply* $x_0 \leqq y_0$.

Indeed, $x \leqq y$ means $x = x \cap y$ and consequently $x_0 = x_0 \cap y_0$ which says that $x_0 \leqq y_0$.

Corollary 3. $x_n \leqq y_n$, $n = 1, 2, \cdots$, $\sum_1^\infty x_n = x$, $\sum_1^\infty y_n = y$ *imply* $x \leqq y$.

8. **Definition of S-ring.** A linear ring with real scalars R is called an S-ring with respect to a real function $\phi(x)$, defined in R, if and only if:

(R1) $\phi(x)$ satisfies conditions (S1)–(S5) and (S7) (see §1);

(R2) $\phi(x^2 + 2\mu x) + \mu^2 \geqq 0$ for $x \in R$ and any real number μ;

(R3) $\phi(x) + \mu \geqq 0$ and $\phi(y) + \nu \geqq 0$ imply $\phi(xy + \nu x + \mu y) + \mu\nu \geqq 0$ for any real numbers μ and ν and $x \in R$, $y \in R$.

Example 1. The set B of all bounded real functions defined in a topological space is an S-ring with respect to the function $\phi(x) = \text{g.l.b.}\ (x_-)$.

PROOF. We have already proved that ϕ satisfies (R1) (see example 1 of §1). Let us prove then that ϕ satisfies (R2) and (R3). In fact: $(x^2+2\mu x)_-+\mu^2 = (x^2+2\mu x+\mu^2)\wedge\mu^2 \geqq 0$, hence $\text{g.l.b.}\ ((x^2+2\mu x)_-+\mu^2) = \text{g.l.b.}\ (x^2+2\mu x)_-+\mu^2 \geqq 0$; $\phi(x)+\mu \geqq 0$ gives $x_-+\mu \geqq 0$, hence $x+\mu \geqq 0$ and $\mu \geqq 0$. Similarly we have $y+\nu \geqq 0$ and $\nu \geqq 0$. Then $xy+\nu x+\mu y \geqq -\mu\nu$ and consequently $(xy+\nu x+\mu y)_- \geqq -\mu\nu$ because $-\mu\nu$ is negative.

EXAMPLE 2. The set C of all continuous real functions defined in a compact Hausdorff space is an S-ring with respect to the function $\phi(x) = \text{g.l.b.}\ (x_-)$.

This is clear from the example 1.

EXAMPLE 3. The set (BC) of all bounded continuous real functions defined in a locally compact Hausdorff space which vanish at infinity in an S-ring with respect to $\phi(x) = \text{g.l.b.}\ (x)$.

PROOF. It is clear that we have to prove only (R2) and (R3); (1) we have $x^2+2\mu x+\mu^2 \geqq 0$, hence $\text{g.l.b.}\ (x^2+2\mu x)+\mu^2 \geqq 0$; (2) $\phi(x)+\mu \geqq 0$ and $\phi(y)+\nu \geqq 0$ give $x+\mu \geqq 0$ and $y+\nu \geqq 0$, hence $xy+\nu x+\mu y+\mu\nu \geqq 0$ and then $\text{g.l.b.}\ (xy+\nu x+\mu y)+\mu\nu \geqq 0$.

COUNTER-EXAMPLE. The set L is not an S-ring with respect to $\phi(x) = \int x_- dt$. Indeed, for $0 \leqq t \leqq 1$, $x = -t/2$, $y = t$, $\mu = 1/4$, $\nu = 1/2$, condition (R3) is not satisfied, since $\phi(x)+\mu = 0$, $\phi(y)+\nu = 1/2$ and $\phi(xy+\nu x+\mu y)+\mu\nu = -1/24$.

9. Partial order and norm for an S-ring.

PROPOSITION 7. *In an S-ring with respect to $\phi(x)$ we have*: (i) $\phi(x^2) = 0$; (ii) $\phi(y) = 0$ *implies* $\phi(xy - y\phi(x)) = 0$.

PROOF. (i) is easily deducible from (R2) by putting $\mu = 0$; (ii) is deducible from (R3) by putting $\mu = -\phi(x)$, $\nu = 0$.

PROPOSITION 8. *An S-ring with respect to $\phi(x)$ is a partially ordered normed linear ring by defining $x \geqq 0$ if and only if $\phi(x) = 0$ and $\|x\| = g(\phi(x), \phi(-x))$, where $g(r, s)$ is the function introduced in* §4.

This is clear from Propositions 1, 3 and 7.

PROPOSITION 9. *In an S-ring, with respect to ϕ, which has a unit "e" we have*:

(E1) $\phi(-e) \geqq -1$;
(E2) $nx+e \geqq 0$ *for any integer* $n > 0$ *implies* $x \geqq 0$;
(E3) $x - \phi(x)e \geqq 0$;
(E4) $\lambda e + x \geqq 0$, $\lambda e - x \geqq 0$ *imply* $\lambda \geqq \max\ (-\phi(x), -\phi(-x))$.

PROOF. (1) (E1) is clear from (R2) by making $x=e$ and $\mu=-1$; (2) we have $nx=-e+nx+e$, hence, by Proposition 2, $\phi(nx)-\phi(-e) \geqq \phi(nx+e)$. Since by hypothesis $\phi(nx+e)=0$ for any integer $n>0$, we get $\phi(nx) \geqq \phi(-e) \geqq -1$ and consequently $\phi(x) \geqq -1/n$ for any $n>0$, which says that $\phi(x)=0$, hence $x \geqq 0$; (3) (E3) is clear from (ii) of Proposition 7; (4) we have $x+\lambda e-\lambda e=x$, hence, by Proposition 2, $\phi(x+\lambda e) \leqq \phi(x)-\phi(-\lambda e)=\phi(x)-\lambda\phi(-e) \leqq \phi(x)+\lambda$; consequently, $\phi(x+\lambda e)=0$ implies $\phi(x)+\lambda \geqq 0$. Similarly we get $\phi(-x)+\lambda \geqq 0$.

10. Sub S-ring of an S-ring.

PROPOSITION 10. *If R' is an S-ring with respect to $\phi(x)$ which has no unit and if R is the set of all pairs (x, μ) $(x \in R'$, μ a real number) with algebraic operations defined by $(x, \mu)+(y, \nu)=(x+y, \mu+\nu)$, $(x, \mu)(y, \nu)=(xy+\nu x+\mu y, \mu\nu)$, $\lambda(x, \mu)=(\lambda x, \lambda\mu)$, then R is an S-space with respect to $\psi(x, \mu)=\min(0, \phi(x)+\mu)$ and R has a unit, precisely the element* $(0, 1)$.

The proof is trivial, and it is easily seen also that R' is a maximal ideal in R.

11. L-space, M-space and ring of continuous functions.

THEOREM 2. *An S-space V with respect to a ϕ such that $\phi(x+y)=\phi(x)+\phi(y)$ for $x \leqq 0$, $y \leqq 0$ is an L-space.*[1]

PROOF. We have seen that an S-space is a Banach space with norm $\|\cdot\|$ defined by $\|x\|=g(\phi(x), \phi(-x))$, where $g(r, s)$ is the function introduced in §4, hence, in particular, V is a Banach space with norm defined by $\|x\|=-\phi(x)-\phi(-x)$; (2) V is a linear lattice (see Proposition 1); $x_n \leqq y_n$, $x_n \to x$, $y_n \to y$ imply $x \leqq y$ because $y-x \geqq y-x+x_n-y_n$, hence $\phi(y-x) \geqq \phi(y-y_n)+\phi(x_n-x) \geqq -\|y-y_n\|-\|x_n-x\|$; finally, $x \geqq 0$, $y \geqq 0$ imply $\|x+y\|=-\phi(-x-y)=-\phi(-x)-\phi(-y)=\|x\|+\|y\|$.

THEOREM 3. *An S-space with respect to a ϕ such that*

$$\text{(M)} \qquad \phi(-x-y)=\min(\phi(x-y), \phi(y-x)) \text{ for } x \cap y=0$$

is an M-space.[2]

PROOF. (1) Let V be the S-space in question; then V is a Banach space with norm defined by $\|x\| = \max(-\phi(x), -\phi(-x))$ and also a linear lattice; (2) $x_n \leqq y_n$, $x_n \to x$, $y_n \to y$ imply $x \leqq y$ because $\phi(y-x) \geqq \phi(y-y_n)+\phi(x_n-x) \geqq -\|y-y_n\|-\|x_n-x\|$; (3) let $x \cap y = 0$, then $\|x+y\| = -\phi(-x-y) = -\min(\phi(x-y), \phi(y-x))$ (see (M)) $= \|x-y\|$; (4) let $x \cap y = 0$, then $-x-y \leqq -x \leqq -x+y$, $-x-y \leqq -y \leqq -y+x$, hence $\phi(-x-y) \leqq \phi(-x) \leqq \phi(y-x)$, $\phi(-x-y) \leqq \phi(-y) \leqq \phi(x-y)$. Consequently, $\phi(-x-y) \leqq \min(\phi(-x), \phi(-y)) \leqq \min(\phi(x-y), \phi(y-x))$, hence $-\|x+y\| \leqq \min(-\|x\|, -\|y\|) \leqq -\|x-y\| = -\|x+y\|$. From this we get $\|x+y\| = \max(\|x\|, \|y\|)$.

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[1] S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. vol. 42 (1941) pp. 523–537.

[2] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. vol. 42 (1941) pp. 994–1024.

COROLLARY. *An S-space (with respect to ϕ) V which has an element* 1 *such that* (i) $\phi(1) = 0$, $\phi(-1) = -1$, (ii) $-\phi(x) \leqq 1$ *implies* $-x \leqq 1$ *is an M-space and the element* 1 *is a Kakutani unit*[3] *for the space V.*

PROOF. All we have to prove is that condition (M) holds in V. Indeed, we have $-x-y \leqq x-y$, $-x-y \leqq y-x$, hence $\phi(-x-y) \leqq \phi(x-y)$, $\phi(-x-y) \leqq \phi(y-x)$ and consequently $\phi(-x-y) \leqq \min(\phi(x-y), \phi(y-x))$. Let us prove now the inverse inequality. This is obvious if $k = \min(\phi(x-y), \phi(y-x)) = 0$, since in this case $x-y=0$ and therefore $x=y=0$ because $x \cap y = 0$ by hypothesis. Suppose then $k<0$. As we have $\phi(x-y) \geqq k$, $\phi(y-x) \geqq k$, we get $-\phi((x-y)/k) \leqq 1$, $-\phi((y-x)/k) \leqq 1$, then, by hypothesis, $(x-y)/k \leqq 1$, $(y-x)/k \leqq 1$, hence $|x-y| \leqq |k| 1$. Consequently $\phi(-|x-y|) \geqq \phi(-|k|1) = |k|\phi(-1) = k$, which means, since $-(x+y) = -|x-y|$, that $\phi(-x-y) \geqq k$.

Finally we have: (1) $\|1\| = -\phi(-1) = 1$, $1 \geqq 0$ because $\phi(1) = 0$; (2) let $\|x\| \leqq 1$, that is to say, $\max(-\phi(x), -\phi(-x) \leqq 1$; then $-\phi(x) \leqq 1$, $-\phi(-x) \leqq 1$, hence $-x \leqq 1$, $x \leqq 1$, then $|x| \leqq 1$.

THEOREM 4. *An S-ring with unit R is a Stone normed ring*[4] *and consequently a linear lattice.*[5]

PROOF: (1) R is a linear ring and is partially ordered (see Proposition 8); (2) if "e" is the unit of R, then $nx+e \geqq 0$ for any integer $n>0$ implies $x \geqq 0$ (see Proposition 9, (E2)); (3) $\lambda e = x \geqq 0$ for some real number λ (see Proposition 9, (E3)); (4) R is complete in the norm

$$\|x\| = \inf(\lambda; \lambda x + e \geqq 0, \lambda x - e \geqq 0),$$

which turns out to be $\|x\| = \max(-\phi(x), -\phi(-x))$ (see Proposition 9, (E4)).

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[3] S. Kakutani, Ibid. p. 997.

[4] M. H. Stone, *A general theory of spectra*, I, Proc. Nat. Acad. Sci. U.S.A. vol. 26(1940) pp. 280–283.

[5] M. H. Stone, Ibid. pp. 280–281.

COROLLARY. *An S-ring R' without unit is a linear lattice.*

In fact, by Proposition 10, R' is a maximal ideal of the ring R defined in §10 (see Proposition 10) and so, by Theorem 4 and Stone's result,[6] R' is a linear lattice, precisely a sublinear lattice of R.

THEOREM 5. *An S-ring without unit R' whose index ϕ is such that $\phi(x)=\phi(x_-)$ is equivalent (=isometric, ring and lattice isomorphic) to the set (BC) of all bounded continuous real functions defined in a locally compact Hausdorff space which vanish at infinity, and $\phi(x)=\inf (f_x(I))$, where* inf *means greatest lower bound of and f_x is the correspondent function of the element $x \in R'$.*

PROOF. (1) R' is a maximal ideal of the S-ring R defined in §10 (see Proposition 10). This S-ring R is a Stone normed ring (see Theorem 4) and so equivalent to the set C of all continuous real functions defined in a compact space.[7] This compact space is the set M of all maximal ideals in R (with an appropriate topology) and the correspondence between R and C is that one which associates to each $w \in R$ the real function $f_w(I)$, defined in the space M, such that $f_w(I)e-w \in I$ (e is the unit of R).[8] Then, if $w \in R'$ we have $f_w(R')e-w \in R'$, hence $f_w(R')=0$, that is to say, if f_x is the correspondent function of an element x of R' then $f_x(R')=0$. We have, therefore, that R' is equivalent to the subset (BC) of C formed by all functions of C which vanish in $R' \in M$. Let us take off R' from M: we get a new topological space M' which is locally compact and has R' as the point at infinity. We have also that now (BC) is the class of all bounded continuous real functions defined in M' and which vanish at infinity; (2) let us prove now that $\phi(x)=\text{g.l.b.}\ (f_x(I))$: we have $\|x\|=\sup_{I\in M} |f_x(I)|$, f_x being the correspondent function of $x \in R$. Suppose now that x belongs to R', then $\|x\|=\max\ (-\phi(x),\ -\phi(-x))$ and so $\max\ (-\phi(x),\ -\phi(-x))=\sup_{I\in M}|f_x(I)|$. If $-x\geqq 0$, then $f_{-x}=-f_x\geqq 0$, $\phi(-x)=0$ and consequently $-\phi(x)=\sup_{I\in M}\ (-f_x(I))$, that is to say $\phi(x)=\inf_{I\in M}\ (f_x(I))$. For any $x \in R'$ we have, by hypothesis, $\phi(x)=\phi(x_-)$ and so $\phi(x)=\inf_{I\in M}\ (f_{x_-}(I))$. But $f_{x_-}=(f_x)_-$, so $\phi(x)=\inf_{I\in M}\ ((f_x(I))_-)$. The function f_x vanishes at least once (at infinity), hence $\inf\ ((f_x)_-)=\inf\ (f_x)$ and therefore $\phi(x)=\inf_{I\in M}\ (f_x(I))$. But $\inf_{I\in M}\ (f_x)=\inf_{I\in M'}\ (f_x)$, and so $\phi(x)=\inf_{I\in M'}(f_x(I))$.

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[6] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 321–364; in particular, Theorem 80.

[7] M. H. Stone, loc. cit. footnote 5.

[8] M. H. Stone, loc. cit. footnote 6, Theorems 76 and 80.

ON THE GROUPS OF REPEATED GRAPHS

ROBERT FRUCHT

In a recently published paper[1] Kagno showed that Pappus' graph, consisting of the 9 vertices $A, B, C, D, E, F, G, H, I$, and the 27 arcs $AD, AE, AF, AG, AH, AI, BD, BE, BF, BG, BH, BI, CD, CE, CF, CG, CH, CI, DG, DH, DI, EG, EH, EI, FG, FH, FI$, has a group of order 1296 which may be generated by the following set of eight substitutions: (ABC), (AB), (DEF), (DE), (GHI), (GH), $(ADG)(BEH)(CFI)$, $(AD)(BE)(CF)$.[2] Kagno's proof of this fact (Theorem 5)[1] is straightforward, but somewhat lengthy, and it seems of interest to note that this theorem follows at once from a more general and almost self-evident theorem on the groups of repeated graphs, if we apply to Pappus' graph the following lemma, also due to Kagno:[1] "If G' is the complement of G, then G and G' have the same group."[3] Indeed the complement Π' of Pappus' graph contains the 9 arcs $AB, AC, BC, DE, DF, EF, GH, GI, HI$; hence Π' is not connected, but consists of three triangles (or complete 3-points) ABC, DEF, GHI; that is, Π' is a threefold repeated triangle. To such a repeated graph we can apply the following theorem, which is of interest in itself apart from the use made of it here.

THEOREM. *If G is a connected graph of n vertices, having no simple loops, with a group $\mathfrak{H}$ of order h, and if Γ is the graph consisting of m copies $G_1, G_2, \cdots, G_m$ of the same graph G, then the group of Γ is Pólya's "Gruppenkranz" $\mathfrak{S}_m[\mathfrak{H}]$, that is, the group of order $m!h^m$ and degree mn, whose substitutions may be described briefly as follows:*[4] *Let*

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[1] I. N. Kagno, *Desargues' and Pappus' graphs and their groups*, Amer. J. Math. vol. 69 (1947) pp. 859–862.

[2] It may be remarked that the same group might be generated by other sets of fewer elements, for example, by the following one containing only 3 substitutions: $(ABC)(DE)$, $(ADG)(BEH)(CFI)$, $(AD)(BE)(CF)$.

[3] Here the complement G' of a graph G (without loops) is to be defined as follows: Let $\Gamma^p_q=\Gamma^q_p=1$, if and only if the vertices p and q are joined by an arc, otherwise let $\Gamma^p_q=\Gamma^q_p=0$. Now, if for any pair of vertices p, q in G, $\Gamma^p_q=1$, then in G' let $\Gamma^p_q=0$, and if $\Gamma^p_q=0$ in G, then in G' let $\Gamma^p_q=1$. In other applications of this lemma some difficulty may arise from the fact that the complement of a graph may contain isolated points; for example, the complement of a triangle (or complete 3-point) consists only of three isolated points with no arcs connecting them.

[4] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. vol. 68 (1937) p. 178. The same groups have also been studied by other authors. We mention only the following papers:

Wilhelm Specht, *Eine Verallgemeinerung der symmetrischen Gruppe*, Schriften des *the vertices of* G_i $(i=1, 2, \cdots, m)$ *be denoted by the letters* $z_{i1}, z_{i2}, \cdots, z_{in}$, *and form the rectangular matrix*

$$\begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix};$$

then the substitutions of $\mathfrak{S}_m[\mathfrak{H}]$ *may be characterized as permutations of the rows of the matrix, followed by permutations of the letters in each row according to some substitution of the group* $\mathfrak{H}$.

As the proof is fairly obvious and can readily be supplied by the reader, we shall omit the details. It is evident that the graph Γ (consisting of m copies of the connected graph G) is mapped into itself by the substitutions of the "Gruppenkranz" $\mathfrak{S}_m[\mathfrak{H}]$; but, since G is a connected graph with group $\mathfrak{H}$, there cannot be substitutions other than those of $\mathfrak{S}_m[\mathfrak{H}]$ which map Γ into itself.

In the case of the complement Π' of Pappus' graph we have $m=3, n=3, \mathfrak{H}=\mathfrak{S}_3$ (symmetric group on three letters) with $h=6$; the group $\mathfrak{S}_3[\mathfrak{S}_3]$ is of order $3!6^3=1296$, and is obviously the same as that obtained by Kagno.

As another example let us consider the graph called H_{18} by Kagno in a previous paper,[5] that is, the graph consisting of six vertices a, b, c, d, e, f, and the nine arcs $ad, ae, af, bd, be, bf, cd, ce, cf$. Its complement, having the arcs ab, ac, bc, de, df, ef, consists of two triangles; hence its group is the "Gruppenkranz" $\mathfrak{S}_2[\mathfrak{S}_3]$ of order $2!6^2=72$. (In Kagno's paper[5]—cf. his Theorem 3.17—this group is called $\mathfrak{H}_{24} \equiv (abcdef)_{72}$.)

A third example[6] is furnished by the graph formed by the vertices and edges of the regular octahedron; its group is $\mathfrak{S}_3[\mathfrak{S}_2]$ of order $3!2^3=48$. (In Kagno's paper[5] this is the graph H_5 with the group $\mathfrak{H}_{32} \equiv (abcdef)_{48}$; see his Theorem 3.4 in the corrected form.)

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Oystein Ore, *Theory of monomial groups*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 15–64.

R. Frucht, *Coronas de grupos y sus subgrupos, con una aplicacion a los determinantes*, Union Matematica Argentina, Publicacion No. 24, 1942.

[5] I. N. Kagno, *Linear graphs of degree* ≦6 *and their groups*, Amer. J. Math. vol. 68 (1946) pp. 505–520. See also *Corrections* on page 872 of vol. 69 (1947) of the same Journal.

[6] Pólya, loc. cit. p. 214.

Finally it may be remarked that a theorem analogous to our theorem would hold for the automorphism groups of "connected systems" other than graphs, for example, for connected partially ordered systems, for certain functions of several variables, and so on. (Thus the group $\mathfrak{S}_3[\mathfrak{S}_3]$ might also be characterized as the largest permutation group on 9 variables $x_1, x_2, \cdots, x_9$, under which the function $x_1x_2x_3+x_4x_5x_6+x_7x_8x_9$ is invariant.)

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A CLASS OF TOPOLOGICAL SPACES

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1. Introduction. It is a classical theorem of set-theoretical topology that a one-to-one continuous mapping of a bicompact Hausdorff space onto a Hausdorff space is a homeomorphism.[1] Stated in somewhat different terms, this theorem asserts that any bicompact Hausdorff topology on a given set E is a *minimal* Hausdorff topology. If $\mathcal{B}$ is the family of open sets in this topology, then no proper subfamily of $\mathcal{B}$ can be the family of open sets for a Hausdorff topology on E. When this phenomenon is observed, a number of questions immediately present themselves:

(1) Under what conditions will a minimal Hausdorff space be bicompact?

(2) Are there any minimal Hausdorff spaces which are not bicompact?

(3) Is there any simple way of characterizing those topological spaces which have one-to-one continuous images which are bicompact Hausdorff spaces?

Question (1) was answered completely by Katětov [6, p. 40], who proved that a Hausdorff space is bicompact if and only if it is minimal and satisfies the Urysohn separation axiom (that is, every pair of distinct points possess neighborhoods whose closures are disjoint).

Question (2) can be answered in the affirmative by modifying a space constructed by Urysohn (see [2, p. 22]). Let the space X_0 be defined as the set of all points (x, y) in the Euclidean plane such that $0<x^2+y^2\leqq 1$, together with two adjoined points p^+ and p^-. Let neighborhoods of all points (x, y) in X_0 be the usual Euclidean neighborhoods; let $U_n(p^+)$ be $p^+\cup E[(x, y), 0<x^2+y^2<1/n, y>0]$; and let $U_n(p^-)$ be $p^-\cup E[(x, y), 0<x^2+y^2<1/n, y<0]$. As the index n assumes all positive integral values, the neighborhoods $U_n(p^+)$ and $U_n(p^-)$ describe a complete family of neighborhoods for p^+ and p^-. It is obvious that, under this definition, X_0 forms a Hausdorff space which fails to satisfy the Urysohn separation axiom. It is easy to prove, moreover, that X_0 is a minimal Hausdorff space. Katětov [6,

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[1] See, for example [1, p. 95, Satz III]. It is interesting to observe that this result was stated in 1893 by Jordan for bounded closed subsets of n-dimensional Euclidean space (see [5, p. 53]). Numbers in brackets refer to the references cited at the end of the paper.

p. 40] has proved that a Hausdorff space is minimal if and only if it is semi-regular (that is, sets of the form $A^{-\prime-\prime}$ form a basis for the open sets) and H-closed (that is, closed in any Hausdorff space which contains it as a subspace). H-closure is equivalent to the assertion that every covering by a family of open sets admits a finite subfamily whose closures form a covering [1, p. 90, Satz X]. Since the space X_0 is obviously semi-regular and H-closed, it must be minimal. It is obviously not bicompact, however, being irregular. One may observe also that the set of points $\{(1/n, 0)\}_{n=1}^{\infty}$ has no limit point.

The purpose of the present note is to answer question (3).

2. **Rings of continuous functions.** If X is any topological space, we designate the set of all real-valued continuous functions on X by the symbol $\mathfrak{C}(X, R)$ and the set of all bounded continuous real-valued functions on X by the symbol $\mathfrak{C}^*(X, R)$. These sets of functions are algebraic rings of a very special type, and have been studied in detail by a number of authors. (See, for example, [7, pp. 453 *et. seq.*] and [4, *passim*].) We propose to use properties of these function rings in solving our present problem. In considering rings $\mathfrak{C}^*(X, R)$, we may limit ourselves to bicompact Hausdorff spaces, since, to every completely regular space X, one may assign a uniquely determined bicompact Hausdorff space βX which contains X as a dense subspace and which has the property that $\mathfrak{C}^*(\beta X, R)$ is algebraically isomorphic to $\mathfrak{C}^*(X, R)$. A subring of a ring $\mathfrak{C}(X, R)$ is said to be an analytic subring if it is closed in the uniform topology for the ring $\mathfrak{C}(X, R)$ and contains the constant functions. In comparing topological properties of a bicompact Hausdorff space with algebraic properties of its function ring $\mathfrak{C}^*(X, R)$, the following facts become evident.

(4) Every analytic subring of $\mathfrak{C}^*(X, R)$ is isomorphic to some ring $\mathfrak{C}^*(Y, R)$, where Y is a continuous image of X which is a (necessarily bicompact) Hausdorff space. Conversely, if Y is a bicompact Hausdorff space which is a continuous image of X, then the function ring $\mathfrak{C}^*(Y, R)$ is isomorphic to an analytic subring of $\mathfrak{C}^*(X, R)$.

(5) If X is any completely regular space, and if Y is a completely regular space which is a continuous image of X, then the ring $\mathfrak{C}(X, R)$ contains an analytic subring which is isomorphic to the ring $\mathfrak{C}(Y, R)$.

(6) If X is a bicompact Hausdorff space, every maximal ideal in $\mathfrak{C}^*(X, R)$ is the set of all functions in $\mathfrak{C}^*(X, R)$ vanishing at some point of X.

Statements (4) and (6) were proved by Stone [7, Theorems 81 and 80]. Statement (5) is obviously true. It may be noted that the converse of (5) is false. As an example, consider the space R consisting of the real numbers in their usual topology, and let $\mathfrak{P}$ be the analytic subring of $\mathfrak{C}(R, R)$ generated by the constant functions and the function $f(x)=x$. It is clear that $\mathfrak{P}$ is a proper subring of $\mathfrak{C}(R, R)$, since it does not contain, for example, the function e^x. It can readily be shown, using methods developed in [4], that $\mathfrak{P}$ cannot be isomorphic to the ring $\mathfrak{C}(X, R)$ for any topological space X. If $\mathfrak{P}$ were isomorphic to such a ring, then one could obtain a space X such that $\mathfrak{P}$ is isomorphic to $\mathfrak{C}(X, R)$ by considering all maximal ideals $\mathfrak{M}$ in $\mathfrak{P}$ such that $\mathfrak{P}/\mathfrak{M}$ is algebraically isomorphic to R and topologizing them in accordance with [4, Theorem 9]. Upon carrying out this process, one simply obtains R in its usual topology, which produces an evident contradiction.

3. **Limitation of spaces considered.** Since every bicompact Hausdorff space X is normal, it is clear that for distinct points p and q in X, there is a function $g\in\mathfrak{C}(X, R)$ such that $g(p)=1$ and $g(q)=0$. This property being preserved upon passage to any stronger topology, we may evidently limit ourselves to spaces having this separation property. On the other hand, it is well known that any space with this property has a one-to-one continuous image which is completely regular. (See for example [4, Theorem 4].) We may thus make a second limitation, and consider only completely regular spaces.

Among completely regular spaces, we single out a certain class, namely, those spaces X whose topology can be completely described in terms of the algebraic structure of the ring $\mathfrak{C}(X, R)$. This statement can be rendered more precise. An ideal $\mathfrak{J}$ in $\mathfrak{C}(X, R)$ is said to be real if $\mathfrak{C}(X, R)/\mathfrak{J}$ is R, and $\mathfrak{J}$ is said to be fixed if $\prod_{f\in\mathfrak{J}} E[f(x)=0]\neq 0$. The topology of a completely regular space X can be described in terms of the algebraic structure of $\mathfrak{C}(X, R)$ if and only if every real ideal in $\mathfrak{C}(X, R)$ is fixed. We call such spaces Q-spaces, and have investigated their properties in [4]. We find it essential in the present discussion to limit our attention to Q-spaces.

4. **Main theorem.** Our technique is to translate a topological property into a property of function rings and then to retranslate this algebraic property into a topological property.

THEOREM A. *A Q-space X has the property that there exists a bicompact Hausdorff space Y which is a one-to-one continuous image of X if and only if the ring $\mathfrak{C}^*(X, R)$ contains an analytic subring $\mathfrak{A}$ such that every maximal ideal in $\mathfrak{A}$ can be imbedded in precisely one real ideal of $\mathfrak{C}(X, R)$.*

Suppose that Y is a bicompact Hausdorff space which is a one-to-one continuous image of X under a mapping Φ. Then the functions $f\Phi(x)$, where $f\in\mathfrak{C}^*(Y, R)$, form an analytic subring $\mathfrak{A}$ of $\mathfrak{C}^*(X, R)$ which is isomorphic to the ring $\mathfrak{C}^*(Y, R)$. By remark (6), the points of Y are in one-to-one correspondence with the maximal ideals of $\mathfrak{C}^*(Y, R)$, and hence of $\mathfrak{A}$. Since Φ is a one-to-one mapping, it is plain that the functions in any maximal ideal $\mathfrak{M}$ of $\mathfrak{A}$ all vanish at exactly one point of X. The ideal $\mathfrak{M}_p$ in $\mathfrak{C}(X, R)$ consisting of all functions in $\mathfrak{C}(X, R)$ vanishing at p is the unique extension of $\mathfrak{M}$ to a real ideal in $\mathfrak{C}(X, R)$.

Conversely, suppose that $\mathfrak{C}^*(X, R)$ contains a subring of the kind specified above. Then the set of all maximal ideals of $\mathfrak{A}$ may be topologized (see [4, Theorem 9]) to form a bicompact Hausdorff space Y which is a continuous image of the space βX. βX is to be considered as the space of all maximal ideals in $\mathfrak{C}(X, R)$ (see [4, Theorem 46]), and the continuous mapping of βX onto Y is obtained by mapping a given maximal ideal of βX onto its intersection with $\mathfrak{A}$. X being a Q-space, all of the real ideals in $\mathfrak{C}(X, R)$ are fixed, and the hypotheses on $\mathfrak{A}$ make it clear that the mapping just described produces a one-to-one continuous mapping of X onto Y.

THEOREM B. *A Q-space X has a one-to-one continuous image Y which is a bicompact Hausdorff space if and only if the family*[2] $\mathfrak{Z}(X)$ *contains a subfamily $\mathcal{A}$ such that:* (1) *given $p\neq q$ in X, there is a set $B\in\mathcal{A}$ containing exactly one of p and q; $\prod_{A\in\mathcal{A}}A$ is void;* (2) *any subfamily of $\mathcal{A}$ with the finite intersection property has total intersection nonvoid.*

We use Theorem A to prove the present theorem, showing that the conditions stated above are equivalent to the existence of a subring $\mathfrak{A}$ of $\mathfrak{C}^*(X, R)$ having the properties set forth in Theorem A. Suppose that such a subring exists. Let $\mathfrak{B}$ be the set of functions in $\mathfrak{A}$ which vanish somewhere in X. It is plain that $\mathfrak{B}$ is precisely the set of functions in $\mathfrak{A}$ not having inverses. The family $\mathcal{A}=\{Z(f)\}_{f\in\mathfrak{B}}$ is a family with the properties required. We may obviously exclude the trivial case in which $\mathfrak{A}$ contains exactly one maximal ideal. Assume that condition (1) fails for points p and q in X. Then, the maximal ideal in $\mathfrak{A}$ consisting of all functions in $\mathfrak{A}$ which vanish at q is contained in the distinct maximal real ideals $\mathfrak{M}_p$ and $\mathfrak{M}_q$ of $\mathfrak{C}(X, R)$, a contradiction. Next, let $\mathcal{J}$ be any subfamily of $\mathcal{A}$ with the finite intersection property. The functions f in $\mathfrak{A}$ such that $Z(f)\in\mathcal{J}$ generate a proper ideal in $\mathfrak{A}$, which is contained in a maximal real ideal of $\mathfrak{C}(X, R)$. Since X is a Q-space, it follows that $\mathcal{J}$ has total intersection nonvoid.

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[2] Let $Z(f)$ be the set of points where the function f vanishes. Then $\mathfrak{Z}(X)$ is the family of all $Z(f)$, for $f\in\mathfrak{C}(X, R)$.

The converse is easily established. Let $\mathcal{A}$ be a subfamily of $Z(X)$ of the kind described, and let $\mathfrak{A}$ be the analytic subring of $\mathfrak{C}^*(X, R)$ generated by all bounded continuous real-valued functions such that $Z(f)\in\mathcal{A}$. It is clear that every maximal ideal in $\mathfrak{A}$ is contained in precisely one real ideal in $\mathfrak{C}(X, R)$, and this observation completes the present proof.

5. **Examples.** It is instructive to observe Theorems A and B in relation to certain common spaces.

(7) No completely regular space of cardinal number less than $2^{\aleph_0}$ and devoid of isolated points can have a one-to-one continuous mapping onto a bicompact Hausdorff space. This is obvious from the fact that such a bicompact Hausdorff space would have to be dense in itself, and such spaces are known to have cardinal number not less than $2^{\aleph_0}$. (See [2, p. 30, corollaire 1].)[3]

(8) Any locally bicompact Hausdorff space X has a one-to-one continuous mapping onto a bicompact Hausdorff space. Let p be any point of X, and let new neighborhoods of p be $U(p)\cup F'$, where $U(p)$ is a generic neighborhood of p and F is any bicompact subset of X. This new topology establishes the desired result.

(9) Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of topological spaces each having a one-to-one continuous map onto a bicompact Hausdorff space Y_λ. Then the Cartesian product $P_{\lambda\in\Lambda}X_\lambda$ admits an obvious one-to-one continuous mapping onto the bicompact Hausdorff space $P_{\lambda\in\Lambda}Y_\lambda$.

We next illustrate Theorem A by examining a simple space.

(10) Let Γ be the rational numbers in the closed interval $[0, 1]$ in their usual topology. Theorem A and remark (7) imply that any analytic subring $\mathfrak{A}$ of $\mathfrak{C}^*(\Gamma, R)$, such that for all $r_1\neq r_2$ in Γ there exists $f\in\mathfrak{A}$ with $f(r_1)\neq f(r_2)$, contains maximal ideals contained in no real ideal of $\mathfrak{C}(\Gamma, R)$. (Γ is a Q-space, like every separable metric space.) Let $\mathfrak{P}$ be the analytic subring of $\mathfrak{C}^*(\Gamma, R)$ generated by the constants and the function $f(r)=r$. Plainly this ring is isomorphic to $\mathfrak{C}^*(I, R)$, where I is the closed interval $[0, 1]$. All ideals $\mathfrak{M}_t$ in this ring, where t is any irrational number, are contained in no real ideal of $\mathfrak{C}(\Gamma, R)$. Next, let t be any irrational number in $(0, 1)$, and let ϕ_t be that function on Γ such that $\phi_t(r)=1$ for $r<t$ and $\phi_t(r)=0$ for $r>t$. Let $\mathfrak{Q}$ denote the analytic subring of $\mathfrak{C}^*(\Gamma, R)$ generated by the functions ϕ_t and the constant functions. It can then be proved that $\mathfrak{Q}=\mathfrak{C}^*(\Gamma, R)$. Let F be any closed subset of Γ. It is easy to see that F' is the union

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[3] Katětov has observed (in a letter to the writer) that no countable Hausdorff space dense in itself can be H-closed, which is a stronger statement than the present remark, for countable spaces.

of disjoint open intervals (a_λ, b_λ) in Γ, where a_λ and b_λ can be either rational or irrational. If a_λ and b_λ are irrational, let g_λ be the charactetistic function of (a_λ, b_λ), defined only on Γ, of course. If a_λ and b_λ are rational, let g_λ be

$$e^{-\tan^2[\pi(x-b_\lambda)/(b_\lambda-a_\lambda)+\pi/2]}$$

in (a_λ, b_λ) and zero elsewhere. If exactly one of a_λ and b_λ is rational, say a_λ, let g_λ be equal to $(x-b_\lambda)/(b_\lambda-a_\lambda)+1$ in (a_λ, b_λ) and zero elsewhere. The function $f=\sum_\lambda g_\lambda$ is clearly in $\mathfrak{Q}$, and has the property that $f=0$ exactly on F. If F_1 and F_2 are disjoint closed subsets of Γ, with functions f_1 and f_2 as just defined, then $f_1/(f_1+f_2)$ is a function in $\mathfrak{Q}$ which vanishes on F_1 and equals 1 throughout F_2. By a generalized approximation theorem [3], it follows that $\mathfrak{Q}=\mathfrak{C}^*(\Gamma, R)$. Clearly $\mathfrak{Q}$ fails to meet the requirements of Theorem A, since the space of maximal ideals in $\mathfrak{Q}$ can be considered as the space $\beta\Gamma$, which contains $2^{2^{\aleph_0}}$ points.

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THE UNIVERSITY OF CHICAGO

REMARKS ON CYCLIC ADDITIVITY

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1. Introduction. For the purposes of this discussion suppose that X and Y are topological spaces while G is a commutative, topological semi-group (with zero element) which, as a space, is Hausdorff. In other words, each pair (g_1, g_2) of elements in G uniquely determines an element (g_1+g_2) in G; the operation $+$ is associative and commutative; there is a unique element 0 such that $g\in G$ implies $g+0=g$; finally, G is a Hausdorf space and the operation $+$ provides a mapping ($=$continuous transformation) from the product space $G\times G$ into G. Obviously, topological groups, and the space of non-negative real numbers compactified by the addition of ∞, with the operation $+$ meaning addition, and the convention that $a+\infty=\infty+a=\infty$, provide examples of such semi-groups.

It will be said that lm is a *Peanian factorization* of a mapping $f:X\rightarrow Y$ if and only if there are mappings $m:X\rightarrow\mathfrak{X}$ and $l:\mathfrak{X}\rightarrow Y$ such that $\mathfrak{X}$ is a Peano space and the composition lm is f. The space $\mathfrak{X}$ is called the *middle space* of the Peanian factorization lm of f.

Let F be the class of mappings $f:X\rightarrow Y$ each of which has at least one Peanian factorization, and suppose that γ is a transformation from F into G.

For each Peano space $\mathfrak{X}$ let $\mathcal{E}(\mathfrak{X})$ be the class of true cyclic elements of $\mathfrak{X}$. (See Whyburn [6] for the Peano space theory involved in this paper.)[2] If $\mathfrak{C}\in\mathcal{E}(\mathfrak{X})$ there is a unique monotone retraction $r_{\mathfrak{C}}:\mathfrak{X}\rightrightarrows\mathfrak{C}$. (The double arrow indicates that $r_{\mathfrak{C}}(\mathfrak{X})=\mathfrak{C}$.) If $f\in F$ and lm is a Peanian factorization of f with middle space $\mathfrak{X}$, while $\mathfrak{C}\in\mathcal{E}(\mathfrak{X})$, then define $f_{\mathfrak{C}}=lr_{\mathfrak{C}}m$.

It is the object of this paper to investigate the statement

$$\gamma(f)=\sum\gamma(f_{\mathfrak{C}}),\qquad \mathfrak{C}\in\mathcal{E}(\mathfrak{X}) \tag{1}$$

where the equality means that for each neighborhood U of $\gamma(f)$ there is a finite subclass $\mathcal{J}(U)$ of $\mathcal{E}(\mathfrak{X})$ such that if $\mathcal{J}$ is any finite subclass of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U)$ then $U\ni\sum\gamma(f_{\mathfrak{C}})$, $\mathfrak{C}\in\mathcal{J}$, it being understood that addition over an empty class yields 0.

In the event that (1) holds for each $f\in F$ and for each Peanian factorization of f, then γ is said to be *cyclicly additive.*

Cyclic additivity theorems of a weaker type have been considered

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

by Helsel [3], Radó [5] and others in connection with Lebesgue area.

2. **The theorems.** *The first object is to investigate consequences of the assumption that γ is cyclicly additive.*

I. If $f\in F$ and has a Peanian factorization in which the middle space is a dendrite, then $\gamma(f)=0$.

This is an obvious consequence of the fact that a dendrite has no true cyclic elements.

I_a. If $f\in F$ and has a Peanian factorization in which the middle space is an arc, then $\gamma(f)=0$.

(It is understood that an arc may consist of a single point.)

Before considering more substantial necessary conditions, suppose that $\mathfrak{X}$ is a Peano space and define $\mathfrak{P}\in\mathcal{P}(\mathfrak{X})$ if and only if $\mathfrak{P}$ is a Peano subspace of $\mathfrak{X}$ such that $\mathcal{E}(\mathfrak{P})\subset\mathcal{E}(\mathfrak{X})$. Let $\mathfrak{Q}\in\mathcal{Q}(\mathfrak{X})$ if and only if $\mathfrak{Q}$ is a Peano subspace of $\mathfrak{X}$ having a finite cyclic chain approximation $\mathfrak{E}_1,\cdots,\mathfrak{E}_q$ such that either $\mathfrak{E}_k$ is an arc, or $\mathfrak{E}_k\in\mathcal{E}(\mathfrak{X})$, $k=1,\cdots,q$. (See Whyburn [6, p. 73].)

It is easy to see that $\mathcal{Q}(\mathfrak{X})\subset\mathcal{P}(\mathfrak{X})\supset\mathcal{A}(\mathfrak{X})$, where $\mathcal{A}(\mathfrak{X})$ is the class of A-sets in $\mathfrak{X}$.

Given $\mathfrak{P}\in\mathcal{P}(\mathfrak{X})$, let $\mathfrak{A}=\mathfrak{P}\cup\bigcup\mathfrak{E}$, where the union is taken over those elements $\mathfrak{E}\in\mathcal{E}(\mathfrak{X})$ having the property that $\mathfrak{P}\cap\mathfrak{E}$ consists of at least two points. It follows that $\mathfrak{A}\in\mathcal{A}(\mathfrak{X})$, that $x\in\mathfrak{A}-\mathfrak{P}$ implies there is a unique $\mathfrak{E}\in\mathcal{E}(\mathfrak{A})$ such that $\mathfrak{x}\in\mathfrak{E}$, and if $\mathcal{E}(\mathfrak{P})\not\ni\mathfrak{E}\in\mathcal{E}(\mathfrak{A})$ then $\mathfrak{E}\cap\mathfrak{P}$ is a dendrite. Since a dendrite is an absolute retract if $\mathcal{E}(\mathfrak{P})\not\ni\mathfrak{E}\in\mathcal{E}(\mathfrak{A})$, then there is a retraction $\rho_{\mathfrak{E}}:\mathfrak{E}\rightrightarrows(\mathfrak{E}\cap\mathfrak{P})$. (See Borsuk [1].) Define

$$\theta_{\mathfrak{P}}(\mathfrak{x})=\begin{cases}\mathfrak{x} & \text{if}\quad \mathfrak{x}\in\mathfrak{P},\\ \rho_{\mathfrak{E}}(\mathfrak{x}) & \text{if}\quad \mathfrak{x}\in\mathfrak{E}\quad\text{and}\quad \mathcal{E}(\mathfrak{P})\not\ni\mathfrak{E}\in\mathcal{E}(\mathfrak{A}).\end{cases}$$

It follows that $\theta_{\mathfrak{P}}$ is a retraction from $\mathfrak{A}$ onto $\mathfrak{P}$, but $\theta_{\mathfrak{P}}$ is not uniquely defined in terms of $\mathfrak{A}$ and $\mathfrak{P}$. If $r_{\mathfrak{A}}:\mathfrak{X}\rightrightarrows\mathfrak{A}$ is the unique monotone retraction, and $r_{\mathfrak{P}}=\theta_{\mathfrak{P}}r_{\mathfrak{A}}$, then $r_{\mathfrak{P}}:\mathfrak{X}\rightrightarrows\mathfrak{P}$ is a retraction.

The retraction $r_{\mathfrak{P}}$ depends upon the retraction $\theta_{\mathfrak{P}}$ which is not unique, however, if $\mathfrak{E}\in\mathcal{E}(\mathfrak{P})$, while $s_{\mathfrak{E}}:\mathfrak{P}\rightrightarrows\mathfrak{E}$ and $r_{\mathfrak{E}}:\mathfrak{X}\rightrightarrows\mathfrak{E}$ are retractions, then

$$r_{\mathfrak{E}}=s_{\mathfrak{E}}r_{\mathfrak{P}}. \tag{2}$$

There may be retractions from $\mathfrak{X}$ onto $\mathfrak{P}$ defined in other ways; however, the notation $r_{\mathfrak{P}}$ will always be used to indicate a retraction $\theta_{\mathfrak{P}}r_{\mathfrak{A}}$ defined as above, hence $r_{\mathfrak{P}}$ is unique modulo the factor $\theta_{\mathfrak{P}}$.

In this connection notice that if α is the maximum of the diameters of the components of $\mathfrak{X}-\mathfrak{A}$, and β is the maximum of the diameters of $\mathfrak{C}$ for $\mathcal{E}(\mathfrak{P})\not\ni\mathfrak{C}\in\mathcal{E}(\mathfrak{A})$, then

$$\rho\{\mathfrak{x}, r_{\mathfrak{P}}(\mathfrak{x})\} \leqq \alpha+\beta. \tag{3}$$

If $f\in F$ and lm is a Peanian factorization of f with middle space $\mathfrak{X}$ while $\mathfrak{P}\in\mathcal{P}(\mathfrak{X})$, then $f_{\mathfrak{P}}=lr_{\mathfrak{P}}m$ is a mapping from X into Y having a Peanian factorization $l(r_{\mathfrak{P}}m)$ with middle space $\mathfrak{P}$. In other words, $f_{\mathfrak{P}}\in F$. Different selections of $r_{\mathfrak{P}}$ will produce different mappings $f_{\mathfrak{P}}$; however, if $\mathfrak{C}\in\mathcal{E}(\mathfrak{P})$ then in view of (2) it is true that $f_{\mathfrak{C}}\equiv lr_{\mathfrak{C}}m$ $=ls_{\mathfrak{C}}r_{\mathfrak{P}}m\equiv(f_{\mathfrak{P}})_{\mathfrak{C}}\equiv f_{\mathfrak{P}\mathfrak{C}}$. Hence $f_{\mathfrak{P}\mathfrak{C}}$ is independent of the retraction $r_{\mathfrak{P}}$ in spite of the fact that $f_{\mathfrak{P}}$ is not.

II. If $f\in F$ has a Peanian factorization lm with middle space $\mathfrak{X}$, while $\mathfrak{A}_i\in\mathcal{A}(\mathfrak{X})$, $i=1, 2$, $\mathfrak{A}_1\cap\mathfrak{A}_2$ is a point, and $\mathfrak{A}_1\cup\mathfrak{A}_2=\mathfrak{X}$, then $\gamma(f_{\mathfrak{A}_1})+\gamma(f_{\mathfrak{A}_2})=\gamma(f)$.

PROOF. If the theorem is false suppose that the sum on the left above is $\eta\neq\gamma(f)$, and select neighborhoods U and U_0 for η and $\gamma(f)$, respectively, such that

$$U\cap U_0=\varnothing. \tag{4}$$

Let U_i be a neighborhood of $\gamma(f_{\mathfrak{A}_i})$, $i=1, 2$, such that

$$U_1+U_2\subset U.$$

Note that $\mathcal{E}(\mathfrak{A}_1)\cup\mathcal{E}(\mathfrak{A}_2)=\mathcal{E}(\mathfrak{X})$, and $\mathcal{E}(\mathfrak{A}_1)\cap\mathcal{E}(\mathfrak{A}_2)=\varnothing$. Since $\mathcal{A}(\mathfrak{X})$ $\subset\mathcal{P}(\mathfrak{X})$ and γ is cyclicly additive, there is a finite subcollection $\mathcal{J}(U_i)$ of $\mathcal{E}(\mathfrak{A}_i)$ such that if $\mathcal{J}_i$ is a finite subcollection of $\mathcal{E}(\mathfrak{A}_i)$ containing $\mathcal{J}(U_i)$, then (understanding that $\mathfrak{A}_0=\mathfrak{X}$)

$$U_i\ni\sum\gamma(f_{\mathfrak{C}}),\qquad \mathfrak{C}\in\mathcal{J}_i,\qquad\qquad i=0, 1, 2.$$

If $\mathcal{J}$ is any finite subcollection of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U_0)\cup\mathcal{J}(U_1)$ $\cup\mathcal{J}(U_2)$ let $\mathcal{J}_i=\mathcal{J}\cap\mathcal{E}(\mathfrak{A}_i)$, $i=1, 2$, and note that $\mathcal{J}_1\cap\mathcal{J}_2=\varnothing$ while $\mathcal{J}_1\cup\mathcal{J}_2=\mathcal{J}$. Hence

$$\begin{aligned} U_0 &\ni [\textstyle\sum\gamma(f_{\mathfrak{C}}), \mathfrak{C}\in\mathcal{J}] \\ &= [\textstyle\sum\gamma(f_{\mathfrak{C}}), \mathfrak{C}\in\mathcal{J}_1]+[\textstyle\sum\gamma(f_{\mathfrak{C}}), \mathfrak{C}\in\mathcal{J}_2] \\ &\in U_1+U_2\subset U. \end{aligned}$$

Therefore $U\cap U_0\neq\varnothing$ in contradiction to (4).

II$_a$. The same as II except that the hypothesis is strengthened by the condition $\mathfrak{X}\in\mathcal{Q}(\mathfrak{X})$.

III. If $f\in F$ and has a Peanian factorization lm with middle space $\mathfrak{X}$, then for each neighborhood U of $\gamma(f)$ there is a $\mathfrak{Q}\in\mathcal{Q}(\mathfrak{X})$ such that $\mathfrak{Q}\subset\mathfrak{P}\in\mathcal{P}(\mathfrak{X})$ implies that $\gamma(f_{\mathfrak{P}})\in\mathfrak{U}$.

PROOF. There is a finite subcollection $\mathcal{J}(U)$ of $\mathcal{E}(\mathfrak{X})$ such that if $\mathcal{J}$ is any finite subcollection of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U)$, then $U \ni \sum \gamma(f_{\mathfrak{C}})$, $\mathfrak{C} \in \mathcal{J}$. The theorem will be proved if it can be shown that there is a $\mathfrak{Q} \in \mathcal{Q}(\mathfrak{X})$ such that $\mathcal{E}(\mathfrak{Q}) = \mathcal{J}(U)$.

If $\mathcal{J}(U) = \emptyset$ let $\mathfrak{Q}$ be any point in $\mathfrak{X}$. If $\mathcal{J}(U) \neq \emptyset$, suppose $\mathfrak{C}_1, \cdots, \mathfrak{C}_n$ are the elements in $\mathcal{J}(U)$. Define $\mathfrak{C}_1^* = \mathfrak{C}_1$ and join $\mathfrak{C}_2$ to $\mathfrak{C}_1^*$ by an arc $\mathfrak{K}_1$ with end points only in $\mathfrak{C}_1^* \cup \mathfrak{C}_2$. The arc $\mathfrak{K}_1$ may be a single point. Let $\mathfrak{C}_2^* = \mathfrak{C}_2 \cup \mathfrak{K}_1 \cup \bigcup \mathfrak{C}$ where the union is taken over the true cyclic elements $\mathfrak{C}$ in $\mathcal{J}(U)$ which have at least two points in common with $\mathfrak{K}_1$.

There is a first subscript n_1, if any, for which $\mathfrak{C}_{n_1}$ is not in $\mathfrak{C}_1^* \cup \mathfrak{C}_2^*$. Join $\mathfrak{C}_{n_1}$ to $\mathfrak{C}_1^* \cup \mathfrak{C}_2^*$ by an arc $\mathfrak{K}_2$ with end points only in $\mathfrak{C}_1^* \cup \mathfrak{C}_2^* \cup \mathfrak{C}_{n_1}$. Let $\mathfrak{C}_3^* = \mathfrak{C}_{n_1} \cup \mathfrak{K}_2 \cup \bigcup \mathfrak{C}$ where the union is taken over the true cyclic elements $\mathfrak{C}$ in $\mathcal{J}(U)$ which have at least two points in common with $\mathfrak{K}_2$.

This process stops after a finite number of steps m for want of a true cyclic element in $\mathcal{J}(U)$ not in $\mathfrak{C}_1^* \cup \cdots \cup \mathfrak{C}_m^* \equiv \mathfrak{Q}$. The Peano space $\mathfrak{Q}$ certainly has a cyclic chain approximation $\mathfrak{C}_1, \cdots, \mathfrak{C}_q$ such that $\mathfrak{C}_k$ is either an arc or $\mathfrak{C}_k \in \mathcal{J}(U) \subset \mathcal{E}(\mathfrak{X})$, $k = 1, \cdots, q$. Hence $\mathfrak{Q} \in \mathcal{Q}(\mathfrak{X})$ and $\mathcal{E}(\mathfrak{Q}) = \mathcal{J}(U)$.

III$_a$. If $f \in F$ and has a Peanian factorization lm with middle space $\mathfrak{X}$, then for each neighborhood U of $\gamma(f)$ there is a $\mathfrak{Q} \in \mathcal{Q}(\mathfrak{X})$ such that $\mathfrak{Q} \subset \mathfrak{N} \in \mathcal{Q}(\mathfrak{X})$ implies $\gamma(f_{\mathfrak{N}}) \in U$.

The question of the sufficiency of these conditions is considered next.

THEOREM. *If γ is a transformation from F into G having the properties* I$_a$, II$_a$ *and* III$_a$, *then γ is cyclicly additive.*

PROOF. Let $f \in F$ and suppose lm is a Peanian factorization of f with middle space $\mathfrak{X}$. Suppose U is a neighborhood of $\gamma(f)$ and let $\mathfrak{Q}$ be the space in $\mathcal{Q}(\mathfrak{X})$ given by III$_a$. Define $\mathcal{J}(U) = \mathcal{E}(\mathfrak{Q})$, and suppose $\mathcal{J}$ is a finite subcollection of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U)$.

If $\mathcal{J} = \mathcal{J}(U)$ let $\mathfrak{N} = \mathfrak{Q}$. If $\mathcal{J} \neq \mathcal{J}(U)$, the construction used in the proof of III shows that there is an $\mathfrak{N} \in \mathcal{Q}(\mathfrak{X})$ such that $\mathcal{E}(\mathfrak{N}) = \mathcal{J}$ and $\mathfrak{N} \supset \mathfrak{Q}$. By III$_a$, $\gamma(f_{\mathfrak{N}}) \in U$.

But $\mathfrak{N}$ has a finite cyclic chain decomposition $\mathfrak{C}_1, \cdots, \mathfrak{C}_r$ such that either $\mathfrak{C}_k$ is an arc or $\mathfrak{C}_k \in \mathcal{E}(\mathfrak{X})$, $k = 1, \cdots, r$. Using the properties of a cyclic chain approximation, together with I$_a$ and II$_a$ it follows directly that

$$\begin{aligned} \gamma(f_{\mathfrak{N}}) &= \sum \gamma(f_{\mathfrak{N}\mathfrak{C}}), && \mathfrak{C} \in \mathcal{E}(\mathfrak{N}), \\ &= \sum \gamma(f_{\mathfrak{C}}), && \mathfrak{C} \in \mathcal{J}. \end{aligned}$$

Therefore,

$$U \ni \sum \gamma(f_{\mathfrak{C}}), \qquad \mathfrak{C} \in \mathcal{J},$$

and hence

$$\gamma(f) = \sum \gamma(f_{\mathfrak{C}}), \qquad \mathfrak{C} \in \mathcal{E}(\mathfrak{X}).$$

3. **An application.** Though greater generality is possible, suppose that X and Y are compacta, and for fixed $q \geqq 0$ consider the Čech cohomology groups $H^q(X)$ and $H^q(Y)$ where the coefficient group is discrete and does not change in the remainder of the discussion. The topology on $H^q(X)$ and $H^q(Y)$ is taken discrete and it should be mentioned that if $q=0$ one considers the reduced cohomology group.

If $f \in F$ then there is an induced homomorphism

$$f^*: H^q(Y) \rightarrow H^q(X).$$

Select any element $y \in H^q(Y)$, let $G = H^q(X)$, and define

$$\gamma(f) = f^*(y).$$

It will be shown that γ *is cyclicly additive*; *that is*, *if* lm *is a Peanian factorization of* $f \in F$ *with middle space* $\mathfrak{X}$, *then for each* $y \in H^q(Y)$,

$$f^*(y) = \sum f_{\mathfrak{C}}^*(y), \qquad \mathfrak{C} \in \mathcal{E}(\mathfrak{X}).$$

In view of the sufficiency theorem it is enough to check conditions I_a, II_a and III_a.

In the event $f \in F$ and has a Peanian factorization lm whose middle space is an arc, then f is clearly homotopic to a point and hence $f^*(y) = 0$.

To check II_a one will need the fact that: If X_1 and X_2 are closed subsets of X, and $X = X_1 \cup X_2$, while $f_i: X \rightarrow Y$ is a mapping for $i = 0, 1, 2$, such that $f_0 = f_i$ on X_i, $i = 1, 2$, and $x \in X_j$ implies $f_i(x) = y_0$, a single point of Y, $i, j = 1, 2$; $i \neq j$; then $f_0^* = f_1^* + f_2^*$. (See Borsuk [2].)

Suppose $\mathfrak{A}_i \in \mathcal{A}(\mathfrak{X})$, $i = 1, 2$, and $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is a point $\mathfrak{x}_0$, while $\mathfrak{A}_1 \cup \mathfrak{A}_2 = \mathfrak{X}$. Define $X_i = m^{-1}(\mathfrak{A}_i)$ and $f_i = l r_{\mathfrak{A}_i} m$, $i = 1, 2$. Now X_1 and X_2 are closed subsets of X while $X_1 \cup X_2 = X$. Moreover, if $x \in X_i$, then $m(x) \in \mathfrak{A}_i$, hence $r_{\mathfrak{A}_i} m(x) = m(x)$ and so $f_i(x) = lm(x) = f(x)$; however, $f_j(x) = l r_{\mathfrak{A}_j} m(x) = l(\mathfrak{x}_0)$ since $r_{\mathfrak{A}_j}(\mathfrak{A}_i) = \mathfrak{x}_0$, $i, j = 1, 2$; $i \neq j$. Hence $f^* = f_1^* + f_2^*$ and $\gamma(f) = \gamma(f_{\mathfrak{A}_1}) + \gamma(f_{\mathfrak{A}_2})$, which shows that condition II and hence II_a is satisfied.

To check condition III_a recall that there is an $\epsilon(y)$ such that if $\phi_1: X \rightarrow Y$ and $\phi_2: X \rightarrow Y$ are two mappings with the property that $\rho\{\phi_1(x), \phi_2(x)\} < \epsilon(y)$, $x \in X$, then $\phi_1^*(y) = \phi_2^*(y)$. (See, for example, Hurewicz-Wallman [4, p. 140].)

Since l is continuous on a compact space there is a δ such that $\rho\{\mathfrak{x}_1, \mathfrak{x}_2\} < \delta$ implies that $\rho\{l(\mathfrak{x}_1), l(\mathfrak{x}_2)\} < \epsilon(y)$.

Let $\{\mathfrak{C}_k\}$ be a cyclic chain approximation to $\mathfrak{X}$, and select a so large that $n \geqq a$ implies that each component of $\mathfrak{X} - \mathfrak{A}_n$ has a diameter less than $\delta/2$, where $\mathfrak{A}_n = \cup \mathfrak{C}_k$, $k = 1, \cdots, n$. Consider $\mathfrak{A}_a$ and let $\mathfrak{K}_k$ be an arc whose end points are "end points" of $\mathfrak{C}_k$, $k = 1, \cdots, a$. Let $\mathfrak{E} \in \mathcal{E}$ if and only if $\mathfrak{E} \in \mathcal{E}(\mathfrak{A}_a)$, $d(\mathfrak{E}) < \delta/2$ and $\mathfrak{E} \cap \mathfrak{C}_k \neq \emptyset$ for exactly one integer $k = 1, \cdots, a$. Define $\mathfrak{Q} = [\mathfrak{A}_a - \cup \mathfrak{E}, \mathfrak{E} \in \mathcal{E}] \cup [\cup \mathfrak{K}_k, k = 1, \cdots, a]$.

It is easy to see that $\mathfrak{Q} \in \mathcal{Q}(\mathfrak{X})$, in fact $\mathfrak{A}_a$ is the smallest A-set containing $\mathfrak{Q}$, and if $\mathcal{E}(\mathfrak{Q}) \not\ni \mathfrak{E} \in \mathcal{E}(\mathfrak{A}_a)$, then $\mathfrak{E} \cap \mathfrak{Q}$ is an arc.

Now suppose that $\mathfrak{Q} \supset \mathfrak{R} \in \mathcal{Q}(\mathfrak{X})$. If $\mathfrak{A}$ is the smallest A-set containing $\mathfrak{R}$, then $\mathfrak{A} \supset \mathfrak{A}_a$ and hence each component of $\mathfrak{X} - \mathfrak{A}$ is in some component of $\mathfrak{X} - \mathfrak{A}_a$. Moreover, if $\mathcal{E}(\mathfrak{R}) \not\ni \mathfrak{E} \in \mathcal{E}(\mathfrak{A})$, then $\mathfrak{E} \in \mathcal{E}$ or $\mathfrak{E}$ is in some component of $\mathfrak{X} - \mathfrak{A}_a$. In either event $d(\mathfrak{E}) < \delta/2$. Hence by (3), $\rho\{x, r_{\mathfrak{R}}(x)\} < \delta$.

Since $f_{\mathfrak{R}} = l r_{\mathfrak{R}} m$, the selection of δ shows that $\rho\{f(x), f_{\mathfrak{R}}(x)\} < \epsilon(y)$, $x \in X$. Consequently $f^*(y) = f^*_{\mathfrak{R}}(y)$; that is, $\gamma(f) = \gamma(f_{\mathfrak{R}})$.

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ON FIBERING SPHERES BY TORUSES

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1. **Introduction.** In this note we answer a question which was recently asked by H. Hopf, namely whether there exist fiber decompositions of the n-dimensional sphere S^n, or of the n-dimensional Euclidean space E^n, with an s-dimensional torus T^s (the product of s factors S^1) as fiber. The answers are given in Theorems 1 and 2 below. The fiber decompositions are understood in the sense of fiber bundle [1],[1] with the group of all homeomorphisms as structure group, that is, we are given three spaces X, F and M (the bundle, the fiber and the base space), and a map $\phi: X \to M$, the projection; and for every point p of M there is given a neighborhood V_p of p and a homeomorphism of $\phi^{-1}(V_p)$ with the product $V_p \times F$, such that $\phi^{-1}(q)$ is mapped onto $q \times F$ for every point $q \in V_p$. The sets $\phi^{-1}(p)$ are called the fibers.

2. **Results.** We now consider the case where X is either S^n or E^n $(n>0)$, and where F is T^s $(s>0)$. We shall establish the following results:

THEOREM 1. *A fiber decomposition (in the sense of* §1), *with a (locally finite) polyhedron as base space, of the n-sphere* S^n *with the s-torus* T^s *as fiber exists if and only if n is odd and* $s=1$.

THEOREM 2. *For no n and s does there exist a fiber decomposition (in the sense of* §1), *with a (locally finite) polyhedron as base space, of Euclidean n-space* E^n *with the s-torus* T^s *as fiber.*

REMARK. We show in §7 (a) that Theorem 1 holds also if arbitrary separable metric spaces (instead of polyhedra only) are admitted as base spaces. The corresponding statement for Theorem 2 can be made only modulo a theorem concerning singular homology groups, which is probably true, but for which, as far as the authors are aware, there is no proof in the literature.

We note for the if part of Theorem 1 that there exists a well known fiber decomposition of S^{2l+1} with circles T^1 as fiber; the base space is the complex projective space K^l (of dimension $2l$) (see [2, p. 55]).

The proofs of Theorems 1 and 2 are based on the consideration of

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[1] Numbers in brackets refer to the list of references at the end of the paper.

homotopy groups and singular homology groups and their connections, in particular on the results of Hurewicz-Steenrod [3], Eckmann [4] and Eilenberg-MacLane [5].

3. **Homotopy groups.** Let (X, F, M, ϕ) be a fiber bundle as in §1. With a fixed fiber F_0 and base point q_0 in F_0 we form the absolute homotopy groups $\pi_k(X, q_0)$ or in short $\pi_k(X)$, and the relative homotopy groups $\pi_k(X, F_0, q_0)$ or in short $\pi_k(X, F_0)$; we set up the homotopy sequence of X mod F_0 [6, p. 463]:

$$\begin{aligned}(1)\qquad \cdots \pi_k(F_0, q_0) \rightarrow \pi_k(X, q_0) \rightarrow \pi_k(X, F_0, q_0) \rightarrow \pi_{k-1}(F_0, q_0) \\ \rightarrow \cdots \rightarrow \pi_1(X, q_0).\end{aligned}$$

This is an exact sequence; the kernel of each homomorphism is the image under the preceding homomorphism. Furthermore, from Steenrod's results in [1, Theorem 9, p. 302] on the existence of a covering homotopy in fiber bundles, and from the reasoning of [3, Theorem 2], or from [4, p. 162, (4)], it follows that the groups of X mod F_0 and the absolute groups of M are isomorphic (we put $p_0 = \phi(q_0)$):

$$(2)\qquad \pi_k(M, p_0) \approx \pi_k(X, F_0, q_0).$$

We now consider the special case of Theorem 1 and assume $F = T^s$ and $X = S^n$ (for the time being we make the restriction $n > 2$); we denote the base space by M_1 (by hypothesis, M_1 is a polyhedron). The first step in the proof of Theorem 1 is the determination, in equations (5), of the homotopy groups of M_1 up to dimension $n-1$.

Let Z_j stand for the free abelian group of j generators. The following facts are well known:

$$(3)\qquad \begin{aligned}\pi_1(T^s) &= Z_s, \\ \pi_k(T^s) &= 0 \qquad \text{for } k \neq 1;\end{aligned}$$

$$(4)\qquad \pi_k(S^n) = 0 \qquad \text{for } 0 < k < n.$$

The connectedness of T^s implies that $\pi_1(M_1)$ vanishes: every closed path in M_1 is, by the covering homotopy theorem, image of a path in S^n with initial and terminal point in the same fiber, and therefore also image of a closed path in S^n; moreover $\pi_1(S^n)$ vanishes. For $2 \leq k < n$ it follows from the exactness of (1) and from (4) that the groups $\pi_k(S^n, T_0^s)$ and $\pi_{k-1}(T^s)$ are isomorphic (if in a "section of length 4" in an exact sequence the two outer groups vanish, then the two inner groups are isomorphic, cf [7, p. 687]). By (2), the groups $\pi_k(S^n, T_0^s)$ and $\pi_k(M_1)$ are isomorphic. It follows that $\pi_k(M_1)$ and $\pi_{k-1}(T^s)$ are isomorphic for $2 \leq k < n$; in view of (3) we can therefore state the following equations:

$$\pi_k(M_1) = 0 \qquad \text{for } 0 < k < n,\ k \neq 2, \tag{5}$$
$$\pi_2(M_1) = Z_s.$$

4. **Homology groups.** Let H^i be the ith singular integral homology group, and let Σ^i be the subgroup of spherical cycles. To prove our Theorems 1 and 2 we shall make use of Theorem II^m of Eilenberg-MacLane [5] which says: If X is a space such that $\pi_i(X)=0$ for $i<m$ and $m<i<n$ $(m<n)$ then $H^i(X)$, for $i<n$, and $H^n(X)/\Sigma^n(X)$ are determined by the structure of the group $\pi_m(X)$; our equations (5) show that this theorem is applicable to the space M_1 with $m=2$. In order to compute the groups $H^i(M_1)$, $i<n$, and $H^n(M_1)/\Sigma^n(M_1)$, we use a geometrical method; we construct a space P whose homotopy groups $\pi_i(P)$ for $i<n$ are isomorphic with the corresponding groups of M_1, and whose homology groups are known; by the theorem just mentioned the groups of M_1 will be isomorphic to those of P, for the appropriate dimensions. The space P is defined as the topological product of s complex projective spaces K^l, with l sufficiently large, for example, $l>n$. For the homotopy groups of K^l the following relations are known:

$$\pi_2(K^l) = Z_1; \quad \pi_i(K^l) = 0, \qquad 0 < i < 2l+1,\ i \neq 2. \tag{6}$$

These relations are immediate consequences of equation (5), for $n=2l+1$ and $s=1$, since, as mentioned in §2, K^l is the base space of a fiber decomposition of S^{2l+1} with fiber T^1. The homotopy groups of a topological product are the direct sums of the homotopy groups of the factors; the relations

$$\pi_i(P) \approx \pi_i(M_1), \qquad 0 < i < n \tag{7}$$

follow then from (5) and (6).

The homology groups of K^l are given by the equations [8, p. 83]:

$$H^{2k}(K^l) = Z_1, \text{ for } 0 \leqq k \leqq l;\ H^{2k-1}(K^l) = H^{2l+k}(K^l) = 0 \text{ for } k > 0. \tag{8}$$

The subgroups of spherical cycles are given by

$$\Sigma^2(K^l) = H^2(K^l); \quad \Sigma^i(K^l) = 0, \qquad i \neq 2, \tag{9}$$

as can be seen easily from (6) and from the fact that the generators of $H^2(K^l)$ and of $\pi_2(K^l)$ are both given by the subspace K^1 of K^l. The homology groups of P can now be found from (8) by Künneth's theorem; we note that all groups of odd dimensions vanish, that the groups of even dimensions (not greater than dim P) do not vanish, and that nonbounding spherical cycles exist only in dimension 2:

$$(10)\qquad \begin{aligned} &H^{2t-1}(P) = 0,\ t > 0;\quad H^{2t}(P) \neq 0 \text{ for } 0 \leqq t \leqq 2^{-1} \dim P;\\ &\qquad\qquad \Sigma^{t}(P) = 0 \qquad\qquad \text{for } t \neq 2. \end{aligned}$$

5. **Proof of Theorem 1.** As noted above, equations (5) and (7) make it possible to apply Theorem II^m of [5] with $m=2$; the following relations between the homology groups of M_1 and P hold therefore:

$$(11)\quad H^i(M_1) \approx H^i(P) \text{ for } 0 \leqq i < n;\ H^n(M_1)/\Sigma^n(M_1) \approx H^n(P)/\Sigma^n(P).$$

This amounts to a partial determination of the homology groups of M_1. On the other hand, the homology groups of M_1 vanish for dimensions above $n-s$: since every point p of M_1 has a neighborhood V such that $V \times T^s$ is homeomorphic with the open subset $\phi^{-1}(V)$ of E^n it is clear that the dimension of the polyhedron M_1 is $n-s$, and therefore the homology groups in higher dimensions vanish. We state the equation:

$$(12)\qquad H^i(M_1) = 0 \qquad \text{for } i > n - s.$$

Theorem 1 follows now quickly from the properties of the homology groups of M_1, given by (10), (11), (12):

By equation (12), the group $H^n(M_1)$ vanishes. From (11) we see that therefore the subgroup $\Sigma^n(P)$ equals the whole group $H^n(P)$. Comparing this with (10) we conclude that n must be odd, since obviously $n < \dim P$ (notice that we assumed $n > 2$). But then $H^{n-1}(P)$ does not vanish, again by (10). By (11) $H^{n-1}(M_1)$ does not vanish; by (12) we have $n-1 \leqq n-s$; it follows that $s=1$. This concludes the proof of Theorem 1 for $n>2$. For $n=2$ the only case of interest is $s=1$; it is easily seen that then the Jordan curve theorem for S^2 would imply that every point of the compactum M_1 is a cut point, which is impossible [9, p. 54]. For $n=1$ there is nothing to prove.

6. **Proof of Theorem 2.** We now assume that we have a fiber decomposition of E^n ($n>0$) with fiber T^s ($s>0$); we denote the base space by M_2; by hypothesis, M_2 is a polyhedron. The reasoning which leads to a contradiction is similar to that of §5. Equation (4) is replaced by

$$(4')\qquad \pi_k(E^n) = 0, \qquad k > 0.$$

$\pi_1(M_2)$ vanishes, for the same reason as $\pi_1(M_1)$. The exactness of (1), and (4′) show now that $\pi_k(E^n, T^s_0)$ and $\pi_{k-1}(T^s_0)$ are isomorphic for all $k \geqq 2$. We find, therefore, applying (2) and (3),

$$(5')\qquad \pi_2(M_2) = Z_s, \qquad \pi_k(M_2) = 0 \qquad \text{for } k \neq 2.$$

We again consider the comparison space P, the product of s factors

K^l, with $l>n$. From (5′) and (6) it follows, as in §4, that

$$\pi_i(P) \approx \pi_i(M_2) \qquad \text{for } i < 2l+1; \tag{7′}$$

Theorem II^m of [5], which applies again with $m=2$ because of (5′) and (7′), shows that

$$H^i(P) \approx H^i(M_2) \qquad \text{for } i < 2l+1. \tag{11′}$$

By the reasoning which we used to establish (12), we see that the dimension of M_2 is $n-s$ and that the homology groups in dimensions above $n-s$ vanish:

$$H^i(M_2) = 0 \qquad \text{for } i > n-s. \tag{12′}$$

We now choose an even i_0 with $n-s<i_0<2l+1$; this is possible since $l>n$; from (10) we see that $H^{i_0}(P)$ does not vanish, but $H^{i_0}(M_2)$ vanishes because of (12′). This contradiction establishes Theorem 2.

7. **Remarks.** (a) It would be desirable to prove Theorems 1 and 2 with arbitrary separable metric spaces (and not only polyhedra) as admissible base spaces for the fiber decompositions. Now the only point where the polyhedral character of M_1 and M_2 has been made use of is in establishing equation (12) and (12′). We shall show how to derive (12), and so to prove Theorem 1, if M_1 is assumed to be only separable metric: M_1 is compact, as image of S^n. Consider a neighborhood V_p of a point $p\in M_1$, for which $V_p\times T^s$ is homeomorphic with $\phi^{-1}(V_p)$, under a homeomorphism h. Since T^s is a product of one-dimensional factors, it follows from a theorem of Hurewicz [10] that V_p, and therefore M_1, is of dimension $n-s$. In addition, M_1 is locally contractible: Consider an arbitrary $t\in T^s$; then $h(V_p\times t)$ is a set in S^n which is mapped by ϕ homeomorphically onto V_p. Consider a small spherical neighborhood U of $h(p\times t)$, contained in $\phi^{-1}(V_p)$, and let $W=U\cap h(V_p\times t)$. Then $\phi(W)$ is a neighborhood of p, contained in V_p. The set W is obviously contractible in U, and ϕ maps the contraction of W into a contraction of $\phi(W)$, taking place in $\phi(U)$, and so in V_p. It follows that the singular groups of M_1 are isomorphic with the Čech groups of M_1 (cf. [11, p. 107]), and so they vanish in dimensions above that of M_1. We have found:

$$H^i(M_1) = 0 \qquad \text{for } i > n-s, \tag{12″}$$

and as noted above, this suffices to prove Theorem 1 under the weakened hypothesis. In the case of Theorem 2, assuming M_2 separable metric only, one sees, by the same reasoning as above, that M_2 is locally compact, locally contractible, and of dimension $n-s$. In order to establish equation (12′) (and with it Theorem 2 under the weakened hypothesis) one would have to have a theorem to the effect that in a separable metric, locally compact, locally contractible space the singular homology groups and the compact Čech groups are isomorphic, and so vanish for dimensions above the dimension of the space.

(b) Theorem 2 verifies in a special case the conjecture that E^n cannot be fibered with any compact fiber (cf. [**12**; **13**]).

(c) Theorem 1 gives a topological proof for the fact (which is useful in determining which spheres are group manifolds (cf. [**14**])): If a sphere is a Lie group then its rank is 1. The rank is the dimension of a maximal abelian subgroup; such a subgroup is homeomorphic with a torus T^r, and the cosets of the subgroup give a fiber decomposition of the group.

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ON CONTINUOUS CURVES IRREDUCIBLE ABOUT COMPACT SETS

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Some years ago, Leo Zippin [4][1] raised the following question: Given a compact set K in a locally connected complete metric space, S, it is known that S contains a compact continuum M irreducible about K. What conditions on K are necessary and sufficient that M may be chosen locally connected? He showed that a necessary condition is that K be a curve set; that is, the nondegenerate components of K are locally connected and form a null sequence. For K 1-dimensional, he proved the condition is also sufficient, and it was conjectured that it is always sufficient. Later Martin Ettlinger [3] gave other results tending to support this. In this note I give a partial solution.

THEOREM. *If K is a compact curve set in a convex metric space S, then S contains a locally connected compact continuum irreducible about K.*

The recent announcement by R. H. Bing [1] of a proof that every finite-dimensional Peano space has a convex metric makes this result appear less special than it might seem otherwise.

PROOF. Zippin has shown in [4] that the proposition need be proved only for the case where K is a special curve set, which is a curve set with only a countable number of components, all but one being points forming a sequence converging to that one. His argument is stated for S compact, but, however, is valid under our hypothesis. Hence, we may suppose K is the sum of a sequence $\{x_n\}$ of distinct points and a locally connected continuum C. The sequence x_n can be chosen so that $d(x_n, C) \geq d(x_{n+1}, C)$. For each n, let y_n denote a point of C such that $d(x_n, y_n) = d(x_n, C)$. Let T_1 denote a straight-line interval from x_1 to y_1. Let x_{n_1} be the first point of x_n not in T_1, and let T_2 denote a straight-line interval joining x_{n_1} and y_{n_1}. If $T_1 \cdot T_2$ is not empty, replace the interval of T_2 irreducible about $y_{n_1} + T_1 \cdot T_2$ by the arc of T_1 irreducible about $y_1 + T_1 \cdot T_2$, and let T_2 now denote this new arc. It is easy to see from the triangle inequality and the convexity of T_1 and the original T_2 that this does not change the length of T_2. We now have that $T_1 \cdot T_2$ is either empty or is connected and contains y_1. Let x_{n_2} be the first point of x_n not in

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[1] Numbers in brackets refer to the bibliography.

T_1+T_2, and let T_3 denote a straight-line interval from x_{n_2} to y_{n_2}. If $(T_1+T_2)\cdot T_3$ is not empty, a modification of T_3 similar to that of T_2, without changing the length of T_3, permits us to assume that $(T_1+T_2)\cdot T_3$ is connected and contains one of y_1 and y_{n_1}. Continue the process indefinitely. We obtain a sequence T_n of straight-line intervals such that (1) each point x_k is contained in one; (2) each has a point of $\{x_n\}$ for one end point, and a point of $\{y_n\}$ for the other, that being its only point in C; (3) each is a shortest path between its end point in $\{x_n\}$ and C; and (4) the intersection of each two is either empty or is connected and intersects C. Let M denote $C+\sum T_n$; then M is obviously connected, and irreducibly connected about $C+\sum x_n$. We need only verify local connectivity. If x is a point of $M-C$, then only a finite number of sets T_n have points more than $2^{-1}d(x, C)$ from C, so that x is in an open subset of M which is the sum of a finite number of locally connected sets. Hence M is locally connected in $M-C$. If U is an open subset of M intersecting C, and H is a component of U, then $H\cdot C$ is open in $U\cdot C$, being the sum of components of $U\cdot C$. Also $H\cdot(M-C)$ is open in U, being the sum of components of $U\cdot(M-C)$. If a point y of $H\cdot C$ were a limit point of $(U-H)\cdot M$, it would be in the limiting set of a convergent sequence T_n' of sets T_n. But then it is the sequential limiting point of $T_n'\cdot C$, which contradicts the fact that $H\cdot C$ is open in C. Hence every component of U is open in M, which completes our proof.

COROLLARY. *If the compact curve set K is finite-dimensional and lies in a connected and locally arcwise connected metric space, S, then S contains a compact locally connected continuum irreducible about C.*

PROOF. It is well known[2] that under our hypothesis S contains a compact locally connected continuum M containing K which is obtained by adding a countable number of arcs to K, and which is therefore also finite-dimensional. The corollary then follows from Bing's result and the theorem.

The reader interested in metric geometry will notice that the hypotheses of the theorem could be weakened or altered by assumptions about length, quasi-convexity, and so on. Such changes would have required more space without altering the essential nature of the argument. For such concepts, the reader is referred to Blumenthal [2]. In this connection, it seems to me that a large part of the possible interest in this note is due to its exploitation of metric methods in topology, which has not yet been systematically studied.

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[2] See the references in [4], or the proof for Moore spaces in [3].

Added in proof. Since this note was submitted, E. E. Moise and Bing have each announced solutions of the convexification problem, so that this note settles Zippin's question.

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UNIVERSITY OF MICHIGAN

A SIMPLE SOLUTION OF THE DIOPHANTINE EQUATION $x^3+y^3=z^2+t^2$

R. C. CAMPBELL

1. **Introduction.** The problem of finding integral solutions for the equation $x^3+y^3=z^2+t^2$ was evidently proposed by E. Miot [1].[1] A. Cunningham [2] noted the obvious solutions $x=a^2$, $y=b^2$, $z=a^3$, $t=b^3$. Additional solutions, some of which were obtained by setting $x+y=l^2$, or $x+y=m^2+n^2$, and $x^2-xy+y^2=p^2+q^2$, were also given by Cunningham.

The equation has the obvious solutions $x=-y$, $z=t=0$. These are considered of little interest and are not discussed here. However, should it be desired to include these solutions, they may be given by $x=n$, $y=-n$, $z=t=0$. These same remarks apply to the general solution of $x^3+y^3=z^2$ given here as a direct consequence of the solution of $x^3+y^3=z^2+t^2$.

2. **Solution.** Let the equation first be written in the form

$$(x+y)[(2x-y)^2+3y^2]=4(z^2+t^2).$$

Hence, $(x+y)(2x-y+i\,3^{1/2}y)(2x-y-i\,3^{1/2}y)=4(z+it)(z-it)$.

Consider the system of equations,

$$n(x+y)=4s, \tag{1}$$

$$s(2x-y+i3^{1/2}y)=(p+mi+q3^{1/2}+ir3^{1/2})(z+it), \tag{2}$$

$$(p+mi+q3^{1/2}+ir3^{1/2})(2x-y-i3^{1/2}y)=n(z-it). \tag{3}$$

This system of equations will be satisfied in rational numbers if the following system, obtained from the system (1), (2), (3) by equating like components, is so satisfied. Hence, consider the system of equations,

$$nx+ny=4s, \tag{4}$$

$$2sx-sy-pz+mt=0, \tag{5}$$

$$sy-rz-qt=0, \tag{6}$$

$$mz+pt=0, \tag{7}$$

$$qz-rt=0, \tag{8}$$

$$2px+(3r-p)y-nz=0, \tag{9}$$

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[1] Numbers in brackets refer to bibliography given at end of paper.

$$2mx - (3q + m)y + nt = 0, \tag{10}$$

$$2qx + (m - q)y = 0, \tag{11}$$

$$2rx - (p + r)y = 0. \tag{12}$$

Now, consider the equations (5), (6), (7). All solutions of this system of equations are given by:

$$x = \mu_1 \begin{vmatrix} -s & -p & m \\ s & -r & -q \\ 0 & m & p \end{vmatrix} = \mu_1(prs + m^2s - mqs + p^2s), \tag{13}$$

$$y = \mu_1 \begin{vmatrix} 2s & m & -p \\ 0 & -q & -r \\ 0 & p & m \end{vmatrix} = 2\mu_1(prs - mqs), \tag{14}$$

$$z = \mu_1 \begin{vmatrix} 2s & -s & m \\ 0 & s & -q \\ 0 & 0 & p \end{vmatrix} = 2\mu_1(ps^2), \tag{15}$$

$$t = \mu_1 \begin{vmatrix} 2s & -p & -s \\ 0 & -r & s \\ 0 & m & 0 \end{vmatrix} = -\, 2\mu_1(ms^2). \tag{16}$$

In order that (7) and (8) may have nontrivial solutions for z and t it is clear that

$$m = -\, pq/r. \tag{17}$$

Hence

$$x = \mu \frac{(p + r)(r^2 + q^2)}{r^2}, \tag{18}$$

$$y = 2\mu \frac{(r^2 + q^2)}{r}, \tag{19}$$

$$z = 2\mu s, \tag{20}$$

$$t = 2\mu\, qs/r. \tag{21}$$

For (9) to be satisfied by the values of x, y, z, given by (18), (19), and (20), direct substitution shows that,

$$n = \frac{(r^2 + q^2)(p^2 + 3r^2)}{sr^2}. \tag{22}$$

With the condition (22), it may now be verified that the values of x, y, z, t given by (18), (19), (20) and (21) satisfy equations (5) through (12).

If equation (4) is to be satisfied by (18) and (19), it is found that

$$\mu = \frac{4s^2r^4}{(p+3r)(p^2+3r^2)(r^2+q^2)^2}.$$

Hence,

$$x = \frac{4s^2r^2(p+r)}{(p+3r)(p^2+3r^2)(r^2+q^2)}, \tag{23}$$

$$y = \frac{8s^2r^3}{(p+3r)(p^2+3r^2)(r^2+q^2)}, \tag{24}$$

$$z = \frac{8s^3r^4}{(p+3r)(p^2+3r^2)(r^2+q^2)^2}, \tag{25}$$

$$t = \frac{8s^3r^3q}{(p+3r)(p^2+3r^2)(r^2+q^2)^2}. \tag{26}$$

It is at once clear that s is completely at our disposal. Therefore, after choosing $s=(p+3r)(p^2+3r^2)(r^2+q^2)$, making the obvious simplifications and introducing the customary λ, we have

$$x = \lambda^2(p+r)(p+3r)(p^2+3r^2)(r^2+q^2), \tag{27}$$

$$y = 2\lambda^2r(p+3r)(p^2+3r^2)(r^2+q^2), \tag{28}$$

$$z = \lambda^3r(p+3r)^2(p^2+3r^2)^2(r^2+q^2), \tag{29}$$

$$t = \lambda^3q(p+3r)^2(p^2+3r^2)^2(r^2+q^2). \tag{30}$$

3. **Completeness.** Let A, B, C, D be any values which satisfy the original equation; that is, $A^3+B^3=C^2+D^2$, and set

$$A = \lambda^2(p+r)(p+3r)(p^2+3r^2)(r^2+q^2), \tag{31}$$

$$B = 2\lambda^2r(p+3r)(p^2+3r^2)(r^2+q^2), \tag{32}$$

$$C = \lambda^3r(p+3r)^2(p^2+3r^2)^2(r^2+q^2), \tag{33}$$

$$D = \lambda^3q(p+3r)^2(p^2+3r^2)^2(r^2+q^2). \tag{34}$$

It will now be shown that there exist values for p, q, r, and λ that will produce the given solution. In particular, if A, B, C, and D be integral, p, q, r are integral and λ rational. To this end consider the ratio,

$$C/D = r/q,$$

and choose $r=\lambda_1 C$, $q=\lambda_1 D$.

It now follows from the ratio A/B that

$$p = \frac{\lambda_1 C(2A - B)}{B}.$$

Let $\lambda_1 = B$. Hence,

$$r = BC, \tag{35}$$

$$q = BD, \tag{36}$$

$$p = C(2A - B). \tag{37}$$

Substituting (35), (36), and (37) in (31), (32), (33), and (34) we have

$$\begin{aligned} A &= 16\lambda^2 AB^2C^4(C^2 + D^2)^2, \\ B &= 16\lambda^2 B^3C^4(C^2 + D^2)^2, \\ C &= 64\lambda^3 B^3C^7(C^2 + D^2)^3, \\ D &= 64\lambda^3 B^3C^6D(C^2 + D^2)^3. \end{aligned}$$

Choosing $\lambda = 1/4BC^2(C^2+D^2)$, $B\neq 0$, $C\neq 0$, completes the proof. Moreover, it is clear that B and C may be chosen different from zero; for, if there be given solutions involving zeros, it is manifestly possible to set B and C equal to the nonzero members of these solutions, while A or D, or both A and D, may be chosen as zero. Hence, excluding the trivial solutions noted in the introduction, all solutions of the equation $x^3+y^3=z^2+t^2$ are given by (27), (28), (29), and (30).

It is worthwhile to note that the general solution of the equation $x^3+y^3=z^2$ now follows immediately. In (27), (28), (29) and (30) set $q=0$, and suppress the powers of r. Then,

$$\begin{aligned} x &= \lambda^2(p + r)(p + 3r)(p^2 + 3r^2), \\ y &= 2\lambda^2 r(p + 3r)(p^2 + 3r^2), \\ z &= \lambda^3(p + 3r)^2(p^2 + 3r^2)^2. \end{aligned}$$

The generality is at once clear, for (27), (28), (29), and (30) must give, in particular, all solutions of the form $x^3+y^3=z^2+0^2$. For a history of this case see [3, pp. 578–581].

The well known solution for the equation $y^3=z^2+t^2$ also follows as a special case. In (27), (28), (29) and (30) set $p=-r$, and make the indicated reductions. Then,

$$y = \lambda^2(r^2 + q^2), \qquad z = \lambda^3 r(r^2 + q^2), \qquad t = \lambda^3 q(r^2 + q^2).$$

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SQUARE ROOTS IN LOCALLY EUCLIDEAN GROUPS

A. M. GLEASON

A possible attack on the fifth problem of Hilbert is to demonstrate the existence of one-parameter subgroups in any locally Euclidean group.[1] It is known that, provided there are no "small" subgroups, some one-parameter subgroups exist. One would like to prove, however, that in a suitable neighborhood of the identity, every element is on one and only one one-parameter subgroup. If this is true, it is possible to extract square roots (that is, solve $x^2=a$ for given a) uniquely in this neighborhood, and the sequence of successive square roots a, $a^{1/2}$, $(a^{1/2})^{1/2}$, $\cdots$ converges to the identity. Conversely, it is easily seen that, if unique square roots exist, and if the sequence of square roots converge to the identity, then the one-parameter subgroups can be found. In this paper we give a new proof[2] that square roots exist in a suitable neighborhood of the identity and show, in addition, that either they are unique or small subgroups exist.

Throughout this paper we shall deal only with locally Euclidean topological groups of dimension n; consequently we may speak of an n-cell neighborhood of a point p, meaning a homeomorphic image of a Euclidean n-simplex containing the point p in its interior.

The proofs of Theorems 1 and 2 use the group property so sparingly that they can easily be restated as theorems on the involutions of a manifold.

THEOREM 1. *In a locally Euclidean group there exists a neighborhood*

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[1] The arguments outlined here are those of B. von Kerekjarto in his paper *Geometrische Theorie der zweigliedrigen kontinuierlichen Gruppen*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 107–114.

[2] Cf. B. de Kerekjarto, *Sur l'existence de racines carrées dans les groupes continus*, C. R. Acad. Sci. Paris vol. 193 (1931) pp. 1384–1385.

of the identity containing no elements of order two.[3]

PROOF. Let C_n be an n-cell neighborhood of the identity, e. We can easily construct a sequence, $C_{n-1}, \cdots, C_0$ (the subscripts are merely indices and do not refer to dimension), of n-cell neighborhoods of e such that $C_k^2 \subset C_{k+1}$ $(k=0, 1, \cdots, n-1)$. We claim that there are no elements of order two in C_0.

Suppose, on the contrary, that there is an element $a \in C_0$ such that $a \neq e$ and $a^2 = e$. We shall construct by induction a singular n-sphere in the cell C_n (that is, a continuous function f mapping the unit sphere S^n of $n+1$ dimensional space into the cell C_n) satisfying the relation

$$T \circ f = f \circ A, \tag{1}$$

where T is the group translation $T(x) = ax$, and A is the antipodal map of S^n into itself. S^n contains in a natural fashion the sequence of spheres $S^0, S^1, \cdots, S^{n-1}, S^n$, each being the "equator" of the next. Define f on S^0 by letting e and a be the images of the two points of S^0. Evidently (1) holds and $f(S^0) \subset C_0$. Suppose that $k<n$, and that f is defined on S^k and $f(S^k) \subset C_k$. Since C_k is aspherical in all dimensions we can extend f over one hemisphere of S^{k+1}, the image being in C_k. Then we may use the relation (1) to define f over the other hemisphere and, since $T(C_k) \subset C_{k+1}$, we shall have $f(S^{k+1}) \subset C_{k+1}$. Thus we obtain the desired singular n-sphere. However, the Ulam-Borsuk theorem asserts, for any singular n-sphere in an n-cell, that there is a point x for which $f(x) = f \circ A(x)$. By (1) we have $T \circ f(x) = f(x)$, a contradiction since T has no fixed points.

THEOREM 2. *If C is a Euclidean n-cell containing the identity, e, then there exists a singular $(n-1)$-sphere f in $C - \{e\}$ such that*

$$I \circ f = f \circ A, \tag{2}$$

where I is the inversive mapping, $I(x) = x^{-1}$. Moreover, if f is any such singular $(n-1)$-sphere, and if $C \cup C^{-1} \subset D$, where D is an n-cell containing no elements of order two, then f is an essential singular sphere in $C - \{e\}$.

PROOF. We proceed as in Theorem 1, choosing first the n-cells $C_{n-1} = C, C_{n-2}, \cdots, C_0$ where each is a neighborhood of the identity and $C_k \cup C_k^{-1} \subset C_{k+1} C_k^{-1} \subset C_{k+1}$. We can define f on S^0 so that (2) holds and $f(S^0) \subset C_0 - \{e\}$. Since the sets $C_k - \{e\}$ are aspherical in the dimensions $0, 1, \cdots, n-2$ the inductive extension of f can be carried through $n-1$ steps, using (2) in place of (1), to give the required singular $(n-1)$-sphere.

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[3] This theorem was first given by M. H. A. Newman, *A theorem on periodic transformations of spaces*, Quart. J. Math. Oxford Ser. vol. 2 (1931) pp. 1–8.

Now suppose that the cell D exists and that f is a singular $(n-1)$-sphere in $C-\{e\}$ satisfying (2). If f is not essential, extend f over a hemisphere of S^n and complete the extension over S^n by the relation (2). Then $e\notin f(S^n)\subset D$ and, the latter being an n-cell, the Ulam-Borsuk theorem tells us that, for some x, $f(x)=f\circ A(x)$. By (2) we have $f(x)=[f(x)]^{-1}$, but D contains no elements of order two and $f(x)\neq e$; we have found a contradiction.

Corollary. *The degree of the inversive mapping about the identity is* $(-1)^n$.

Proof. With f as in the theorem above we have $\partial I\times\partial f=\partial(I\circ f)=\partial(f\circ A)=\partial f\times\partial A$, and we know that $\partial f\neq 0$. Hence $\partial I=\partial A=(-1)^n$.

Theorem 3. *If N is a given neighborhood of the identity, there is a neighborhood M of the identity such that every element of M has a square root in N.*

Proof. We can choose n-cell neighborhoods of the identity, B, C, and D, such that $D\subset N$, D contains no elements of order two, $C\cup C^{-1}\subset D$, and $B^2\subset C$. By Theorem 2 we can construct a singular $(n-1)$-sphere f in $B-\{e\}$ satisfying (2). Let Q be the quadratic mapping $Q(x)=x^2$. Since I and Q permute, $I\circ Q\circ f=Q\circ I\circ f=Q\circ f\circ A$. Hence $Q\circ f$ is a singular $(n-1)$-sphere in $C-\{e\}$ satisfying (2). Applying the second part of Theorem 2, $Q\circ f$ is essential in $C-\{e\}$ and this implies that Q is an essential mapping of $B-\{e\}$ into $C-\{e\}$. Hence every point of a sufficiently small neighborhood M of the identity is the image under Q of some point of B; that is, every point of M has a square root in $B\subset N$.

Corollary. *The same theorem is true for roots of any order.*

Definition. A topological group is said to have small subgroups if every neighborhood of the identity contains a nontrivial subgroup (that is, a subgroup with more than one element).

Theorem 4. *If a locally Euclidean group does not have small subgroups, then there are neighborhoods M and N of the identity such that every element of M has a unique square root in N.*

Proof. Let C and N be n-cell neighborhoods of the identity such that C contains no nontrivial subgroups, and $NN^{-1}N\subset C$. Since $C-\operatorname{Int} N$ is compact and contains no elements of order two, its image under Q is closed and does not contain e. With the aid of

Theorem 3 we can choose a neighborhood M of the identity such that $M\cap Q(C-\operatorname{Int} N)$ is void and such that every element of M has at least one square root in N. The construction shows that, *if $a\in M$, any square root of a lying in C is actually in N.*

Suppose that $x\in N$ and $y\in N$ are distinct square roots of $a\in M$. It is clear (without using the hypothesis about small subgroups) that x and y do not commute, for then $z=xy^{-1}$ would be an element of order two. We shall show that all powers of z are in C. Since $x^2=y^2$, $y^{-1}x\cdot xy^{-1}=y^{-1}y^2y^{-1}=e$, hence $z^{-1}=y^{-1}x$, and $zx=xz^{-1}$. Then $z^mx\cdot z^mx=xz^{-m}z^mx=a$; that is, all the elements z^mx are square roots of a. We shall show by induction that all of these elements are in N. If $m>0$ and $z^{m-1}x\in N$, $z^mx=(z^{m-1}x)y^{-1}x\in NN^{-1}N\subset C$, hence $z^mx\in N$. Then $z^{\pm m}\in NN^{-1}\subset C$ for all m, and we have found a nontrivial subgroup entirely contained in C, a contradiction.

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NOTE ON A THEOREM OF DICKSON

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Though the problems concerning perfect and multiply perfect numbers are among the oldest of number theory, very little progress has been made in the investigation of these numbers. A k-ply perfect number is one for which $\sigma(n)=kn$ where $\sigma(n)$ is the sum of the divisors of n. The case $k=2$ is that of the perfect numbers. Though the form of the even perfect numbers is completely determined [1],[1] the question of whether or not there exists any odd perfect numbers is still a complete mystery. Sylvester [2] has shown that an odd perfect number must have at least five distinct prime factors, and Dickson [3] has proved that there are at most a finite number of odd perfect numbers having any given number of distinct prime factors. More generally, defining "primitive non-deficient" numbers to be those integers n for which

$$\sigma(n)/n \geqq 2 \tag{1}$$

and such that (1) does not hold for any proper divisor of n, Dickson showed that there are at most a finite number of odd primitive non-deficient numbers having a given number of distinct prime factors. In this note we shall give a simpler proof of Dickson's theorem; in fact prove a more general theorem which includes Dickson's as a special case.

DEFINITION 1. An integer n shall be called k-non-deficient (k any positive real number) if

$$\sigma(n)/n \geqq k \tag{2}$$

and k-deficient otherwise.

DEFINITION 2. An integer n shall be called primitive k-non-deficient if n is k-non-deficient, and all proper divisors of n are k-deficient.

Our generalization of Dickson's theorem may then be stated as:

THEOREM 1. *There are at most a finite number of primitive k-non-deficient numbers n such that*

(a) *if k is rational, n is relatively prime to the numerator of k; and*
(b) *the number of distinct prime factors of n is fixed.*

PROOF. We assume that there are an infinite number of such primi-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

tive k-non-deficient numbers which have s distinct prime factors. Writing each of these in the form

$$n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}, \qquad p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_s^{\alpha_s},$$

we note that at least $p_s^{\alpha_s} \to \infty$ over this sequence. Hence from these n can extract an infinite subsequence n_j such that

$$n_j = p_1^{\alpha_1} \cdots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1,j}} \cdots p_t^{\alpha_{tj}} p_{t+1,j}^{\alpha_{t+1,j}} \cdots p_{sj}^{\alpha_{sj}}$$

where

(I) $p_i^{\alpha_i}$ are fixed independently of j for $i=1, \cdots, r$;

(II) p_i are fixed independently of j, and

$$\lim_{j\to\infty} \alpha_{ij} = \infty$$

for $i=r+1, \cdots, t$; and

(III) $$\lim_{j\to\infty} p_{ij} = \infty$$

for $i=t+1, \cdots, s$.

It is to be noted that any one of these sets may actually be empty, and all that we know a priori is that at least one of the two groups (II), (III) is not empty.

Now first we shall show that the group (III) cannot be empty. For if it were, the primes dividing each n_j are the same as those dividing n_0. Thus by taking j sufficiently large we obtain $\alpha_{ij} \geqq \alpha_{i0}$, $i=1, \cdots, s$, where the strict inequality holds for at least one i; so that n_0 is a proper divisor of n_j. But this contradicts the assumption that both n_0 and n_j are primitive k-non-deficient.

Since (III) is non-empty, and the n_j primitive k-non-deficient we have

$$\prod_{(\mathrm{I})} \prod_{(\mathrm{II})} \prod_{(\mathrm{III})} \frac{p_{ij}^{\alpha_{ij}+1} - 1}{(p_{ij} - 1)p_{ij}^{\alpha_{ij}}} \geqq k > \prod_{(\mathrm{I})} \prod_{(\mathrm{II})} \prod_{i=t+1}^{s-1} \frac{p_{ij}^{\alpha_{ij}+1} - 1}{(p_{ij} - 1)p_{ij}^{\alpha_{ij}}}.$$

Then taking the limit as $j \to \infty$, we get

$$\prod_{(\mathrm{I})} \frac{p_i^{\alpha_i+1} - 1}{(p_i - 1)p_i^{\alpha_i}} \prod_{(\mathrm{II})} \frac{p_i}{p_i - 1} \geqq k \geqq \prod_{(\mathrm{I})} \frac{p_i^{\alpha_i+1} - 1}{(p_i - 1)p_i^{\alpha_i}} \prod_{(\mathrm{II})} \frac{p_i}{p_i - 1}$$

which in turn yields

$$\prod_{(\mathrm{I})} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \prod_{(\mathrm{II})} p_i = k \prod_{(\mathrm{I})} p_i^{\alpha_i} \prod_{(\mathrm{II})} (p_i - 1). \tag{3}$$

If k is irrational, this gives an immediate contradiction. If k is rational and group (II) is not empty, the fact that $\prod_{(\mathrm{II})} p_i$ is relatively prime to the numerator of k implies that $\prod_{(\mathrm{II})} p_i$ divides $\prod_{(\mathrm{II})} (p_i-1)$. This is clearly impossible. On the other hand, if (II) is empty (3) becomes

$$\prod_{(\mathrm{I})} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = k \prod_{(\mathrm{I})} p_i^{\alpha_i}.$$

But this in turn is also impossible since $\prod_{(\mathrm{I})} p_i^{\alpha_i}$ is a proper divisor of n_0 and must be k-deficient. This contradiction completes the proof of the theorem.

By an entirely analogous argument the following generalization of another theorem of Dickson [4] may be obtained.

THEOREM 2. *For k rational, there are at most a finite number of primitive k-non-deficient integers n such that*

(a) *$p \mid$ (numerator of k), and $p^{\alpha} \| n$ implies $\alpha < C$, a fixed constant*; *and*

(b) *the number of distinct prime factors of n is fixed.*

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COMBINATORIAL HOMOTOPY. II

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1. Introduction. This paper is concerned with problems of realizability, which were discussed in (I) (i.e. [1]).[1] In studying the realizability of chain mappings we use the system of relative homotopy groups, $\rho_n=\pi_n(K^n, K^{n-1})$, of a (connected) complex, K, where $n\geqq 1$ and $\rho_1=\pi_1=(K^1)$. The chain groups are defined as the relative homology groups, $C_n=H_n(\tilde{K}^n, \tilde{K}^{n-1})$, where $\tilde{K}$ is a universal covering complex of K. The groups ρ_n appear to be more useful than C_n in problems concerning geometrical realizability. On the other hand the chain groups are convenient in studying concrete problems. Moreover they provide a means of applying Theorem 3 in (I). A large part of the paper deals with the relations between the two systems of groups.

The system of relative homotopy groups, $\{\rho_n\}$, is a "group system," as defined in [8]. It is a special kind of group system because each group is "free" in one of three different senses. More precisely, ρ_1 is a free group, ρ_n is a free $\pi_1(K)$-module if $n>2$ and ρ_2 is what we call a free crossed module. These conditions of freedom are important in realizability problems. We tentatively describe $\{\rho_n\}$ as a *homotopy system.* It is essentially the same as what was called a "natural system" on p. 1216 of [3], redefined in terms of relative homotopy groups and free crossed modules.

2. Crossed modules. It will be convenient to have a name for groups with the algebraic properties of relative homotopy groups, and to have proved some lemmas concerning them. We shall call such a group a *crossed module*, or, more explicitly, a *crossed γ-module* or a *crossed (γ, d)-module.*[2] By this we mean an additive, but not necessarily commutative, group, ρ, which is related as follows to a multiplicative group γ:

(2.1) (a) *ρ admits γ as a group of operators.*[3]

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Anne Cobbe has pointed out to me that a crossed (γ, d)-module determines a Q-kernel, with $Q=\gamma/d\rho$ (see [10]), and that any Q-kernel has a representation as a crossed module. I learn from Saunders MacLane that crossed modules, under a different name, are defined in a forthcoming sequel to [10].

[3] I.e. to each $x\in\gamma$ corresponds an automorphism, $x:\rho\rightarrow\rho$, such that $x'(xa)=(x'x)a$ and $xa=a$ if $x=1$ ($x, x'\subset\gamma$, $a\in\rho$). We allow trivial operators (i.e. elements $x\in\gamma$ such that $xa=a$ for every $a\in\rho$) other than $1\in\gamma$.

(b) *There is a homomorphism,* $d:\rho\rightarrow\gamma$, *such that* $d(xa)=x(da)x^{-1}$ $(a\in\rho,\ x\in\gamma)$.

(c) $a+b-a=(da)b$ *for every pair,* a, $b\subset\rho$.

Notice that any group, ρ, is a crossed (γ, d)-module, with a suitable choice of γ and d. For example we may take γ to be the group of inner automorphisms of ρ, and d to be defined by (2.1c). Notice also that any Abelian group, ρ, which admits γ as a group of operators is a crossed (γ, d)-module with $d\rho=1$.

Let C be the group ρ made Abelian and let h be the natural homomorphism $h:\rho\rightarrow C$. Then $h^{-1}(0)$ is generated by the commutators, $a+b-a-b$, or, using (2.1c), $(da)b-b$. Therefore C may also be regarded as ρ with the operators in $d\rho$ "neutralized." It follows from (2.1b) that $d\rho$ is an invariant sub-group of γ. Let $\bar{\gamma}=\gamma/d\rho$ and let $\bar{x}\in\bar{\gamma}$ be the co-set containing a given element $x\in\gamma$. Given $ha\in C$, we define $\bar{x}(ha)=h(xa)$. The transformation $\bar{x}: C\rightarrow C$, thus defined, is single-valued since the commutator sub-group of ρ is transformed into itself by x (and indeed by any automorphism). It is obvious that $\bar{x}: C\rightarrow C$ is an endomorphism; also that $1ha=ha$ and $\bar{x}(\bar{x}'ha)=(\bar{x}\bar{x}')ha$. Therefore C is a $\bar{\gamma}$-module, i.e. an Abelian group which admits $\bar{\gamma}$ as a group of operators. Moreover h is an operator homomorphism, in the sense that $h(xa)=\bar{x}ha$.

It follows from (2.1c) that $(da)b=b$ if b is in the center of ρ. That is to say the elements in $d\rho$ operate trivially on the centre of ρ, which may therefore be regarded as a $\bar{\gamma}$-module. It also follows from (2.1c), with $da=1$, that $d^{-1}(1)$ is in the centre of ρ. Moreover it follows from (2.1b) that $d(xa)=1$ if $da=1$, whence $d^{-1}(1)$ is transformed into itself by the operators in γ. Therefore $d^{-1}(1)$ may also be regarded as a $\bar{\gamma}$-module.

Since $d^{-1}(1)$ is in the centre of ρ the group ρ is a central extension of $d^{-1}(1)$ by $d\rho$. If it is also a trivial extension, i.e. if d has a right inverse, $\theta:d\rho\rightarrow\rho$ $(d\theta=1)$, it follows that ρ is the direct sum $\rho=d^{-1}(1)+\theta d\rho$. Since h makes ρ Abelian and since $d^{-1}(1)$ is already Abelian it follows that $h|d^{-1}(1)$ is then an isomorphism (into). Therefore we have:

LEMMA 1. *If* ρ *is a trivial extension of* $d^{-1}(1)$ *by* $d\rho$, *then* $da=1$ *and* $ha=0$ *together imply* $a=0$.

In particular $h|d^{-1}(1)$ is an isomorphism (into) if $d\rho$ is a free group, since any extension by a free group is trivial. Therefore $h|d^{-1}(1)$ is an isomorphism (into) if γ is a free group.

Let ρ' be a crossed (γ', d')-module and let $g:\gamma\rightarrow\gamma'$ be a given homomorphism. By an *operator homomorphism,* $f:\rho\rightarrow\rho'$, *associated with* g

we mean a homomorphism such that

$$(2.2)\qquad \begin{aligned} &\text{(a)}\quad d'f = gd, \\ &\text{(b)}\quad f(xb) = (gx)fb \qquad\qquad (x \in \gamma,\ b \in \rho). \end{aligned}$$

We shall describe f as an *operator isomorphism*[4] if, and only if, it and g are both isomorphisms. Notice that this definition applies to ordinary modules, with $d\rho = 1$, $d'\rho' = 1$, in which case (2.2b) is the operative condition. Notice also that, if there is no $x' \in \gamma'$ such that $x'fb = fb$ for every $b \in \rho$, then it follows from (2.2b) that the same operator homomorphism, $\rho \rightarrow \rho'$, cannot be associated with each of two different homomorphisms $\gamma \rightarrow \gamma'$. This is the case, for example, if ρ' is a free γ'-module and $f\rho$ contains a basis element of ρ'.

We now define what we call a *free crossed γ-module.*[5] Let $\{\alpha_i\}$ be any indexed[6] aggregate, let γ be a given group and let an arbitrary element $x_i \in \gamma$ be associated with each α_i. We define an additive group, ρ, by means of "symbolic" generators and relations. The generators shall be all triples $(+, x, \alpha_i)$, where $x \in \gamma$, together with their "negatives" $(-, x, \alpha_i)$, We write $(+, x, \alpha_i)$ as (x, α_i) and $(-, x, \alpha_i)$ as $-(x, \alpha_i)$. The relations shall be

$$(2.3)\qquad (x, \alpha_i) + (y, \alpha_j) - (x, \alpha_i) = (xx_ix^{-1}y, \alpha_j),$$

for all pairs $x, y \subset \gamma$ and all values of i, j, together with the "trivial" relations, $(x, \alpha_i) - (x, \alpha_i) = 0$ and $-(x, \alpha_i) + (x, \alpha_i) = 0$.

Since

$$\begin{aligned} (zx, \alpha_i) + (zy, \alpha_j) - (zx, \alpha_i) &= (zxx_ix^{-1}z^{-1}zy, \alpha_j) \\ &= (zxx_ix^{-1}y, \alpha_j) \end{aligned}$$

the transformation $(x, \alpha_i) \rightarrow (zx, \alpha_i)$, of generators, determines an endomorphism, $z: \rho \rightarrow \rho$ $(z \in \gamma)$. Obviously $za = a$ if $z = 1$, $z'(za) = (z'z)a$, where $a \in \rho$. Therefore ρ admits γ as a group of operators. Similarly a homomorphism $d: \rho \rightarrow \gamma$, which satisfies (see [2, p. 421]) (2.1b), is defined by $(x, \alpha_i) \rightarrow xx_ix^{-1}$. Then (2.1c) follows (see [2, p. 421]) from (2.3) and the trivial relations. Therefore ρ is a crossed (γ, d)-module.

Let $a_i \in \rho$ be the element which corresponds to the generator $(1, \alpha_i)$. We shall call the set of elements $\{a_i\}$ a *basis* for ρ. Any element in

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[4] An isomorphism, without qualification, will always mean an isomorphism onto.

[5] This is a group of the sort defined on p. 421 of [2]. A correspondence with Saunders MacLane has helped to clarify the presentation.

[6] The set of values taken by i may have any cardinal number. The elements α_i are not elements of the group, ρ, which is to be defined. An element of ρ is an equivalence class of "words," written as sums, of the "symbolic" generators. $\alpha_i \neq \alpha_j$ if $i \neq j$.

ρ is of the form

$$\epsilon_1 x_1 a_{i_1} + \cdots + \epsilon_n x_n a_{i_n}, \tag{2.4}$$

where $\epsilon_i = \pm 1$, $x_i \in \gamma$, and repetitions are allowed.

We shall describe a crossed module, ρ', as *free* if, and only if, it is operator isomorphic to a crossed module, ρ, of the sort just defined. This being so, a set of elements will be called a *basis* for ρ' if, and only if, it is the image of $\{a_i\}$ in an operator isomorphism $\rho \to \rho'$. In general there will be many operator isomorphisms, $\rho \to \rho'$, and hence many bases for ρ'.

We now prove a lemma, which is fundamental in realizability problems. Let ρ be a free crossed (γ, d)-module and ρ' an arbitrary crossed (γ', d')-module. Let $g: \gamma \to \gamma'$ be an arbitrary homomorphism and let $\{a_i\}$ be a basis for ρ. With each basis element a_i we associate an element $b_i' \in \rho'$, which satisfies the condition $d'b_i' = gda_i$, but is otherwise arbitrary.

LEMMA 2. *There is a unique operator homomorphism, $f: \rho \to \rho'$, associated with g, such that $fa_i = b_i'$ for each value of i.*

We assume, as we obviously may, that ρ is defined, as above, in terms of symbolic generators (x, α_i) and that a_i is the element corresponding to $(1, \alpha_i)$. Then $d'b_i' = gda_i = gx_i$ and it follows from (2.1c), in ρ', that

$$\begin{aligned}(gx)b_i' + (gy)b_j' - (gx)b_i' &= (gx)d'b_i'(gx)^{-1}(gy)b_j' \\ &= g(xx_ix^{-1}y)b_j'.\end{aligned}$$

Therefore the transformation $(x, \alpha_i) \to (gx)b_i'$, of generators, determines a homomorphism, $f: \rho \to \rho'$, which obviously satisfies (2.2b). Also $fa_i = b_i'$ and $d'fa_i = gda_i$. Therefore it follows from (2.1b) that

$$\begin{aligned}d'fxa_i = d'\{(gx)fa_i\} &= (gx)(d'fa_i)(gx)^{-1} \\ = g\{x(da_i)x^{-1}\} &= gdxa_i.\end{aligned}$$

Since ρ is generated by $\{xa_i\}$ it follows that (2.2a) is satisfied. Similarly f is uniquely determined by the condition $fa_i = b_i'$, and the lemma is proved.

We conclude the section with two "extreme" examples of a free crossed module, ρ. First let $d\rho = 1$. Then (2.3) merely expresses the fact that ρ is commutative, and ρ will be called a *free γ-module*. In this case we can collect all the terms $\epsilon_\lambda x_\lambda a_{i_\lambda}$ in (2.4), which have the same i_λ, and write (2.4) as

$$r_1 a_{j_1} + \cdots + r_k a_{j_k},$$

where $j_1, \cdots, j_k$ are distinct and $r_1, \cdots, r_k$ are in the group ring, R, of γ. This expression is unique since, obviously, $\{xa_\iota\}$ are free generators of ρ. Therefore ρ is a free R-module, having $\{a_\iota\}$ as a basis. In any case $h\rho$ is obviously a free $\bar{\gamma}$-module, having $\{ha_\iota\}$ as a basis, where $h:\rho\rightarrow h\rho$ makes ρ Abelian. Notice that, in consequence of this fact, there is no relation of the form $m_1a_{i_1}+m_2a_{i_2}+ \cdots =0$, in ρ, where $m_1, m_2, \cdots$ are nonzero integers and $i_1, i_2, \cdots$ are distinct.

A free Abelian group,[7] which admits γ as a group of operators, is not necessarily a free γ-module. Therefore a crossed (γ, d)-module is not necessarily free just because (2.1c) is a complete set of relations between a set of generating elements $a, b, \cdots$.

Secondly let γ be a free group and, in the original definition of a free crossed module, let $\{x_i\}$ be a set of free generating elements of γ (inverses not included). Let $x=x_{i_1}^{e_1} \cdots x_{i_n}^{e_n}$ $(e_\lambda = \pm 1)$. Then it follows from (2.3), and their "negatives," which are consequences of (2.3) and the trivial relations, that

$$(xx_i, \alpha_j) = (x, \alpha_i) + (x, \alpha_j) - (x, \alpha_i),$$
$$(xx_i^{-1}, \alpha_j) = -(x, \alpha_i) + (x, \alpha_j) + (x, \alpha_i).$$

Therefore it follows by induction on n that ρ is generated by $\{a_\iota\}$, without the help of the operators in γ. Therefore the right inverse, $d^{-1}:\gamma\rightarrow\rho$, of d, which is given by $d^{-1}x_i=a_i$, is onto ρ. Therefore $d:\rho\approx\gamma$. Thus ρ is an ordinary free group, which is freely generated by $\{a_\iota\}$. Also γ written additively and operating on itself according to the rule $xy=x+y-x$ $(x, y\subset\gamma)$ is a free $(\gamma, 1)$-module. A set of free generating elements will be called a *basis* for γ, whether the multiplicative or the additive notation is used.

3. A lemma on crossed homomorphisms. Let G be an additive group, which need not be Abelian, and let γ be a multiplicative group, which operates on G. A crossed homomorphism, $h:\gamma\rightarrow G$, is a map such that

$$h(xy) = hx + xhy \qquad (x, y \subset \gamma). \tag{3.1}$$

We shall generally be presented with a group G, a group, γ', which operates on G, and a homomorphism $f:\gamma\rightarrow\gamma'$. In this case a crossed homomorphism, $h:\gamma\rightarrow G$, will mean the same as before, with $xa=(fx)a$ $(x\in\gamma, a\in G)$, and h will be called a crossed homomorphism *associated with* f. Notice that, writing $x=1$ in (3.1), we have $h(1)=0$ and writing $x=y^{-1}$ we have $hy^{-1} = -y^{-1}hy$.

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[7] E.g. one in which γ operates trivially, or a left ideal in the group ring of γ.

As an example, which occurs in §10 below, let $f, f':\gamma\to G$ be two homomorphisms and let us write $ab=a+b-a$ $(a, b\subset G)$. Let $\theta:G\to G$ be a fixed endomorphism and let $h:\gamma\to G$ be given by

$$hx = \theta f'x - fx.$$

Then

$$\begin{aligned} h(xy) &= \theta f'(xy) - f(xy) \\ &= \theta f'x + \theta f'y - fy - fx \\ &= (\theta f'x - fx) + fx + (\theta f'y - fy) - fx \\ &= hx + (fx)hy. \end{aligned}$$

Therefore h is a crossed homomorphism associated with f.

Let γ be a free group, which operates on G, and let $\{x_i\}$ be a basis for γ. With each x_i we associate an arbitrary element $a_i\in G$.

LEMMA 3. *There is a unique crossed homomorphism, $h:\gamma\to G$, such that $hx_i=a_i$ for each value of i.*

Any element $y\in\gamma$ is uniquely represented as a product

$$y = x_{i_1}^{\epsilon_1}\cdots x_{i_n}^{\epsilon_n} \qquad (\epsilon_i = \pm 1),$$

which is reduced, in the sense that $\epsilon_{j+1}=\epsilon_j$ if $i_{j+1}=i_j$ (i.e. no cancellation is possible). We define $h:\gamma\to G$ by

$$hy = \epsilon_1 y_1 a_{i_1} + \cdots + \epsilon_n y_n a_{i_n}, \tag{3.2}$$

where $y_1=1$ if $\epsilon_1=1$ and

$$\begin{aligned} y_j &= x_{i}^{\epsilon_1}\cdots x_{i_{j-1}}^{\epsilon_{j-1}} && \text{if} \quad \epsilon_j = +1 \\ &= x_{i_1}^{\epsilon_1}\cdots x_{i_j}^{\epsilon_j} && \text{if} \quad \epsilon_j = -1. \end{aligned} \tag{3.3}$$

Then $hx_i=a_i$, $hx_i^{-1}=-x_i^{-1}a_i$. It follows from (3.2) and (3.3) that

$$\begin{aligned} h(zx_i^{\epsilon}) &= hz + za_i && \text{if} \quad \epsilon = +1 \\ &= hz - zx_i^{-1}a_i && \text{if} \quad \epsilon = -1 \\ &= hz + zhx_i^{\epsilon} && \text{if} \quad \epsilon = \pm 1, \end{aligned}$$

for any $z\in\gamma$. Hence, assuming that $h(xy)=hx+xhy$, for a given $x\in\gamma$, we have

$$\begin{aligned} h(xyx_i^{\epsilon}) &= h(xy) + xyhx_i^{\epsilon} \\ &= hx + x(hy + yhx_i^{\epsilon}) \\ &= hx + xh(yx_i^{\epsilon}). \end{aligned}$$

It follows from induction on n that (3.1) is satisfied. Therefore h is a crossed homomorphism.

Let $h:\gamma \rightarrow G$ be any crossed homomorphism such that $hx_i = a_i$. Then it follows from (3.1) and induction on n that h is given by (3.2) and (3.3). Therefore h is unique and the lemma is proved.

4. **Homotopy systems.** Let $\rho = \{\rho_n\}$ $(n = 1, 2, \cdots)$ be a family of groups, together with a "boundary operator," d, which is a family of homomorphisms, $d_r:\rho_r \rightarrow \rho_{r-1}$ $(r \geqq 2)$, such that $dd = 0$ (i.e. $d_r d_{r+1}\rho_{r+1}$ is the neutral element in ρ_{r-1}). The group ρ_n shall be multiplicative if $n = 1$ and additive if $n > 1$, though possibly noncommutative if $n = 2$. It follows from (4.1b) below and (2.1b) that $d\rho_2$ is an invariant subgroup of ρ_1. We write $\bar{\rho}_1 = \rho_1/d\rho_2$, and $\bar{x} \in \bar{\rho}_1$ will always denote the co-set containing a given element $x \in \rho_1$. The system ρ and the operator d shall satisfy the conditions:

(4.1) (a) *ρ_1 is a free group,*

(b) *ρ_2 is a free crossed (ρ_1, d_2)-module,*

(c) *ρ_n $(n > 2)$ is a free $\bar{\rho}_1$-module,*

(d) *d_r is an operator homomorphism for each $r = 2, 3, \cdots$. That is to say, d_2 satisfies* (2.1b), *as required in the definition of a crossed module, and*[8] *$d_n\bar{x} = \bar{x}d_n$ $(n > 2)$, for each operator $\bar{x} \in \bar{\rho}_1$.*

We describe such a system as a *homotopy system.* If $\rho_r = 0$ for $r > n \geqq 0$ but $\rho_n \neq 0$, in case $n > 0$, we shall describe ρ as ***n*-dimensional** and shall write $n = \dim \rho$ (if $r = 1$ then $\rho_r = 0$ means $\rho_1 = 1$).

For the sake of uniformity it will be convenient to let ρ_1 operate on ρ_n $(n > 2)$ according to the rule $xa = \bar{x}a$, where $a \in \rho_n$, $x \in \rho_1$. Also, in order to include the case $n = 1$ in such formulae as (4.3) below, we shall sometimes think of ρ_1 as an additive group, which operates on itself according to the rule $xy = x + y - x$. With these conventions (4.1d) reduces to $dx = xd$. When considered as a group of operators ρ_1 will always be multiplicative.

Let $\rho = \{\rho_n\}$ and $\rho' = \{\rho_n'\}$ be two homotopy systems and let $f_1:\rho_1 \rightarrow \rho_1'$ be a homomorphism such that $f_1 d_2\rho_2 \subset d_2\rho_2'$, where d denotes the boundary operator in both ρ and ρ'. Then f_1 induces a homomorphism, $\bar{f}_1:\bar{\rho}_1 \rightarrow \bar{\rho}_1'$, which is given by $\bar{f}_1\bar{x} = \overline{f_1 x}$. Let $f_n:\rho_n \rightarrow \rho_n'$ $(n \geqq 2)$ be an operator homomorphism, which is associated with f_1 if $n = 2$ and with $\bar{f}_1$ if $n > 2$. Then, with the convention explained in the preceding paragraph, we may describe f_n as associated with f_1 for any $n \geqq 2$. Notice that $f_1 d_2\rho_2 \subset d_2\rho_2'$ if there is a homomorphism, $f_2:\rho_2 \rightarrow \rho_2'$, such that $f_1 d_2 = d_2 f_2$.

By a *homomorphism*, $f:\rho \rightarrow \rho'$, we shall mean a family of homomorphisms, $f_n:\rho_n\to\rho_n'$ $(n\geqq 1)$, such that

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[8] $d_2\bar{x} = \bar{x}d_2$ has a meaning since $d_3\rho_3 \in d_2^{-1}(1)$.

(a) $fd=df$ (i.e. $f_{n-1}d_n=d_nf_n$ *for each* $n\geqq 2$),

(b) f_n *is an operator homomorphism associated with* f_1 *if* $n\geqq 2$.

We shall describe f as an *isomorphism* if, and only if, f_n is an isomorphism for each $n=1, 2, \cdots$. We shall describe ρ as *isomorphic* to ρ', and shall write $f:\rho\approx\rho'$, if, and only if, there is an isomorphism $f:\rho\to\rho'$. It is obvious that the resultant of homomorphisms (isomorphisms), $f:\rho\to\rho'$ and $f':\rho'\to\rho''$, is a homomorphism (isomorphism), $f'f:\rho\to\rho''$; also that the inverse of an isomorphism is itself an isomorphism.

We now define a relation of equivalence, $\rho\equiv\rho'$, which is the algebraic analogue of "being of the same homotopy type" between complexes. *By a deformation operator*, $\xi:\rho\to\rho'$, *associated* with a homomorphism, $f_1:(\rho_1, d\rho_2)\to(\rho_1', d\rho_2')$, we shall mean a family of maps, $\xi_n:\rho_{n-1}\to\rho_n$ $(n\geqq 2)$, such that:

(a) $\xi_2:\rho_1\to\rho_2'$ *is a crossed homomorphism associated with* f_1,

(b) $\xi_n:\rho_{n-1}\to\rho_n'$ $(n>2)$ *is an operator homomorphism associated with* f_1.

We shall describe a homomorphism, $f:\rho\to\rho'$, as *equivalent* to a homomorphism, $g:\rho\to\rho'$, and shall write $f\simeq g$, if, and only if, there is a deformation operator, $\xi:\rho\to\rho'$, associated with f_1, and an element $w'\in\rho_1'$, such that

$$w'g_n - f_n = d_{n+1}\xi_{n+1} + \xi_n d_n \qquad (n\geqq 1), \tag{4.3}$$

where $\xi_1 d_1=0$. When $n=1$ this is another way of writing

$$w'(g_1x)w'^{-1}(f_1x)^{-1} = d_2\xi_2 x \qquad (x\in\rho_1). \tag{4.4}$$

We shall describe a homomorphism, $f:\rho\to\rho'$, as an *equivalence*, if, and only if, there is a homomorphism, $f':\rho'\to\rho$, such that $f'f\simeq 1$, $ff'\simeq 1$, where 1 denotes the identical isomorphism both in ρ and in ρ'. We shall describe ρ as *equivalent* to ρ', and shall write $\rho\equiv\rho'$, if, and only if, there is an equivalence, $f:\rho\to\rho'$.

THEOREM 1. *The relations*, $\simeq$, *between homomorphisms* $\rho\to\rho'$, *and* $\equiv$, *between homotopy systems, are equivalence relations.*

It is not difficult to prove this directly. However we shall deduce it from theorems proved in §§10, 11 below.

Let $f\simeq g:\rho\to\rho'$. Then it follows from (4.4) that

$$\bar f_1 = \alpha\bar g_1:\bar\rho_1\to\bar\rho',$$

where α is an inner automorphism of $\bar\rho_1'$. Let $f:\rho\to\rho'$ be an equivalence and let $f':\rho'\to\rho$ be such that $f'f\simeq 1$, $ff'\simeq 1$. Then $\bar f_1'\bar f_1=\alpha$, $\bar f_1\bar f_1' =\alpha'$, where α, α' are inner automorphisms of $\bar{p}_1$, $\bar{p}_1'$. Hence it follows that $\bar{f}_1$ is an isomorphism.

5. The homotopy system of a complex. By a complex we shall mean a connected, CW-complex, as defined in (I), which may be finite or infinite. The characteristic maps of the cells in a complex will now be maps $I^n \to \bar{e}^n$ (not $\sigma^n \to \bar{e}^n$), where I^n is the n-cube in Hilbert space, which is given by $0 \leqq t_1, \cdots, t_n \leqq 1$, $t_i = 0$ if $i > n \geqq 0$, with $I^0 = (0, 0, \cdots)$.

Our conventions concerning homotopy groups are as follows. We treat I^n as $I \times I^{n-1}$ $(n \geqq 1)$ and define

$$I_1^{n-1} = 1 \times I^{n-1}, \qquad E^{n-1} = \partial I^n - (I_1^{n-1} - \partial I_1^{n-1}).$$

Thus I_1^{n-1} is the face of I^n in which $t_1 = 1$. Let $k: I^{n-1} \to I_1^{n-1}$ be given by $kp = (1, p)$ $(p \in I^{n-1})$. Let spaces P, $P_0 \subset P$ and a point $p_0 \in P$ be given. Then an element $a \in \pi_r(P, P_0, p_0)$ $(r \geqq 2)$ shall be a homotopy class of maps of the form

$$\theta: (I^r, \partial I^r, E^{r-1}) \to (P, P_0, p_0).$$

Let

$$\beta_r: \pi_r(P, P_0, p_0) \to \pi_{r-1}(P_0, p_0, p_0) = \pi_{r-1}(P_0, p_0)$$

be the boundary homomorphism, which we define as follows. If $\theta \in a$, then $\beta_r a$ shall be the element defined by the map

$$\phi k: I^{r-1} \to P_0,$$

where $\phi = \theta \mid I_1^{r-1}: I_1^{r-1} \to P_0$.

Let

$$\delta_t: (I^r, \partial I^r, I^0) \to (I^r, \partial I^r, I^0)$$

be a fixed homotopy such that $\delta_0 = 1$, $\delta_1 E^{r-1} = I^0$. Then we say that a map

$$\phi: (I^r, \partial I^r, I^0) \to (P, P_0, p_0)$$

represents the element of $\pi_r(P, P_0, p_0)$, which is defined by the map $\phi\delta_1: I^r \to P$. Similarly a map

$$\psi: (\partial I^r, I^0) \to (P_0, p_0)$$

will be said to represent the element of $\pi_{r-1}(P_0, p_0)$, which is defined by the map

$$(\psi\delta_1 \mid I_1^{r-1}) k: I^{r-1} \to P_0.$$

A path

$$\lambda:(I, 0, 1) \to (P_0, p_0, p_0')$$

determines an isomorphism

$$\bar{\lambda}:\pi_r(P, P_0, p_0') \to \pi_r(P, P_0, p_0),$$

and also its inverse, $\bar{\lambda}^{-1}$. Let

$$\theta':(I^r, \partial I^r, I^0) \to (P, P_0, p_0') \tag{5.1}$$

be a map which represents an element $a' \in \pi_r(P, P_0, p_0')$. Then we describe $\bar{\lambda}a'$ as the element which is represented by the map θ', *joined to* p_0 *by the path* λ. If $p_0' = p_0$ we write $\bar{\lambda}a' = xa'$, where $x \in \pi_1(P_0, p_0) = \pi_1^0$, say, is the element represented by the path λ. When xa' is thus defined, $\pi_r(P, P_0, p_0)$ is a π_1^0-module, if $r > 2$, and a crossed (π_1^0, β_2)-module[9] if $r = 2$.

Let spaces Q, $Q_0 \subset Q$ and a point $q_0 \in Q_0$ be given. Then a map

$$\phi:(P, P_0, p_0) \to (Q, Q_0, q_0)$$

induces a homomorphism

$$f:\pi_r(P, P_0, p_0) \to \pi_r(Q, Q_0, q_0).$$

Let $a \in \pi_r(P, P_0, p_0)$ be represented by the map (5.1), joined to p_0 by the path λ. Then it is easily verified that fa is represented by the map $\phi\theta'$, joined to q_0 by the path $\phi\lambda$. Taking $p_0' = p_0$ it follows that f is an operator homomorphism associated with the homomorphism, $\pi_1(P_0, p_0) \to \pi_1(Q_0, q_0)$, which is induced[10] by ϕ.

Similar remarks apply to $\pi_{r-1}(P_0, p_0)$ and $\pi_{r-1}(P_0, p_0')$, related to each other by the path λ.

Let K be a given complex and let a 0-cell, $e^0 \in K^0$, be taken as base-point for all groups $\pi_n(K^n, K^{n-1})$, $\pi_n(K)$, etc. Let

$$\rho_r = \rho_r(K) = \pi_r(K^r, K^{r-1}) \qquad (r \geqq 2),$$

$$\rho_1 = \rho_1(K) = \pi_1(K^1).$$

Let

$$j_n:\pi_n(K^n) \to \rho_n \qquad (n \geqq 1;\ j_1 = 1)$$

be the homomorphism induced by the identical map $(K^n, e^0, e^0) \to (K^n, K^{n-1}, e^0)$ and let[11]

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[9] See [12, p. 486], which may be taken as a general reference for the elementary propositions stated here.

[10] I.e. induced by $\phi | P_0 : P_0 \to Q_0$.

[11] Cf. [8, p. 429], with $X_i = K^i$ $(i > 0)$.

$$d_r = j_{r-1}\beta_r : \rho_r \rightarrow \rho_{r-1}.$$

Then $d_2=\beta_2$. Since $\beta_r j_r=0$ we have $d_r d_{r+1}=0$. If $r \geqq 3$ the group ρ_r is a free $\pi_1(K^{r-1})$-module[12] and we identify[13] $\pi_1(K^n)$ $(n>1)$, as well as $\pi_1(K)$, with $\bar{\rho}_1=\rho_1/d_2\rho_2$. Also ρ_2 is a free crossed[12] (ρ_1, d_2)-module and d_r is an operator homomorphism in the sense of (4.1d). Therefore $\{\rho_n\}$ is a homotopy system, which we call the ***homotopy system***, $\rho(K)$, of K.

Let $\{e_i^n\}$ $(n \geqq 2)$ be the n-cells in K and let $a_i \in \rho_n$ be the element represented by a characteristic map for e_i^n, joined to e^0 by a path in K^{n-1}. Then $\{a_i\}$ is a basis, which we describe as a ***natural basis***, for ρ_n. Let $T \subset K^1$ be a tree (cf. [4, p. 322]) which is a sub-complex containing K^0, and let $\{e_i^1\}$ be the 1-cells in K^1-T. Let $g_i : I \rightarrow \bar{e}_i^1$ be a characteristic map for e_i^1 and let $x_i \in \rho_1$ be the element represented by a path in T, which leads from e^0 to $g_i(0)$, followed by g_i, followed by a path in T, which leads from $g_i(1)$ back to e^0. Then $\{x_i\}$ is a basis for ρ_1.

A (cellular) map,[14] $\phi : K \rightarrow L$, in a complex L, with base point $e'^0=\phi e^0$, obviously induces a homomorphism, $f:\rho(K) \rightarrow \rho(L)$. If $g:\rho(L) \rightarrow \rho(M)$ is the homomorphism induced by a map, $\psi : L \rightarrow M$, into a complex M, then $gf:\rho(K) \rightarrow \rho(M)$ is obviously the homomorphism induced by $\psi\phi : K \rightarrow M$. If $f:\rho(K) \rightarrow \rho(L)$ is a given homomorphism and if $\phi : K \rightarrow L$ is a map which induces it we shall describe ϕ as a ***realization*** of f.

Let $f:\rho(K) \rightarrow \rho(L)$ be a given homomorphism and let $K_0 \subset K$ be a connected sub-complex, which contains e^0. We shall describe a map, $\phi_0 : K_0 \rightarrow L$, as a ***partial realization*** of f if, and only if,

$$f^0 = fi : \rho(K_0) \rightarrow \rho(L),$$

where f^0 and $i:\rho(K_0) \rightarrow \rho(K)$ are the homomorphisms induced by ϕ_0 and by the identical map $K_0 \rightarrow K$. We shall call i the ***injection*** homomorphism.

Let $K_1 \subset K$ be a (connected) sub-complex, which contains[15] K^{n-1}

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[12] See §16 below.

[13] I.e. we use $\bar{\rho}_1$ and $\bar{x} \in \bar{\rho}_1$ to denote $\pi_1(K)$ and $i\bar{x} \in \pi_1(K)$, etc., where $i:\bar{\rho}_1 \rightarrow \pi_1(K)$ is the ismorphism in which corresponding elements are represented by the same map $I \rightarrow K^1$.

[14] It is to be understood that all our maps and homotopies are cellular (see L in §5 of (I)) and, except when the contrary is stated, that a given map, $K \rightarrow L$, carries the base point of K into the base point of L.

[15] The condition $K^{n-1} \subset K_1$ is not necessary to Lemma 4 below but it helps to simplify the notation.

($n\geqq 2$). Let $K_1=K_0\cup\{e_\sigma^n\}$, where $\{e_\sigma^n\}=K_1-K_0$ is a set of n-cells, each of which is a principal cell (i.e. a cell which is an open sub-set) of K_1. Let $\phi_0:K_0\to L$ be a partial realization of a given homomorphism, $f:\rho(K)\to\rho(L)$, and let $g:\pi_{n-1}(K^{n-1})\to\pi_{n-1}(L^{n-1})$ be the homomorphism which is induced by ϕ_0. Let $a_\sigma\in\rho_n(K)$ be the element which is represented by a characteristic map, $h_\sigma:I^n\to\bar{e}_\sigma^n$, for e_σ^n, joined to e^0 by a path

$$\lambda_\sigma:(I,0,1)\to(K^{n-1},e^0,p_\sigma)\qquad(p_\sigma=h_\sigma\dot{I}^0).$$

LEMMA 4. *If*[16] $\beta fa_\sigma=g\beta a_\sigma$, *for each* σ, *then* ϕ_0 *can be extended to a partial realization*, $\phi_1:K_1\to L$, *of* f.

This is essentially the same as Lemma 7 in §9 of [6], except that we must now pay attention to base points. Let $\beta fa_\sigma=g\beta a_\sigma$, let $q_\sigma=\phi_0p_\sigma$ and let

$$\mu_\sigma=\phi_0\lambda_\sigma:(I,0,1)\to(L^{n-1},e'^0,q_\sigma).$$

Let $\mu_\sigma':\rho_n(L)\to\pi_n(L^n,L^{n-1},q_\sigma)$ be the isomorphism determined by the path μ_σ and let

$$k_\sigma:(I^n,\partial I^n,I^0)\to(L^n,L^{n-1},q_\sigma)$$

be a map which represents the element $\mu_\sigma'fa_\sigma$. Then $k_\sigma|\partial I^n$ and $\phi_0h_\sigma|\partial I^n$ both represent the image of $\beta fa_\sigma=g\beta a_\sigma$ in the isomorphism,

$$\pi_{n-1}(L^{n-1},e'^0)\to\pi_{n-1}(L^{n-1},q_\sigma),$$

which is determined by the path μ_σ. Therefore it follows from the proof of Lemma 7 in [6], applied to each cell e_σ^n, that ϕ_0 can be extended to a map, $\phi_1:K_1\to L$, such that fa_σ is represented by the map $\phi_1h_\sigma:I^n\to L^n$, joined to e'^0 by the path μ_σ.

Let $f^r:\rho(K_r)\to\rho(L)$ $(r=0, 1)$ be the homomorphism induced by ϕ_r and let $a_\sigma^1\in\rho_n(K_1)$ be the element represented by h_σ, joined to e^0 by the path λ_σ. Then $a_\sigma=i^1a_\sigma^1$, where i^1 is the injection $i^1:\rho(K_1)\to\rho(K)$, and $f^1a_\sigma^1=fa_\sigma=fi^1a_\sigma^1$. Let i and i^0 be the injections $i:\rho(K_0)\to\rho(K_1)$ and $i^0=i^1i:\rho(K_0)\to\rho(K)$. Since $\phi_1|K_0=\phi_0$ it is obvious that $f^1i=f^0$. Also $f^0=fi^0=fi^1i$, since ϕ_0 is a partial realization of f. Therefore $f^1ia^0=fi^1ia^0$ if $a^0\in\rho_n(K_0)$. If $\{a_j^0\}$ is a natural basis for $\rho_n(K_0)$ then $\{ia_j^0, a_\sigma^1\}$ is a natural basis for $\rho_n(K_1)$. Since $f^1b=fi^1b$ if $b=ia_j^0$ or a_σ^1 and also (obviously) if $b\in\rho_r(K_1)$ with $r\neq n$ it follows that $f^1=fi^1$. Therefore ϕ_1 is a partial realization of f and the lemma is proved.

In general a homomorphism $f:\rho(K)\to\rho(L)$ cannot be realized by a map $K\to L$. For example a complex projective plane and the union of a 2-sphere and a 4-sphere with a single common point may be covered by complexes, K and L, each of which consists of a 0-cell, a 2-cell and a 4-cell. Then $\rho_n(K) \approx \rho_n(L)$, both groups being cyclic infinite if $n=2$ or 4 and zero otherwise. By considering the cohomology rings, or the groups $\pi_3(K)$, $\pi_3(L)$, one sees that an isomorphism $\rho(K) \rightarrow \rho(L)$ cannot be realized by a map $K \rightarrow L$.

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[16] We shall sometimes write β, f, etc. for β_n, f_n, etc.

6. Realizability. We shall describe a complex, K, as a *(geometrical) realization* of a given homotopy system, ρ, if, and only if, $\rho \approx \rho(K)$. In §15 below we given an example of a homotopy system which has no geometrical realization. Thus it is not always possible to realize either a homormophism $\rho(K) \rightarrow \rho(L)$ by a map $K \rightarrow L$ or a homotopy system ρ by a complex K. In this section we prove three theorems which state sufficient conditions for both kinds of realizability to be possible.

THEOREM 2. *A given homotopy system, ρ, has a geometrical realization if* $\dim \leqq 4$.

Let $n = \dim \rho$ and first let $n=1$. Then ρ is obviously realized by a complex $K = e^0 \cup \{e_i^1\}$, where $\{e_i^1\}$ is a set of 1-cells in (1-1) correspondence with a basis, $\{x_i\}$, for ρ_1.

We proceed by induction on n. Let ρ^r be the homotopy system

$$\rho^r = (\rho_1, \cdots, \rho_r, 0, 0, \cdots),$$

d_s being the same in ρ^r as in ρ if $2 \leqq s \leqq r$. Assume that ρ^{n-1} is realized by a complex K^{n-1} $(1 < n \leqq 4)$ and let $f^{n-1}: \rho^{n-1} \approx \rho(K^{n-1})$. Let $\{a_i\}$ be a basis for ρ_n. If $n > 2$ we have $d_{n-1} d_n = 0$, whence

$$d_{n-1} f_{n-1}^{n-1} d_n a_i = f_{n-2}^{n-1} d_{n-1} d_n a_i = 0,$$

or $f_{n-1}^{n-1} d_n a_i \in d_{n-1}^{-1}(0)$. Since $n - 1 \leqq 3$

$$j_{n-1}: \pi_{n-1}(K^{n-1}) \rightarrow \rho_{n-1}(K^{n-1})$$

is onto[17] $d_{n-1}^{-1}(0)$. Also $f_1^1 d_2 a_i \in \pi_1(K^1)$ if $n=2$. Therefore there is an element $b_i \in \pi_{n-1}(K^{n-1})$, such that

$$f_{n-1}^{n-1} d_n a_i = j_{n-1} b_i \qquad (n \geqq 2).$$

Let $g_i: (\partial I^n, I^0) \rightarrow (K^{n-1}, e^0)$ be a map which represents b_i. Let $\{E_i^n\}$ be a set of n-elements, which are disjoint from each other and from K^{n-1} and let $h_i: \partial E_i^n \rightarrow \partial I^n$ be a homeomorphism. We attach $e_i^n = E_i^n - \partial E_i^n$ to K^{n-1} by means of the map $g_i h_i: \partial E_i^n \rightarrow K^{n-1}$. The result is a complex, $K^n = K^{n-1} \cup \{e_i^n\}$, in which e_i^n has a characteristic map,

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[17] See Theorem 7 in §3 of (I).

$g_i' : I^n \rightarrow \bar{e}_i^n$, such that $g_i' | \partial I^n = g_i$. Let $a_i' \in \rho_n(K^n)$ be the element represented by g_i'. Then $\beta_n a_i' = b_i$, whence

$$d_n a_i' = j_{n-1} b_i = f_{n-1}^{n-1} d_n a_i. \tag{6.1}$$

Hence, and from Lemma 2 if $n=2$, an operator homomorphism, $f_n : \rho_n \rightarrow \rho_n(K^n)$, which is obviously an isomorphism, is defined by $f_n a_i = a_i'$. Moreover it also follows from (6.1) that $d_n f_n = f_{n-1}^{n-1} d_n$. Therefore we have defined an isomorphism $f : \rho \rightarrow \rho(K^n)$, where $f_r = f_r^{n-1}$ if $r < n$, and the theorem follows by induction on n.

THEOREM 3. *A homotopy system, ρ, has a geometrical realization if* $\bar{p}_1 = 1$.

Let $\bar{p}_1 = 1$. Then ρ_r $(r \geqq 3)$ and hence any sub-group of ρ_r is a free Abelian group.[18] Therefore $d_n \rho_n$ is a free Abelian group $(n \geqq 4)$. Let $\{b_i\}$ be a basis for $d_n \rho_n$, let $c_i \in d_n^{-1} b_i$ and let $C_n \subset \rho_n$ be the sub-group generated by $\{c_i\}$. A relation between the elements $\{c_i\}$ would imply the corresponding relation between $\{b_i\}$, whence $\{c_i\}$ is a basis for C_n. Therefore, as in elementary homology theory, ρ_n is the direct sum $\rho_n = Z_n + C_n$, where $Z_n = d_n^{-1}(0)$, and $d_n | C_n$ is an isomorphism onto $d_n \rho_n$. Let $\{z_\lambda^n\}$ be a basis for Z_n.

By Theorem 2, ρ^3 has a realization K^3, where ρ^r means the same as before. Let $n \geqq 4$ and assume that there is a complex, K^{n-1}, such that

$$f^{n-1} : \rho^{n-1} \approx \rho(K^{n-1})$$

and

$$j_{n-1} : \pi_{n-1}(K^{n-1}) \rightarrow \rho_{n-1}(K^{n-1})$$

is onto $d_{n-1}^{-1}(0)$. Then the proof of Theorem 2 shows that there is a complex,

$$K^n = K^{n-1} \cup \{e^n\},$$

such that $f^n : \rho^n \approx \rho(K^n)$ and $f^n | \rho^{n-1} = f^{n-1}$. Moreover we may suppose that $\partial e_\lambda^n = e^0 \in K^0$, for each value of λ, where e_λ^n is the cell which corresponds to $z_\lambda^n \in Z_n$. Let this be so and let P^n be the union of the n-spheres $\bar{e}_\lambda^n$. Then any element in Z_n is represented by a map, $(I^n, \partial I^n) \rightarrow (P^n, e^0)$, which also represents an element in $\pi_n(K^n)$. Therefore j_n is onto Z_n. Therefore complexes $K^3 \subset K^4 \subset \cdots$ and isomorphisms $f^n:\rho^n \approx \rho(K^n)$, such that $f^n | \rho^{n-1} = f^{n-1}$, are defined inductively for every value of $n \geqq 3$. Let

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[18] We assume here that ρ_r has a well-ordered basis. Then this assertion follows by transfinite induction in much the same way as when the basis is finite. I do not know if this can be proved without the axiom of choice.

$$K = \bigcup_n K^n$$

with the topology defined by the condition that $X \subset K$ is closed if, and only if, $X \cap K^n$ is a closed sub-set of K^n, for each $n = 3, 4, \cdots$. Then K is a complex, whose n-section is K^n and $f:\rho \rightarrow \rho(K)$ is obviously an isomorphism, where $f | \rho^n = f^n$. This proves the theorem.

Let K, L be given complexes, let $f:\rho(K) \rightarrow \rho(L)$ be a given homomorphism and let $\phi_0:K_0 \rightarrow L$ be a partial realization of f, where $K_0 \subset K$ is a connected sub-complex, which contains e^0.

THEOREM 4. *If $K_0 \cup K^{m+1} = K$ and if L is a J_m-complex, then ϕ_0 can be extended to a complete realization,*[19] *$\phi:K \rightarrow L$, of f.*

Let a basis for $\rho_1(K)$ be defined, as in §5, in terms of a tree, $T \subset K^1$, such that $K^0 \subset T$. We assume that T consists of a tree,[20] $T_0 \subset K_0$, which contains K_0^0, together with a set of trees, $\{T_i\}$, such that $T_i \cap K_0$ is a single 0-cell e_i^0. We map T_i on $\phi_0 e_i^0$, thus extending ϕ_0 throughout $K_0 \cup T$. It now follows from a simplified version of Lemma 4 that ϕ_0 can be extended to a partial realization, $\phi_1:K_0 \cup K^1 \rightarrow L$, of f.

Let $K_r = K_0 \cup K^r$ $(r \geqq 1)$ and assume that ϕ_0 has been extended to a partial realization, $\phi_{n-1}:K_{n-1} \rightarrow L$, of f, where $2 \leqq n \leqq m+1$. Let $g:\pi_{n-1}(K^{n-1}) \rightarrow \pi_{n-1}(L^{n-1})$ be the homomorphism induced by ϕ_{n-1}. Then it is easily verified that

$$f_{n-1}j_{n-1} = j_{n-1}g:\pi_{n-1}(K^{n-1}) \rightarrow \rho_{n-1}(L).$$

Therefore

$$j_{n-1}\beta_n f_n = d_n f_n = f_{n-1}d_n = f_{n-1}j_{n-1}\beta_n = j_{n-1}g\beta_n.$$

Let L be a J_m-complex. Then $j_{n-1}^{-1}(0) = 0$, since $n-1 \leqq m$. Therefore $\beta_n f_n = g\beta_n$ and it follows from Lemma 4 that ϕ_{n-1} can be extended to a partial realization, $\phi_n:K_n \rightarrow L$, of f. The theorem now follows by induction on n.

7. Homotopy and equivalence. We state three theorems, which will be proved in §§13, 14 below. Let K, L be given complexes and let $f, f':\rho(K) \rightarrow \rho(L)$ be the homomorphisms induced by maps $\phi, \phi':K \rightarrow L$.

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[19] K_0 may consist of the single 0-cell e^0, in which case the theorem asserts that f has a realization if dim $K \leqq m+1$. J_m-complexes are defined in (I), §3.

[20] In constructing T we can start with T_0.

THEOREM 5. *If* $\phi \simeq \phi'$ *then* $f \simeq f'$.

THEOREM 6. *A map* $\phi: K \rightarrow L$ *is a homotopy equivalence if, and only if, the induced homomorphism,* $f: \rho(K) \rightarrow \rho(L)$, *is an equivalence.*

As a corollary to Theorems 4 and 6 we have:

COROLLARY. *If* $\dim K \leqq m+1$ *and if* L *is a* J_m*-complex, then* $K \equiv L$ *if, and only if,* $\rho(K) \equiv \rho(L)$. *More precisely, an equivalence,* $\rho(K) \rightarrow \rho(L)$, *can be realized by a homotopy equivalence,* $K \rightarrow L$, *and conversely.*

Since every complex is a J_2-complex it follows from this and Theorem 2 that an equivalence class of 3-dimensional homotopy systems is an algebraic equivalent of the homotopy type of a 3-dimensional complex. Moreover, if $\dim \rho \leqq 3$ and if K, L are any complexes such that $\rho(K) \equiv \rho(L) \equiv \rho$, then $K \equiv L$, even if $\dim K$, $\dim L > 3$. For it follows from Theorem 2 that ρ can be realized by a complex, P^3, and it follows from Theorem 4 that equivalences, $\rho(P^3) \rightarrow \rho(K)$, $\rho(P^3) \rightarrow \rho(L)$ can be realized by maps $P^3 \rightarrow K$, $P^3 \rightarrow L$, which are homotopy equivalences by Theorem 6. Therefore $K \equiv P^3 \equiv L$. In a later paper we shall prove that if $\rho \equiv \rho'$, $\dim \rho$, $\dim \rho' < \infty$, where ρ, ρ' are given homotopy systems, and if ρ can be realized geometrically, so can ρ'. Therefore, if an equivalence class of homotopy systems contains one, ρ, such that $\dim \rho \leqq 3$, then every homotopy system in the class can be realized geometrically and all the resulting complexes are of the same homotopy type.

By taking $K = S^3$, $L = S^2$ we see that the converse of Theorem 5, namely $f \simeq f'$ implies $\phi \simeq \phi'$, does not hold, even if L is a J_m-complex and $\dim K \leqq m+1$. However we shall prove:

THEOREM 7. *If* L *is a* J_m*-complex and* $\dim K \leqq m$, *then* $f \simeq f'$ *implies* $\phi \simeq \phi'$.

It follows that the homotopy theory of 2-dimensional complexes, including the homotopy classification of mappings, is equivalent to the purely algebraic theory of free crossed modules, whose groups of operators are free groups.

8. Chain groups. With each homotopy system, $\rho = \{\rho_n\}$, which need not have a geometrical realization, we associate a system of *chain groups*, $C = \{C_r\}$ $(r = 0, 1, \cdots)$. Each group C_r shall be a free $\bar{\rho}_1$-module and C_n shall be operator isomorphic to ρ_n if $n > 2$ and to ρ_2 made Abelian if $n = 2$. The group C_1 shall have a basis $\{c_i\}$ which is in a (1-1) correspondence, $x_i \rightarrow c_i$, with a basis $\{x_i\}$ for ρ_1. The group C_0 shall have a basis consisting of a single element c^0. The element c^0 is to be a "preferred" or special element in C_0. We allow ρ_1 to operate on C_n in the same way as on ρ_n $(n>2)$.

The system C shall be related to ρ by a preferred family of maps,

$$h_n : \rho_n \rightarrow C_n.$$

The map h_n shall be an operator isomorphism if $n>2$ and h_2 shall be an operator homomorphism (onto), whose kernel is the commutator sub-group of ρ_2. The map h_1 shall be the crossed homomorphism which is uniquely determined by $hx_\iota = c_\iota$, according to Lemma 3.

We now define a boundary operator, ∂, which is a family of operator homomorphisms, $\partial_n : C_n \rightarrow C_{n-1}$ $(n \geqq 1)$, such that $\partial_n \partial_{n+1} = 0$. We shall define it in such a way that

$$(8.1) \qquad \begin{aligned} &\text{(a)} \quad \partial_1 h_1 x = (\bar{x} - 1)c^0 && (x \in \rho_1), \\ &\text{(b)} \quad \partial_n h_n = h_{n-1} d_n && (n \geqq 2). \end{aligned}$$

If $n>2$ we define $\partial_n = h_{n-1} d_n h_n^{-1}$. We postpone the definition of ∂_2 and define ∂_1 by $\partial_1 c_\iota = (\bar{x}_\iota - 1)c^0$. Let $g : \rho_1 \rightarrow C_0$ be the (principal) crossed homomorphism, which is given by $gx = (\bar{x} - 1)c^0$. The map $\partial_1 h_1 : \rho_1 \rightarrow c^0$ is the resultant of the crossed homomorphism h_1 and the operator homomorphism ∂_1. Therefore it is a crossed homomorphism. Moreover g and $\partial_1 h_1$ are both associated with the same homomorphism, $x \rightarrow \bar{x}$, of ρ_1 onto $\bar{\rho}_1$. Since $\partial_1 h_1 x_\iota = g x_\iota$, from the uniqueness part of Lemma 3 $\partial_1 h_1 = g$, which is expressed by (8.1a).

THEOREM 8. *The crossed homomorphism h_1 has the properties:*

(a) *$h_1 | d_2\rho_2$ is a homomorphism onto $\partial_1^{-1}(0) \subset C_1$.*

(b) *$h_1 x = h_1 y$ $(x, y \subset \rho_1)$ if, and only if, $xy^{-1} \in d_2 h_2^{-1}(0)$. In particular $h_1^{-1}(0) = d_2 h_2^{-1}(0)$.*

(c) *If $\partial_1 c = (\bar{x} - 1)c^0$, where $c \in C_1$, $\bar{x} \in \bar{\rho}_1$, then $c = h_1 x'$, for some $x' \in \bar{x}$.*

In order to prove this we introduce a complex, $K^1 = e^0 \cup \{e_\iota^1\}$, where $\{e_\iota^1\}$ is a set of 1-cells which is in a (1-1) correspondence, $x_\iota \rightarrow e_\iota^1$, with $\{x_\iota\}$. We identify x_ι with the element, $x_\iota \in \pi_1(K^1)$, which is represented by a characteristic map for e_ι^1. Let $\tilde{K}^1$ be a covering complex of K^1, which is associated with the (invariant) sub-group $d_2\rho_2 \subset \rho_1$, and let $p : \tilde{K}^1 \rightarrow K^1$ be the covering map. Then p induces an isomorphism $p_1 : \tilde{\rho}_1 \rightarrow d_2\rho_2$, where $\tilde{\rho}_1 = \pi_1(\tilde{K}^1)$, with a 0-cell $\tilde{e}^0 \in p^{-1}e^0$ as base point. Following S. Eilenberg and N. E. Steenrod in [11] we take

$$C_0' = H_0(\tilde{K}^0), \qquad C_1' = H_1(\tilde{K}_1, \tilde{K}^0)$$

as chain groups in $\tilde{K}^1$. We denote the boundary homomorphism by $\partial' : C_1' \rightarrow C_0'$.

Let $\lambda:(I, 0, 1)\rightarrow(K^1, e^0, e^0)$ be a map representing a given element, $x\in\rho_1$, and let $\tilde{\lambda}:(I, 0)\rightarrow(\tilde{K}^1, \tilde{e}^0)$ be the lifted map (i.e. $\lambda=p\tilde{\lambda}$). Let $hx\in C_1$ be the element represented by $\tilde{\lambda}$. Clearly hx depends only on $x\in\rho_1$ and not on the choice of the representative map λ. Also

$$(8.2)\qquad \partial' hx = (\bar{x} - 1)c'^0,$$

where $c'^0\in C_0'$ is the 0-chain, which corresponds to $\tilde{e}^0$, and $\bar{\rho}_1$ operates in the usual way on C_0', C_1'. Also $\{hx_i\}$ is a basis for C_1'. Let $y\in\rho_1$ be represented by a map $\mu:(I, 0, 1)\rightarrow(K^1, e^0, e^0)$. Then xy is represented by the circuit λ followed by μ and $h(xy)$ by the path $\tilde{\lambda}$ followed by $\mu':I\rightarrow\tilde{K}^1$, where $\mu=p\mu'$ and $\mu'(0)=\tilde{\lambda}(1)$. Therefore

$$(8.3)\qquad h(xy) = hx + \bar{x}hy.$$

We identify C_0, C_1 with C_0', C_1' in such a way that $c_i=hx_i$, $c^0=c'^0$. Then it follows from (8.2), (8.3) and the uniqueness part of Lemma 3 that $h=h_1$. Hence, and by (8.1a), $\partial'=\partial_1$. Since $\bar{x}=1$ if $x\in d_2\rho_2$, it follows from (8.3) and (8.2) that $h_1|d_2\rho_2$ is a homomorphism into $\partial_1^{-1}(0)$. It follows from the geometrical definition of h_1 that $h_1|d_2\rho_2=h'p^{-1}$, where $h':\tilde{\rho}_1\rightarrow\partial_1^{-1}(0)$ is the homomorphism in which corresponding elements are represented by the same map. Therefore it follows from the well known relation between the fundamental group of a polyhedron and its first homology group that $h_1|d_2\rho_2$ is onto $\partial_1^{-1}(0)$, which establishes (a).

If $h_1x=0$, then $(\bar{x}-1)c^0=\partial_1h_1x=0$, whence $\bar{x}=1$ and $x\in d_2\rho_2$. Therefore, writing G° for the commutator sub-group of a given group, G, we have

$$\overset{-1}{h_1}(0) = (h_1 | d_2\rho_2)^{-1}(0) = p_1h'^{-1}(0) = p_1\overset{\circ}{\tilde{\rho}}_1.$$

But $f(G^\circ)=(fG)^\circ$ if f is any homomorphism of G onto any other group. Therefore

$$\overset{-1}{h_1}(0) = (p_1\tilde{\rho}_1)^\circ = (d_2\rho_2)^\circ = d_2\overset{\circ}{\rho}_2 = d_2\overset{-1}{h_2}(0).$$

Let $xy^{-1}=z\in d_2h_2^{-1}(0)$. Then $h_1z=0$ and $\bar{z}=1$, since $z\in d_2\rho_2$. Therefore

$$h_1x = h_1(zy) = h_1y.$$

Conversely, let $h_1x=h_1y$. Then

$$(\bar{x} - 1)c^0 = \partial_1h_1x = \partial_1h_1y = (\bar{y} - 1)c^0,$$

whence $\bar{x}=\bar{y}$. Therefore

$$h_1(xy^{-1}) = h_1x + \bar{x}h_1y^{-1} = h_1x + \bar{y}h_1y^{-1} = h_1x - h_1y = 0,$$

whence $xy^{-1}\in d_2h_2^{-1}(0)$. This proves (b).

Let $c\in C_1$ be such that $\partial_1 c=(\bar{x}-1)c^0$. Then $\partial_1(c-h_1x)=0$ and it follows from (a) that $c-h_1x=h_1y$ for some $y\in d_2\rho_2$. Let $x'=yx$. Then $\bar{y}=1$, $x'\in\bar{x}$ and

$$h_1x' = h_1y + \bar{y}h_1x = h_1y + h_1x = c,$$

which completes the proof of Theorem 8.

We now define ∂_2 by (8.1b), with $n=2$. Since $d_2h_2^{-1}(0)=1$ this gives a single-valued map, $\partial_2: C_2\to C_1$, which is obviously an operator homomorphism. Thus ∂_n is defined for each $n\geqq 1$. It follows from (8.1b) and Theorem 8(a) that

$$\partial_1\partial_2h_2 = \partial_1h_1d_2 = 0$$

and from (8.1b) that

$$\partial_n\partial_{n+1}h_{n+1} = \partial_nh_nd_{n+1} = h_{n-1}d_nd_{n+1} = 0 \qquad (n \geqq 2).$$

Since h_n is onto C_n $(n\geqq 2)$ it follows that $\partial\partial=0$.

Let $c\in C$ and $x\in\rho_1$ be given. Then we have:

LEMMA 5. *If $\partial_2c=h_1x$ there is a unique element, $a\in\rho_2$, such that $h_2a=c$, $d_2a=x$.*

Since h_2 is onto C_2 there is an $a'\in\rho_2$ such that $h_2a'=c$. It follows from (8.1b) and the relation $\partial_2c=h_1x$ that

$$h_1x = \partial_2c = \partial_2h_2a' = h_1d_2a'.$$

Therefore it follows from Theorem 8(b) that

$$x = (d_2b)(d_2a') = d_2(b + a'),$$

where $b\in h_2^{-1}(0)$. Let $a=b+a'$. Then $h_2a=h_2a'=c$, since $h_2b=0$. Therefore $h_2a=c$, $d_2a=x$. Let $h_2a_1=c$, $d_2a_1=x$. Then $h_2(a-a_1)=0$, $d_2(a-a_1)=1$ and it follows from Lemma 1 that $a_1=a$. Therefore a is unique.

9. Chain mappings and homomorphisms. Let ρ, ρ' be two homotopy systems and C, C' the associated systems of chain groups. By a *chain mapping*, $g: C\to C'$, we shall mean a family of operator homomorphisms, $g_n: C_n\to C_n'$, associated with a homomorphism, $\bar{f}:\bar{\rho}_1\to\bar{\rho}_1'$, such that

$$(9.1) \qquad \begin{aligned} &\text{(a)} \quad g_0c^0 = c'^0, \\ &\text{(b)} \quad g_{n-1}\partial_n = \partial_ng_n \qquad (n \geqq 1), \end{aligned}$$

where c^0, c'^0 are the preferred basis elements in C_0, C_0'. If there is a homomorphism, $f:\rho\rightarrow\rho'$, such that

$$g_n h_n = h_n f_n \tag{9.2}$$

then we shall say that f *induces* or *realizes* the chain mapping g. In this case it follows from (8.1a), (9.2) and the relation $g\bar{x}=(\bar{f}\bar{x})g$ $(\bar{x}\in\bar{\rho}_1)$ that

$$\begin{aligned}(\bar{f}_1\bar{x}-1)c'^0 &= \partial_1 h_1 f_1 x = \partial_1 g_1 h_1 x\\ &= g_0\partial_1 h_1 x = g_0(\bar{x}-1)c^0\\ &= (\bar{f}\bar{x}-1)c'^0.\end{aligned} \tag{9.3}$$

Therefore $\bar{f}=\bar{f}_1$.

Let C, C', C'' be systems of chain groups associated with homotopy systems ρ, ρ', ρ''. Let $g:C\rightarrow C'$, $g':C'\rightarrow C''$ be chain mappings which are induced by homomorphisms $f:\rho\rightarrow\rho'$, $f':\rho'\rightarrow\rho''$. Then it is easily verified that $g'g:C\rightarrow C''$ is a chain mapping induced by $f'f:\rho\rightarrow\rho''$.

THEOREM 9. *A homomorphism, $f:\rho\rightarrow\rho'$, induces a unique chain mapping, $g:C\rightarrow C'$. A chain mapping, $g:C\rightarrow C'$, can be realized by at least one*[11] *homomorphism, $f:\rho\rightarrow\rho'$.*

Let $f:\rho\rightarrow\rho'$ be given. Then we define g_0 by $g_0\bar{x}c^0=(\bar{f}_1\bar{x})c'^0$. Let $\{x_i\}$ be a basis for ρ_1. Then $\{h_1x_i\}$ is a basis for C_1 and we define g_1 by $g_1h_1x_i=h_1f_1x_i$. It is easily verified that g_1h_1, $h_1f_1:\rho_1\rightarrow C_1'$ are both crossed homomorphisms associated with f_1. Therefore $g_1h_1=h_1f_1$, according to Lemma 3.

If $n\geqq 2$ we define g_n by $g_nh_n=h_nf_n$. If $h_2a=0$, then $a\in\rho_2^c$ and $f_2a\in\rho_2'^c$, whence $h_2f_2a=0$. Therefore g_2 is a single-valued map, which is obviously an operator homomorphism associated with f_1. So is $g_n=h_nf_nh_n^{-1}$ $(n>2)$. The family of maps $g_n:C_n\rightarrow C_n'$ satisfies (9.2). Since h_n is onto C_n if $n>1$ and $\{h_1x_i\}$ is a basis for C_1 it follows that (9.2), together with $\bar{f}_1$ and (9.1a), determine $\{g_n\}$ uniquely.

It follows from (9.2), with $n=1$, and (8.1) that

$$\begin{aligned}\partial_1 g_1 h_1 x &= \partial_1 h_1 f_1 x = (\bar{f}_1\bar{x}-1)c_0'\\ &= g_0(\bar{x}-1)c^0\\ &= g_0\partial_1 h_1 x.\end{aligned}$$

Therefore $\partial_1g_1=g_0\partial_1$. A similar, but simpler, argument shows that $\partial_ng_n=g_{n-1}\partial_n$ for any $n>1$. Therefore $g=\{g_n\}$ is a (uniquely determined) chain mapping, which is induced by f.

Now let $g:C\rightarrow C'$ be a given chain mapping, associated with a homomorphism $\bar{f}_1:\bar{\rho}_1\rightarrow\bar{\rho}_1'$. Since

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[11] See the addendum and (10.9) below.

$$\begin{aligned}\partial_1 g_1 h_1 x_i &= g_0\partial_1 h_1 x_i = g_0(\bar{x}_i - 1)c^0\\ &= (\bar{f}_1\bar{x}_i - 1)c'^0\end{aligned}$$

it follows from Theorem 8(c), applied to $\bar{\rho}_1'$, that there is an element, $y_i'\in\rho_1'$, such that $g_1h_1x_i=h_1y_i'$. We define $f_1:\rho_1\rightarrow\rho_1'$ by $f_1x_i=y_i'$. Then it follows from Lemma 3, as in the first half of the proof, that

$$g_1h_1 = h_1f_1. \tag{9.4}$$

Let A be a basis for ρ_2 and let

$$c_a' = g_2h_2a\in C_2'$$

for each $a\in A$. Then

$$\partial_2c_a' = \partial_2g_2h_2a = g_1\partial_2h_2a = g_1h_1d_2a$$

and it follows from (9.4) that

$$\partial_2c_a' = h_1f_1d_2a.$$

By Lemma 5 there is a (unique) $b_a'\in\rho_2'$ such that $h_2b_a'=c_a'$, $d_2b_a'=f_1d_2a$. Since $d_2b_a'=f_1d_2a$ it follows from Lemma 2 that there is a (unique) operator homomorphism, $f_2:\rho_2\rightarrow\rho_2'$, associated with f_1, such that $f_2a=b_a'$ for each $a\in A$. Then $d_2f_2a=f_1d_2a$ and

$$h_2f_2a = h_2b_a' = c_a' = g_2h_2a.$$

Therefore $d_2f_2=f_1d_2$ and $h_2f_2=g_2h_2$.

If $n\geqq 3$ let $f_n=h_n^{-1}g_nh_n$. Then f_n $(n\geqq 2)$ is an operator homomorphism associated with f_1 and $gh=hf$, $d_2f_2=f_1d_2$. It remains to prove that $d_nf_n=f_{n-1}d_n$ when $n>2$. It follows from the relations $hd=\partial h$, $hf=gh$, $\partial g=g\partial$ that

$$\begin{aligned}h_{n-1}d_nf_n &= \partial_nh_nf_n = \partial_ng_nh_n\\ &= g_{n-1}\partial_nh_n = g_{n-1}h_{n-1}d_n\\ &= h_{n-1}f_{n-1}d_n.\end{aligned} \tag{9.5}$$

If $n>3$, then h_{n-1} is an isomorphism and it follows that $d_nf_n=f_{n-1}d_n$. Also $h_2(d_3f_3-f_2d_3)=0$ and

$$d_2(d_3f_3 - f_2d_3) = d_2d_3f_3 - f_1d_2d_3 = 0.$$

Therefore it follows from Lemma 1 that $d_3f_3-f_2d_3=0$, which completes the proof.

There is an element of arbitrariness in the choice of f_1, satisfying (9.4). Let $f_1:\rho_1\rightarrow\rho_1'$ be a given homomorphism such that $h_1f_1=g_1h_1$.

Then the above construction gives $f:\rho\rightarrow\rho'$ uniquely. Therefore we have the addendum to Theorem 9:

ADDENDUM. *The chain mapping g has a unique realization, f, such that $f_1:\rho_1\rightarrow\rho_1'$ is a given homomorphism, which satisfies the condition $h_1f_1=g_1h_1$.*

10. Chain homotopy. Let ρ, ρ' be given homotopy systems and C, C' the associated systems of chain groups. By a *chain deformation* operator: $\eta:C\rightarrow C'$, *associated* with a homomorphism, $\bar{f}_1:\bar{\rho}_1\rightarrow\bar{\rho}_1$, we shall mean a family of operator homomorphisms, $\eta_n:C_{n-1}\rightarrow C_n$ $(n\geqq 1)$, associated with $\bar{f}_1$. Let g, $g':C\rightarrow C'$ be chain mappings associated with homomorphisms $\bar{f}_1$, $\bar{f}_1':\bar{\rho}_1\rightarrow\bar{\rho}_1'$. We shall describe g as *chain homotopic* to g', and shall write $g\simeq g'$, if, and only if, there is an element $\bar{w}'\in\bar{\rho}_1'$ and a chain deformation operator, $\eta:C\rightarrow C'$, associated with $\bar{f}_1$ such that

$$\bar{w}'g_r' - g_r = \partial_{r+1}\eta_{r+1} + \eta_r\partial_r \qquad (r\geqq 0;\ \eta_0\partial_0 = 0). \tag{10.1$_r$}$$

Since $g_0c^0=g_0'c'^0=c'^0$, where $c^0\in C_0$, $c'^0\in C_0'$ are the preferred basis elements, it follows from $(10.1)_0$ that $\partial_1\eta_1c^0=(\bar{w}'-1)c'^0$. It follows from Theorem 8(c) that there is a $w'\in\bar{w}'$ such that

$$\eta_1c^0 = h_1w'. \tag{10.2}$$

I say that, in consequence of $(10.1)_0$,

$$\bar{f}_1'\bar{x} = \bar{w}'^{-1}(\bar{f}_1\bar{x})\bar{w}'. \tag{10.3}$$

For $\partial_1\eta_1:C_0\rightarrow C_0'$ is an operator homomorphism associated with $\bar{f}_1$. Therefore it follows from $(10.1)_0$ that

$$\begin{aligned}(\bar{w}'g_0' - g_0)\bar{x}c^0 &= (\bar{f}_1\bar{x})(\bar{w}'g_0' - g_0)c^0\\ &= (\bar{f}_1\bar{x})(\bar{w}' - 1)c'^0.\end{aligned}$$

But we also have

$$\begin{aligned}(\bar{w}'g_0' - g_0)\bar{x}c^0 &= \left\{\bar{w}'(\bar{f}_1'\bar{x})g_0' - (\bar{f}_1\bar{x})g_0\right\}c^0\\ &= \left\{\bar{w}'(\bar{f}_1'\bar{x}) - \bar{f}_1\bar{x}\right\}c'^0.\end{aligned}$$

Therefore

$$(\bar{f}_1\bar{x})(\bar{w}' - 1) = \bar{w}'(\bar{f}_1'\bar{x}) - \bar{f}_1\bar{x},$$

which reduces to (10.3).

Let g, $g':C\rightarrow C'$ be the chain mappings induced by homomorphisms $f, f':\rho\rightarrow\rho'$. For convenience we display the formula

$$w'f_n' - f_n = d_{n+1}\xi_{n+1} + \xi_nd_n, \tag{10.4}$$

in which $w' \in \rho_1$, and $\xi : \rho \rightarrow \rho'$ is a deformation operator.

THEOREM 10. *$f \simeq f'$ if, and only if, $g \simeq g'$. More precisely, f, f' are related by* (10.4) *if, and only if, g, g' are related by* (10.1), *where*

$$(10.5) \qquad \begin{aligned} &\text{(a)} \quad \eta_1 c^0 = h_1 w', \\ &\text{(b)} \quad h_n \xi_n = \eta_n h_{n-1} \qquad (n \geqq 2). \end{aligned}$$

First let $f \simeq f'$ and let f, f' be related by (10.4). Let $\{x_i\}$ be a basis for ρ_1. Let $\eta_1 : C_0 \rightarrow C_1'$, $\eta_2 : C_1 \rightarrow C_2'$ be the operator homomorphisms, associated with $\bar{f}_1$, which are defined by $\eta_1 c^0 = h_1 w$, $\eta_2 h_1 x_i = h_2 \xi_2 x_i$. Clearly $\eta_2 h_1$, $h_2 \xi_2 : \rho_1 \rightarrow C_2'$ are both crossed homomorphisms associated with the homomorphism $x \rightarrow \bar{f}_1 \bar{x}$. Therefore $\eta_2 h_1 = h_2 \xi_2$. Since ρ_3' is Abelian it follows that $\xi_3 h_2^{-1}(0) = 0$. Therefore a single-valued map, $\eta_3 : C_2 \rightarrow C_3'$, which is obviously an operator homomorphism, is defined by $\eta_3 h_2 = h_3 \xi_3$. Let $\eta_n = h_n \xi_n h_{n-1}^{-1}$ if $n > 3$. Then (10.5) is satisfied.

Let $F_n : \rho_n \rightarrow \rho_n'$, $G_r : C_r \rightarrow C_r'$ be the maps which are given by

$$(10.6) \qquad \begin{aligned} F_n &= w' f_n' - f_n - \xi_n d_n - d_{n+1} \xi_{n+1} \qquad (n \geqq 1), \\ G_r &= \bar{w}' g_r' - g_r - \eta_r \partial_r - \partial_{r+1} \eta_{r+1} \qquad (r \geqq 0). \end{aligned}$$

We shall prove that

$$(10.7) \qquad h_n F_n = G_n h_n,$$

assuming only (10.3), (10.5) and that g, g' are induced by f, f'. Since $hf = gh$ we have

$$h_n(w' f_n' - f_n) = (\bar{w}' g_n' - g_n) h_n \qquad (n \geqq 2)$$

and from (8.1) and (10.5)

$$\begin{aligned} h_n(d_{n+1}\xi_{n+1} + \xi_n d_n) &= \partial_{n+1} h_{n+1} \xi_{n+1} + \eta_n h_{n-1} d_n \\ &= (\partial_{n+1}\eta_{n+1} + \eta_n \partial_n) h_n. \end{aligned}$$

Therefore (10.7) is valid if $n \geqq 2$.

Let $x \in \rho_1$ and let $u = f_1 x$, $u' = f_1' x$. Then $\overline{w' u' w'^{-1}} \doteq \bar{u}$ according to (10.3), and it follows from (3.2)and (3.3) that

$$(10.8) \qquad \begin{aligned} h_1(w' u' w'^{-1} u^{-1}) &= h_1 w' + w' h_1 u' - \bar{u} h_1 w' - h_1 u \\ &= \bar{w}' h_1 u' - h_1 u - (\bar{u} - 1) h_1 w' \\ &= \bar{w}' h_1 f_1' x - h_1 f_1 x - (\bar{f}_1 \bar{x} - 1) \eta_1 c^0 \\ &= (\bar{w}' g_1' - g_1) h_1 x - \eta_1 (\bar{x} - 1) c^0 \\ &= (\bar{w}' g_1' - g_1) h_1 x - \eta_1 \partial_1 h_1 x \\ &= G_1 h_1 x + \partial_2 \eta_2 h_1 x. \end{aligned}$$

But $\partial_2\eta_2h_1=\partial_2h_2\xi_2=h_1d_2\xi_2$. Since

$$(w'u'w'^{-1}u^{-1})^* = 1, \qquad (d_2\xi_2x)^* = 1,$$

where $y^*=\bar{y}$ $(y\in\rho_1')$, it follows that

$$\begin{aligned} h_1F_1x &= h_1\{w'u'w'^{-1}u^{-1}(d_2\xi_2x)^{-1}\} \\ &= h_1(w'u'w'^{-1}u^{-1}) - h_1d_2\xi_2x \\ &= G_1h_1x. \end{aligned}$$

Therefore (10.7) is valid for every $n\geqq 1$. Since $h_n\rho_n=C_n$ if $n>1$ and $\{h_1x_i\}$ is a basis for C_1 it follows that $G_n=0$ in consequence of $F_n=0$. Since $g_0c^0=g_0'c^0=c'^0$ it follows from (10.5a) that $G_0c^0=(\bar{w}'-1)c'^0-\partial_1\eta_1c^0=0$. Therefore (10.4) implies (10.1), with ξ and η related by (10.5).

Conversely let g, g' be related by (10.1). Then (10.2), i.e. (10.5a), and (10.3) are satisfied. Let $u_i=f_1x_i$, $u_i'=f_1'x_i$. Then it follows from (10.8), with $G_1=0$, and Lemma 5 that there is a (unique) element, $a_i'\in\rho_2'$, such that

$$h_2a_i' = \eta_2h_1x_i, \qquad d_2a_i' = w'u'w'^{-1}u^{-1}.$$

Let $\xi_2:\rho_1\rightarrow\rho_2'$ be the crossed homomorphism which is determined by $\xi_2x_i=a_i'$, according to Lemma 3. Then, reverting to the additive notation for ρ_1', we have

$$h_2\xi_2x_i = \eta_2h_1x_i, \qquad d_2\xi_2x_i = w'f_1'x_i - f_1x_i.$$

Clearly $h_2\xi_2$, $\eta_2h_1:\rho_1\rightarrow C_2'$ are crossed homomorphisms associated with the homomorphism $x\rightarrow\bar{f}_1\bar{x}$. Also $d_2\xi_2:\rho_1\rightarrow\rho_1'$ and, as observed in §3, $(w'f_1'-f_1):\rho_1\rightarrow\rho_1'$ are crossed homomorphisms associated with f_1. Therefore

$$h_2\xi_2 = \eta_2h_1, \qquad d_2\xi_2 = w'f_1' - f_1.$$

If $n>2$ we define

$$\xi_n = h_n^{-1}\eta_nh_{n-1}:\rho_{n-1} \rightarrow \rho_n',$$

thus defining a deformation operator ξ, which satisfies (10.5b). Since (10.3) and (10.5) are satisfied, so is (10.7). Since $G_n=0$ it follows that $h_nF_n=0$, whence $F_n=0$ if $n>2$.

Since

$$d_n(w'f_n' - f_n) = (w'f_{n-1}' - f_{n-1})d_n$$

and

$$d_n(d_{n+1}\xi_{n+1} + \xi_nd_n) = d_n\xi_nd_n = (d_n\xi_n + \xi_{n-1}d_{n-1})d_n$$

it follows that

$$d_nF_n = F_{n-1}d_n \qquad (n \geqq 2).$$

We have proved that $F_1=0$, whence $d_2F_2=0$. Since $h_2F_2=0$ it follows from Lemma 1 that $F_2=0$. Therefore $F_n=0$ for $n\geqq 2$ and the theorem is proved.

On taking $g=g'$, $\eta=0$, $w'=1$ we have the corollary:

COROLLARY. *If $f, f':\rho\to\rho'$ induce the same chain mapping, $g: C\to C'$, then $f\simeq f'$.*

In this case it follows from (10.5) that $\xi_n=0$ if $n>2$. Also we may take $w'=1$, so that

$$f_1' - f_1 = d_2\xi_2, \quad f_2' - f_2 = \xi_2 d_2, \quad f_n' = f_n \qquad (n > 2), \tag{10.9}$$

where $h_2\xi_2=\eta_2h_1=0$.

The operator η is determined uniquely by (10.5) if w' and ξ are given. When η is given the element w' may be replaced by $z'w'$, for any $z'\in d_2h_2^{-1}(0)\subset\rho_1'$, according to Theorem 8(b). Once w' has been fixed, so as to satisfy (10.2), then ξ is determined uniquely by the conditions $F_1=0$ and (10.5). Moreover in this statement it need only be assumed that $\xi_n:\rho_{n-1}\to\rho_n'$ is a single-valued transformation. For $\xi_2x\in\rho_2'$ is determined uniquely by the conditions

$$d_2\xi_2x = (w'f_1' - f_1)x, \qquad h_2\xi_2x = \eta_2h_1x$$

and ξ_n by $\xi_n=h_n^{-1}\eta_nh_{n-1}$ $(n\geqq 3)$. But we have proved that there is a deformation operator, $\xi:\rho\to\rho'$, associated with f_1, which satisfies these conditions. Therefore the fact that ξ_2 is a crossed homomorphism associated with f_1, and ξ_n an operator homomorphism if $n>2$, is a consequence of $F_1=0$ and (10.5).

11. Chain equivalence: Proof of Theorem 1. Let C, C' be systems of chain groups associated with homotopy systems ρ, ρ'. A chain mapping, $g: C\to C'$, is called a *chain equivalence* if, and only if, there is a chain mapping, $g': C'\to C$, such that $g'g\simeq 1$, $gg'\simeq 1$. We shall describe C as *equivalent* to C', and shall write $C\equiv C'$, if, and only if, there is a chain equivalence $g: C\to C'$.

THEOREM 11. *The relations $\simeq$, between chain mappings, and $\equiv$, between systems of chain groups, are equivalence relations.*

On taking $\bar{w}'=1$, $\eta=0$ in (10.1) we have $g\simeq g$. In general it follows from (10.1) that

$$\bar{w}'^{-1}g - g' = \partial\eta^* + \eta^*\partial,$$

where $\eta^* = -\bar{w}'^{-1}\eta$. Since $\eta\bar{x} = (\bar{f}_1\,\bar{x})\eta$ it follows from (10.3) that

$$\eta^*\bar{x} = -\,\bar{w}'^{-1}(\bar{f}_1\bar{x})\eta = -\,(\bar{f}_1'\,\bar{x})\bar{w}'^{-1}\eta^*.$$

Therefore η^* is a chain deformation operator associated with $\bar{f}_1'$ and it follows that $g' \simeq g$.

Let $g,\ g' : C \rightarrow C'$ be related by (10.1) and let $g',\ g'' : C \rightarrow C'$ be related by

$$\bar{w}''g'' - g' = \partial\eta' + \eta'\partial,$$

where η' is a chain deformation operator associated with $\bar{f}_1'$. Then

$$\bar{w}'\bar{w}''g'' - g = \partial(\eta + \bar{w}'\eta') + (\eta + \bar{w}'\eta')\partial.$$

Moreover

$$\begin{aligned}(\eta + \bar{w}'\eta')\bar{x} &= (\bar{f}_1\bar{x})\eta + \bar{w}'(\bar{f}_1'\,\bar{x})\eta' \\ &= (\bar{f}_1\bar{x})(\eta + \bar{w}'\eta').\end{aligned}$$

Therefore $\eta + \bar{w}'\eta'$ is a chain deformation operator associated with $\bar{f}_1$. Therefore $g \simeq g'$ and $g' \simeq g''$ together imply $g \simeq g''$. Therefore $\simeq$ is an equivalence relation.

The relation $\equiv$ is obviously reflexive and symmetric. Let $C,\ C',\ C''$ be chain groups associated with homotopy systems $\rho,\ \rho',\ \rho''$. Let $f \simeq g : C \rightarrow C'$, $f' \simeq g' : C' \rightarrow C''$. Then it is easily verified that

$$f'f \simeq f'g \simeq g'g. \tag{11.1}$$

It follows from this, as in the ordinary theory of homotopy or chain homotopy, that $\equiv$ is transitive. This completes the proof.

Let $\rho,\ \rho'$ be given homotopy systems and $C,\ C'$ associated systems of chain groups.

THEOREM 12. *$\rho \equiv \rho'$ if, and only if, $C \equiv C'$. More precisely, a homomorphism, $f : \rho \rightarrow \rho'$, is an equivalence if, and only if, the induced chain mapping, $g : C \rightarrow C'$, is a chain equivalence.*

Let $f : \rho \rightarrow \rho'$ and $f' : \rho' \rightarrow \rho$ be homomorphisms and let $g : C \rightarrow C'$ and $g' : C' \rightarrow C$ be the induced chain mappings. Then $g'g,\ gg'$ are induced by $f'f,\ ff'$. Also the identical chain mappings, $1 : C,\ C' \rightarrow C,\ C'$ are obviously induced by $1 : \rho,\ \rho' \rightarrow \rho,\ \rho'$. Therefore it follows from Theorem 10 that $f'f \simeq 1,\ ff' \simeq 1$ if, and only if, $g'g \simeq 1,\ gg' \simeq 1$. This proves the theorem.

As an obvious corollary of Theorems 11 and 12 we have one half of Theorem 1, namely that $\equiv$ is an equivalence relation between homotopy systems. To prove the other half let $f \simeq f' : \rho \rightarrow \rho'$ and let $g,\ g' : C \rightarrow C'$ be the induced chain mappings. Then it follows from

Theorems 10 and 11 that $g \simeq g'$, $g' \simeq g$, $f' \simeq f$. Therefore the relation $\simeq$ is symmetric. Similarly it is transitive and it is obviously reflexive. Therefore it is an equivalence relation. This proves Theorem 1.

It follows from (10.3), and an argument similar to the one at the end of §4, that the homomorphism, $\bar{f}_1 : \bar{\rho}_1 \to \bar{\rho}_1'$, which is associated with a chain equivalence, is necessarily an isomorphism.

12. Covering complexes. Let $\tilde{K}$ be a universal covering complex of a given complex, K, with base point $e^0 \in K^0$. Let $p: \tilde{K} \to K$ be the covering map and let a 0-cell $\tilde{e}^0 \in p^{-1}e^0$ be taken as base point in $\tilde{K}$. We shall show how a system of chain groups $\{C_n\}$, associated with $\rho(K)$, may be "realized" as a system of chain groups in $\tilde{K}$.

Our conventions concerning homology are as follows. Let X and $X_0 \subset X$ be given spaces. Then $H_n(X, X_0)$ will denote the nth relative homology group, with integral coefficients, which is defined in terms of (finite) singular chains. By an oriented n-element, E^n, we shall mean an n-element, E^n, associated with a generator, c, of $H_n(E^n, \partial E^n)$. Let E^n be thus oriented. Then a map

$$\theta: (E^n, \partial E^n) \to (X, X_0)$$

induces a homomorphism

$$\theta_*: H_n(E^n, \partial E^n) \to H_n(X, X_0)$$

and we shall described $\theta_* c$ as the element *represented* by θ. Similar observations apply to ∂E^n and maps $\partial E^n \to X_0$.

We orient I so that 0 is its first and 1 is its last point. We orient $I^n = I \times I^{n-1}$ $(n \geqq 1)$ and its faces by induction on n and the standard rule for orienting topological products. This gives

$$\partial I^{n+1} = 1 \times I^n - 0 \times I^n - I \times \partial I^n. \tag{12.1}$$

Following S. Eilenberg and N. E. Steenrod we take

$$C_n = C_n(K) = H_n(\tilde{K}^n, \tilde{K}^{n-1}) \qquad (n \geqq 0)$$

as the group of n-chains[22] in $\tilde{K}$, with $C_0(K) = H_0(\tilde{K}^0)$. We shall use

$$\tau(\bar{x}): \tilde{K} \to \tilde{K}$$

to denote the covering transformation (Deckbewegung), which is determined by $\bar{x} \in \bar{\rho}_1 = \pi_1(K)$. We shall also use $\bar{x}$ to denote the automorphism

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[22] See [11] and also [13, §17]. All our chains are finite because of (D) and (N) in in §5 of (I), from which it follows that any compact sub-set is contained in a finite sub-complex of $\tilde{K}$.

$$\bar{x}: C_n \to C_n,$$

which is induced by $\tau(\bar{x})$. It satisfies the condition $\partial\bar{x}=\bar{x}\partial$. The group C_n, with $\bar{x}: C_n \to C_n$ thus defined, is a free $\bar{\rho}_1$-module.

Let $\{e_i^n\}$ be the cells in K and let $\tilde{e}_i^n \in \tilde{K}$ be a cell which covers e_i^n. The cell $\tilde{e}_i^n$ may be any one of those which cover e_i^n unless $n=0$ and e_i^0 is the base point $e^0 \in K^0$. In this case $\tilde{e}_i^0$ shall be the base point $\tilde{e}^0 \in \tilde{K}^0$. Let $c_i^n \in C_n$ be an *elementary n-chain* associated with $\tilde{e}_i^n$, meaning the element represented by a characteristic map for $\tilde{e}_i^n$. Then $\{c_i^n\}$ is a basis, which we call a *natural basis*, for C_n.

We proceed to show that the groups C_n $(n>1)$ are related to the groups $\rho_n=\rho_n(K)$ in the way described in §8. If K^0 consists of a single 0-cell the same is true of C_0, C_1. In general we shall define sub-groups of C_0, C_1 which are thus related to ρ_0, ρ_1. We begin with some preliminary observations on lifted maps and homotopies.

Let $\phi: (K, e^0) \to (L, e'^0)$ be a map into a complex, L, with base point e'^0. Then there is a unique "lifted" map,

$$\tilde{\phi}: (\tilde{K}, \tilde{e}^0) \to (\tilde{L}, \tilde{e}'^0), \tag{12.2}$$

such that $\phi p = p\tilde{\phi}$, where $\tilde{L}$ is a universal covering complex of L and $\tilde{e}'^0 \in \tilde{L}^0$ is the base point in $\tilde{L}$. Similarly a homotopy $\phi_t: K \to L$, such that $\phi_0 e^0 = e'^0$, can be lifted into a unique homotopy, $\tilde{\phi}_t: \tilde{K} \to \tilde{L}$, such that $\phi_t p = p\tilde{\phi}_t$, $\tilde{\phi}_0\tilde{e}^0 = \tilde{e}'^0$. When we refer to a lifted map, $\tilde{\phi}$ or homotopy, $\tilde{\phi}_t$, it is always to be understood that $\tilde{\phi}\tilde{e}^0 = \tilde{e}'^0$ or $\tilde{\phi}_0\tilde{e}^0 = \tilde{e}'^0$. Let ϕ_t be lifted into $\tilde{\phi}_t$, and let $\phi_1 e^0 = e'^0$, in addition to $\phi_0 e^0 = e'^0$. Let $w \in \pi_1(L^1)$ be the element represented by $\theta: I \to L$, where $\theta(t) = \phi_t e^0$. Then $\tilde{\phi}_1\tilde{e}^0 = \tau(\bar{w})\tilde{e}'^0$, whence

$$\tilde{\phi}_1 = \tau(\bar{w})\tilde{\phi}_1', \tag{12.3}$$

where $\tilde{\phi}_1'$ is the map defined by lifting ϕ_1. It is inherent in the lifting process[23] that

$$\tilde{\phi}_t\tau(\bar{x}) = \tau(\bar{f}_1\bar{x})\tilde{\phi}_t, \tag{12.4}$$

where $\bar{x} \in \bar{\rho}_1$ and $\bar{f}_1: \pi_1(K) \to \pi_1(L)$ is the homomorphism induced by ϕ_0.

Let $\lambda: (I^n, I^0) \to (K^n, e^0)$ $(n \geqq 1)$ be a map which represents a given element $a \in \rho_n$. We define $h_n a \in C_n$ as the element represented by the lifted map $\tilde{\lambda}: I^n \to \tilde{K}^n$. If $n>1$, then $h_n: \rho_n \to C_n$ is the resultant of the "lifting" isomorphism, $\rho_n \to \tilde{\rho}_n = \rho_n(\tilde{K})$, followed by the homomorphism $\tilde{h}_n: \tilde{\rho}_n \to C_n$, in which corresponding elements are represented by the same map. The latter is onto and its kernel is the commutator subgroup of $\tilde{\rho}_n$, which is zero if $n>2$. Therefore h_n is onto and its kernel is the commutator sub-group of ρ_n. As in §8 above it follows that $h_1:\rho_1\to C_1$ is a crossed homomorphism and that

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[23] This is obvious if the lifting is defined in terms of paths joining e^0 to arbitrary points in K.

$$\partial_1 h_1 x = (\tilde{x} - 1)c^0, \tag{12.5}$$

where $c^0\in C_0$ is the elementary 0-chain associated with $\tilde{e}^0$.

We prove that h_n $(n\geqq 2)$ is an operator homomorphism. Let

$$\theta:(I, 0, 1) \to (K^1, e^0, e^0)$$

be a map, which represents a given element $x\in\rho$, and let $\tilde{\theta}$ be the lifted map. Then $\tilde{\theta}(1)=\tau(\tilde{x})\tilde{e}^0$. If $a\in\rho_n$ is represented by a map, $\lambda_1:I^n\to K^n$, then xa is represented by a map, $\lambda_0:I^n\to K^n$, which is related to λ_1 by a homotopy,

$$\lambda_t:(I^n, \partial I^n) \to (K^n, K^{n-1}),$$

such that $\lambda_t I^0=\theta(t)$. Let $\tilde{\lambda}_t$ be the lifted homotopy $(\tilde{\lambda}_0 I^0=\tilde{e}^0)$ and let $\tilde{\lambda}'$ be the map defined by lifting λ_1. Then $\tilde{\lambda}'$ represents $h_n a$ and, as in (12.3), $\tilde{\lambda}_1=\tau(\tilde{x})\tilde{\lambda}'$. Therefore $\tilde{\lambda}_1$ represents $\tilde{x}h_n a$. But $h_n(xa)$ is represented by $\tilde{\lambda}_0$ and hence also by $\tilde{\lambda}_1$, since an element of C_n is defined without reference to base points. Therefore

$$h_n(xa) = \tilde{x}h_n a \qquad (n \geqq 2). \tag{12.6}$$

We now prove that $\partial h=hd$. Let

$$\lambda:(I^n, \partial I^n, E^{n-1}) \to (K^n, K^{n-1}, e^0)$$

be a map which defines $a\in\rho_n$, where

$$E^{n-1} = \partial I^n - (I_1^{n-1} - \partial I_1^{n-1}) \qquad (I_1^{n-1} = 1 \times I^{n-1}).$$

Then $\beta_n a$ and $d_n a$ are represented by

$$\mu k:I^{n-1} \to K^{n-1},$$

where $\mu=\lambda\,|\,I_1^{n-1}$ and $kp=(1, p)$, as in §5. Let $\tilde{\lambda}$ be the map defined by lifting λ. Since E^{n-1} is connected it follows that $\tilde{\lambda}E^{n-1}=\tilde{e}^0$, whence $\tilde{\lambda}(\partial I_1^{n-1})=\tilde{e}^0$. Therefore μ lifts into $\tilde{\mu}=\tilde{\lambda}\,|\,I_1^{n-1}$ and μk lifts into $\tilde{\mu}k$. Therefore $\tilde{\mu}k$ represents $h_{n-1}d_n a$. The element $\partial_n h_n a$ is represented by $\tilde{\lambda}\,|\,\partial I^n$ or, since $\tilde{\lambda}E^{n-1}=\tilde{e}^0$ and $n-1>0$, by $\tilde{\mu}$ or by $\tilde{\mu}k$. Therefore

$$\partial_n h_n = h_{n-1}d_n \qquad (n \geqq 2). \tag{12.7}$$

Let $T\subset K^1$ be a tree, which is a sub-complex containing K^0, and let $\tilde{T}\subset\tilde{K}^1$ be the component of $p^{-1}T$ which contains the base point $\tilde{e}^0$. Let $C_r(\tilde{T})\subset C_r$ $(r=0, 1)$ be the group of r-chains carried by $\tilde{T}$.

Since $\pi_1(T)=1$ it follows that $p|\tilde{T}$ is a homeomorphism onto T. Therefore $C_r(\tilde{T})$ is an ordinary free Abelian group. Since $K^0\subset T$, a basis for $C_0(\tilde{T})$ is a basis for C_0. Let $C_1'\subset C_1$ be the (free) $\bar{\rho}_1$-module, which is generated[24] by a basis for $C_1(\tilde{T})$. If $K^1\neq T$ let $\{e_i^1\}$ be the 1-cells in K^1-T and let $g_i:I\to\bar{e}_i^1$ be a characteristic map for e_i^1. Let $\theta_i:I\to K^1$ be the map which consists of a path in T, leading from e^0 to $g_i(0)$, followed by g_i, followed by a path in T, which leads from $g_i(1)$ back to e^0. Let $x_i\in\rho_1$ be the element defined by θ_i. Then $\{x_i\}$ is a basis for ρ_1. Since $h_1x_i\in C_1$ is represented by the lifted map, $\tilde{\theta}_i$, it follows that

$$h_1x_i = c_i + c_i'$$

where c_i is defined by the map $\tilde{g}_i:I\to\tilde{K}^1$, such that $g_i=p\tilde{g}_i$, $\tilde{g}_i(0)\in\tilde{T}$, and $c_i'\in C_1'$. The set $\{c_i\}$, together with a natural basis, $\{c_\lambda'\}$, for C_1' constitutes a natural basis, $\{c_i, c_\lambda'\}$, for C_1. Therefore $\{h_1x_i, c_\lambda'\}$ is a basis for C_1. Hence C_1 is the direct sum

$$(12.8) \qquad C_1 = C_1^* + C_1',$$

where C_1^* is the (free) $\bar{\rho}_1$-module generated by $\{h_1x_i\}$. Let $C_0^*\subset C_0$ be the $\bar{\rho}_1$-module generated by the elementary 0-chain, c^0, associated with $\tilde{e}^0$. Let $C_n^*=C_n$ if $n>1$. Then $h_n\rho_n\subset C_n^*$ and we interpret h_n as a map $h_n:\rho_n\to C_n^*$ $(n\geq 1)$. It follows from (12.5), (12.6) and (12.7) that $\rho(K)$ is related by h_n to the system $C^*=\{C_n^*\}$ in the way described in §8. We shall describe C as the *complete system* and C^* as a *normalized system* of chain groups determined by K. In general C^* depends on the choice of T. If $K^0=T=e^0$ then $C^*=C$.

LEMMA 6. $C_1^*=\partial^{-1}C_0^*$.

It follows from (12.5) that $\partial_1C_1^*\subset C_0^*$. We have to prove that, if $\partial_1c\in C_0^*$ $(c\in C_1)$ then $c\in C_1^*$. Let $\{c_\lambda^0\}$ be the set of elementary 0-chains associated with the 0-cells in $\tilde{T}-\tilde{e}^0$. Let $v_\lambda=c_\lambda^0-c^0$. Then $\{c^0, v_\lambda\}$ is a basis for C_0. Therefore C_0 is the direct sum

$$(12.9) \qquad C_0 = C_0^* + C_0',$$

where C_0' has $\{v_\lambda\}$ for a basis. Since $\tilde{T}$ is a tree there is a basis, $\{c_\lambda'\}$, for $C_1(\tilde{T})$ such that $\partial_1c_\lambda'=v_\lambda$. Then $\{c_\lambda'\}$ is a basis for the summand C_1' in (12.8) and $\partial_1|C_1'$ is an isomorphism onto C_0'. Let $c=c^*+c'$, where $c^*\in C_1^*$, $c'\in C_1'$, and let $\partial_1c\in C_0^*$. Since $\partial_1C_1^*\subset C_0^*$, $\partial_1C_1'\subset C_0'$ it follows that $\partial_1c'=0$. Therefore $c'=0$ and $c\in C_1^*$.

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[24] I.e. generated by $\bar{\rho}_1$, operating on the basis elements in $C_1(\tilde{T})$.

13. Chain mappings: Proof of Theorems 5 and 6. Let $\phi: (K, e^0) \to (L, e'^0)$ be a given map of K in a complex L, with base point e'^0. Let

$$g_n : C(K) \to C(L) \qquad (n = 0, 1, \cdots)$$

be the homomorphism induced by the lifted map, $\tilde{\phi}: \tilde{K} \to \tilde{L}$. It follows from (12.4), with $\tilde{\phi}_t = \tilde{\phi}$ that each g_n is an operator homomorphism associated with the homomorphism, $\bar{f}_1: \pi_1(K) \to \pi_1(L)$, which is induced by ϕ. Also $g\partial = \partial g$ and we shall describe

$$g = \{g_n\} : C(K) \to C(L)$$

as the *chain mapping* induced by $\tilde{\phi}$. We shall also describe it as *induced*, or *realized geometrically* by ϕ. A chain mapping, $g: C(K) \to C(L)$, which need not have a geometrical realization, shall be defined as in §9, with modified restrictions on g_0. We required g_0 to map each elementary 0-chain on an elementary 0-chain and, except when the contrary is stated, c^0 on c'^0, where c^0, c'^0 are the elementary 0-chains associated with the base points. Chain mappings $C(K) \to C^*(L)$, $C^*(K) \to C(L)$, etc., shall satisfy the similar condition, where $C^*(K)$, $C^*(L)$ are normalized systems determined by K, L. Chain homotopy and chain equivalence shall be defined as in §§10, 11. We still have (10.2) and (10.3), with these modified definitions. It follows from (10.3) that the homomorphism, $\pi_1(K) \to \pi_1(L)$, associated with a chain equivalence, $C(K) \to C(L)$, is necessarily an isomorphism.

We now prove a very well known theorem, adapted to our set of definitions. Let

$$g, g' : C(K) \to C(L)$$

be the chain mappings induced by maps

$$\phi, \phi' : (K, e^0) \to (L, e'^0).$$

THEOREM 13. *If* $\phi \simeq \phi'$ *then* $g \simeq g'$.

Let $\phi_t: K \to L$ be a homotopy of $\phi_0 = \phi$ into $\phi_1 = \phi'$. Let $\lambda: I^n \to \tilde{K}^n$ be a map which represents a given element $c \in C_n(K)$ $(n \geqq 0)$ and let $\eta_{n+1} c \in C_{n+1}(L)$ be the element represented by $\tilde{\mu}: I^{n+1} \to \tilde{L}^{n+1}$, where[25]

$$\tilde{\mu}(t, t_1, \cdots, t_n) = \tilde{\phi}_t \lambda(t_1, \cdots, t_n).$$

As in the classical theory of chain homotopy it follows that the map $\eta_{n+1}: C_n(K) \to C_{n+1}(L)$, which is thus defined, is a homomorphism. It follows from (12.4) that it is an operator homomorphism associated with the homomorphism $\bar{f}_1:\pi_1(K)\to\pi_1(L)$, which is induced by ϕ_0. It follows from (12.1), carried into $\tilde{L}$ by the map $\tilde{\mu}$, and from (12.3) that

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[25] $\tilde{\mu} \partial I^{n+1} \subset \tilde{L}^n$ since ϕ_t, and hence $\tilde{\phi}_t$, is cellular.

$$\partial_{n+1}\eta_{n+1}c = (\bar{w}g_n' - g_n - \eta_n\partial_n)c,$$

or that $\bar{w}g'-g=\partial\eta+\eta\partial$, where $\bar{w}$ means the same as in (12.3). This proves Theorem 13.

THEOREM 14. *A map $\phi:K\to L$ is a homotopy equivalence if, and only if, the induced chain mapping, $g: C(K)\to C(L)$, is a chain equivalence.*

Let ϕ be a homotopy equivalence. Then it follows from Theorem 12, by a familiar argument, that g is a chain equivalence.

Conversely, let g be a chain equivalence. Then the associated homomorphism, $\bar{f}_1:\pi_1(K)\to\pi_1(L)$, which is the one induced by ϕ, is an isomorphism. Also the induced homomorphism,[26] $H_n(\tilde{K})\to H_n(\tilde{L})$, is an isomorphism, for each $n\geqq 0$. Therefore Theorem 14 follows from Theorem 3 in (I).

Let $K^0=e^0$, $L^0=e'^0$, so that $C(K)=C^*(K)$, $C(L)=C^*(L)$, and let $f:\rho(K)\to\rho(L)$ be the homomorphism induced by $\phi:K\to L$. Then it is to be expected, and is in fact proved in Lemma 9 below, that the chain mapping, $g: C(K)\to C(L)$, which is induced by ϕ, is the one induced by f. Therefore Theorems 5, 6 follow at once from Theorems 13, 14 and Theorems 10, 12. In general there is a gap between Theorems 13, 14 and Theorems 10, 12, which we bridge by means of three lemmas concerning $C^*(K)$ and $C^*(L)$.

The identical map $\boldsymbol{i}: C^*(K)\to C(K)$ is obviously a chain mapping.

LEMMA 7. *$\boldsymbol{i}: C^*(K)\to C(K)$ is a chain equivalence.*

Using the same notation as in the proof of Lemma 6, let $k: C(K)\to C^*(K)$ be the chain mapping which is given by $k\,|\,C^*(K)=1$, $kc_\lambda'=0$, $kv_\lambda=0$. Then $k\boldsymbol{i}=1$ and $\boldsymbol{i}k-1=\partial\eta+\eta\partial$, where η is given by $\eta v_\lambda=-c_\lambda'$, $\eta c^0=0$, $\eta C_n(K)=0$ if $n>0$. This proves the lemma.

Let $\boldsymbol{i}$, k mean the same as before, both in K and L. Let g, $g': C(K)\to C(L)$ be any chain mappings and let

$$g^* = kg\boldsymbol{i}: C^*(K)\to C^*(L), \qquad g'^* = kg'\boldsymbol{i}. \tag{13.1}$$

LEMMA 8. *$g^*\simeq g'^*$ if, and only if, $g\simeq g'$.*

It follows from (13.1) that $g^*\simeq g'^*$ if $g\simeq g'$. Let $g^*\simeq g'^*$. Then

$$g\simeq \boldsymbol{i}kg\boldsymbol{i}k = \boldsymbol{i}g^*k\simeq \boldsymbol{i}g'^*k\simeq g',$$

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[26] There is an isomorphism, $H_n(\tilde{K})\approx\partial_n^{-1}(0)-\partial_{n+1}C_{n+1}(K)$, which is natural, as the term is used in §38 (p. 815) of [9].

which proves the lemma.

Let $f:\rho(K)\to\rho(L)$ be the homomorphism and $g: C(K)\to C(L)$ the chain mapping induced by a map $\phi: K\to L$. We shall also describe g^*, given by (13.1), as *induced* by ϕ.

Lemma 9. *$g^*: C^*(K)\to C^*(L)$ is the chain mapping induced by $f:\rho(K)\to\rho(L)$.*

Since $gc^0 = c'^0$ it follows that $g_0 C_0^*(K)\subset C_0^*(L)$. Also $\partial_1 c^*\in C_0^*(K)$ if $c^*\in C_1^*(K)$. Therefore

$$\partial_1 g_1 c^* = g_0 \partial_1 c^* \in C_0^*(L)$$

and it follows from Lemma 6 that $g_1 C_1^*(K)\subset C_1^*(L)$. Since i is the identity and $k| C^*(L) = 1$ it follows that g^*, in (13.1), is given by $g_n^* c^* = g_n c^*$ if $c^*\in C_n^*(K)$. Therefore $g_n h_n a = g_n^* h_n a$ for each $a\in\rho_n(K)$ and $n\geq 1$.

Let $\lambda: I^n\to K^n$ be a map representing $a\in\rho_n(K)$. Then $f_n a$ is represented by $\phi\lambda=\mu$, say, and $h_n f_n a$ by the lifted map $\tilde{\mu}$. Obviously $\tilde{\mu}=\tilde{\phi}\tilde{\lambda}$, where $\tilde{\phi}$, $\tilde{\lambda}$ are defined by lifting ϕ, λ. But $\tilde{\phi}\tilde{\lambda}$ represents the element $g_n h_n a = g_n^* h_n a$. Therefore $hf = g^* h$ and the lemma is proved.

Proof of Theorem 5. Let the notation be the same as before and let $\phi\simeq\phi': K\to L$. Then it follows from Theorem 13 and Lemma 8 that $g^*\simeq g'^*$ and from Lemma 9 and Theorem 10 that $f\simeq f'$. This proves Theorem 5.

Proof of Theorem 6. Since i, k are chain equivalences it follows that g^*, given by (13.1), is a chain equivalence if, and only if, g is a chain equivalence. Therefore Theorem 6 follows from Theorem 14, Lemma 9 and Theorem 12.

As an analogue to the corollary to Theorem 6 we have:

Theorem 15. *If* $\dim K\leq m+1$ *and if L is a J_m-complex, then $K\equiv L$ if, and only if, $C(K)\equiv C(L)$. More precisely, any chain equivalence, $C(K)\to C(L)$, can be realized by a homotopy equivalence, $K\to L$, and conversely.*

Since $k: C(K)\to C^*(K)$ and $i: C^*(L)\to C(L)$ are chain equivalences this follows from the corollary to Theorem 6, from Lemma 9 and from Theorem 12.

We conclude the section with a theorem on the realizability of chain mappings, which is analogous to Theorem 4. Let $g: C(K)\to C(L)$ be a chain mapping associated with a homomorphism $\bar{f}:\pi_1(K)\to\pi_1(L)$. Let $K_0\subset K$ be a connected sub-complex, which contains e^0. Let

$$i:\pi_1(K_0)\to\pi_1(K), \qquad j: C(K_0)\to C(K)$$

be the homomorphism and the chain mapping, associated with $\mathbf{i}$, which are induced by the identical map $K_0 \rightarrow K$. We describe a map, $\phi_0 : K_0 \rightarrow L$, as a *partial realization* of g if, and only if, it induces the chain mapping

$$gj : C(K_0) \rightarrow C(L).$$

The chain mapping gj is associated with $\bar{f}^0 = \bar{f}\bar{\imath} : \pi_1(K_0) \rightarrow \pi_1(L)$. Therefore, if ϕ_0 induces gj, the homomorphism $\bar{f}^0$ is the one induced[27] by ϕ_0.

Let $\phi_0 : K_0 \rightarrow L$ be a partial realization of g. Let $\bar{K}_0$, with a base point covering e^0, be a universal covering space of K_0. Then gj is the chain mapping induced by the lifted map $\bar{\phi}_0 : \bar{K}_0 \rightarrow \bar{L}$. Let $\tilde{K}_0 \subset \tilde{K}$ be the component of $p^{-1}K_0$ which contains $\tilde{e}^0$. Then $\tilde{K}_0$ is a covering complex of K_0, associated with the sub-group $\bar{\imath}^{-1}(1) \subset \pi_1(K_0)$, and $\bar{K}_0$ is a universal covering complex of $\tilde{K}_0$, with a covering map $\bar{p} : \bar{K}_0 \rightarrow \tilde{K}_0$. Since $\bar{f}^0\bar{\imath}^{-1}(1) = \bar{f}\bar{\imath}\bar{\imath}^{-1}(1) = 1$ the map ϕ_0 can be lifted[23] into a map $\tilde{\phi}_0 : \tilde{K}_0 \rightarrow \tilde{L}$, and $\bar{\phi}_0$ is the map obtained by lifting $\tilde{\phi}_0$ into $\bar{K}$. Therefore

$$\bar{\phi}_0 = \tilde{\phi}_0\bar{p} : \bar{K}_0 \rightarrow \bar{L}$$

and we may regard gj as the chain mapping which is induced by ϕ_0. We state this, in a slightly different form, as:

LEMMA 10. *If* $c \in C_n(K)$ *has a representative map,* $\bar{\lambda} : I^n \rightarrow \tilde{K}_0^n$, *then* $g_n c$ *is represented by the map* $\tilde{\phi}_0\bar{\lambda} : I^n \rightarrow \tilde{L}^n$.

THEOREM 16. *If* $K_0 \cup K^{m+1} = K$ *and if* L *is a* J_m*-complex, then any partial realization,* $\phi_0 : K_0 \rightarrow L$, *of a given chain mapping,* $g : C(K) \rightarrow C(L)$, *can be extended to a full realization,* $\phi : K \rightarrow L$.

First assume that $K^1 \subset K_0$. Then $K_0^1 = K^1$, $\tilde{K}_0^1 = \tilde{K}^1$ and[28] $\rho_1(K_0) = \rho_1(K)$. Let $f^0 : \rho(K_0) \rightarrow \rho(L)$ be the homomorphism induced by ϕ_0 and let $g^* : C^*(K) \rightarrow C^*(L)$ mean the same as in (13.1). Let

$$\lambda : (I, 0, 1) \rightarrow (K^1, e^0, e^0)$$

be a map representing a given element $x \in \rho_1(K)$ and let $\bar{\lambda} : I \rightarrow \tilde{K}^1 = \tilde{K}_0^1$ be the lifted map. Then it follows from Lemma 10 that $g_1^* h_1 x = g_1 h_1 x$ is represented by the map $\tilde{\phi}_0\bar{\lambda}$, which also represents $h_1 f_1^0 x$. Therefore $h_1 f_1^0 = g_1^* h_1$. Hence it follows from the addendum to Theorem 9 that there is a realization, $f : \rho(K) \rightarrow \rho(L)$, of g^*, such that $f_1 = f_1^0$.

I say that ϕ_0 is a partial realization of f, or that $f^0 = f\mathbf{i}$, where $\mathbf{i} : \rho(K_0) \rightarrow \rho(K)$ is the injection homomorphism. For, by Lemma 9, $\mathbf{i}$ is a realization of the chain mapping, $j^*: C^*(K_0) \to C^*(K)$, which is induced by the identical map $K_0 \to K$. Therefore fi is a realization of g^*j^*. Since g^* is also given by $g^*c^* = gc^*$ $(c^* \in C^*(K))$ it follows that $g^*j^* = (gj)^*$. But $(gj)^*$ is induced by ϕ_0 and therefore by f^0, according to Lemma 9. Therefore both fi and f^0 are realizations of $(gj)^*$. Since $i_1 = 1$ we have $(fi)_1 = f_1 = f_1^0$. Therefore $f^0 = fi$, by the addendum to Theorem 9. Hence it follows from Theorem 4 that ϕ_0 can be extended to a map, $\phi: K \to L$, which realizes f and hence g^*. Let $\tilde{\phi}$ be the lifted map. Then $\tilde{\phi} | \tilde{K}_0 = \tilde{\phi}_0$, since $\phi | K_0 = \phi_0$. Since $\tilde{K}_0^1 = \tilde{K}^1$ it follows from Lemma 10 that $\tilde{\phi}$ induces the homomorphisms g_0, g_1. Therefore ϕ induces g. Thus the theorem is proved on the assumption that $K^1 \subset K_0$.

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[27] Cf. a remark following (2.2).

[28] In general $C_1(K_0) \neq C_1(K)$ since $C_1(K_0)$ refers to $\bar{K}_0$.

In general let $K_1 = K_0 \cup K^1$. The theorem will follow from what we have already proved when we have extended ϕ_0 to a partial realization, $\phi_1: K_1 \to L$, of g. We first extend $\tilde{\phi}_0: \tilde{K}_0 \to \tilde{L}$ to a map $\psi: p^{-1}K_0 \to \tilde{L}$ by writing

$$\psi\tau(\bar{x})p = \tau(\bar{f}\bar{x})\tilde{\phi}_0 p \qquad (p \in \tilde{K}_0),$$

for every $\bar{x} \in \pi_1(K)$. We then extend ψ to a map

$$\psi': (p^{-1}\tilde{K}_0) = \tilde{K}^0 \to \tilde{L},$$

which induces g_0. This is possible since g_0 maps elementary 0-chains into elementary 0-chains. Moreover

$$\psi'\tau(x) \cup \tau(\bar{f}\bar{x})\psi',$$

since $g_0\bar{x} = (\bar{f}\bar{x})g_0$.

Let e^1 be any 1-cell in $K^1 - K_0$. Let $\tilde{e}^1 \in \tilde{K}^1$ be a 1-cell which covers e^1 and let $c^1 \in C_1(K)$ be an elementary 1-chain associated with $\tilde{e}^1$. Then $\partial c^1 = v - u$, where $u, v \subset C_0(K)$ are the elementary 0-chains associated with the end points, $p, q \subset \tilde{K}^0$, of $\tilde{e}^1$. Also

$$\partial g_1 c^1 = g_0 \partial_1 c^1 = g_0 v - g_0 u$$

and $g_0 u$, $g_0 v$ are the elementary 0-chains associated with $\psi' p$, $\psi' q$. Therefore it follows from an argument similar to the proof of Theorem 8(c) that $g_1 c^1$ is represented by a map

$$\mu: (I, 0, 1) \to (\tilde{L}^1, \psi' p, \psi' q).$$

If $p = q$ then $\psi' p = \psi' q$, and it follows that the map

$$\theta^1: \mu\lambda^{-1}: \bar{\tilde{e}}^1 \to \tilde{L}^1$$

is single-valued and continuous, where $\lambda: I \to \bar{\tilde{e}}^1$ is a characteristic map for $\tilde{e}^1$. Therefore ψ' may be extended to a map, $\tilde{\phi}_1: p^{-1}K_1 \to \tilde{L}$, by writing

$$\tilde{\phi}_1 \mid \tau(\bar{x})\bar{e}^1 = \tau(\bar{f}\bar{x})\theta_{e^1},$$

for every 1-cell $e^1 \in K^1 - K_0$ and every $\bar{x} \in \pi_1(K)$. Then $\tilde{\phi}_1$ induces g_0 and g_1. Clearly $\tilde{\phi}_1\tau(\bar{x}) = \tau(\bar{f}\bar{x})\tilde{\phi}_1$ and the required map, $\phi_1 : K \to L$, is defined by $\phi_1 p = p\tilde{\phi}_1$. This completes the proof.

14. Proof of Theorem 7. Let $P = K \times I$. Then $\tilde{P} = \tilde{K} \times I$ is a universal covering complex of P and we take $(e^0, 0)$ and $(\tilde{e}^0, 0)$ as base points in P and $\tilde{P}$. The n-cells in $\tilde{P}$ are the cells $\tilde{e}^n \times 0$, $\tilde{e}^n \times 1$, $\tilde{e}^{n-1} \times (0, 1)$ for all n-cells and $(n-1)$-cells in $\tilde{K}$. Let $\tilde{\theta}_t : \tilde{K} \to \tilde{P}$ be the homotopy, which is given by $\tilde{\theta}_t \tilde{q} = (\tilde{q}, t)$ $(\tilde{q} \in \tilde{K})$. Let α^0, $\alpha^1 : C(K) \to C(P)$ be the chain mappings[29] induced by $\tilde{\theta}_0$, $\tilde{\theta}_1$ and let

$$\gamma : C_n(K) \to C_{n+1}(P)$$

be the chain deformation operator which is defined as in Theorem 13, with $\tilde{L} = \tilde{P}$, $\tilde{\phi}_t = \tilde{\theta}_t$. Then $(\alpha^0 c^n_\lambda, \alpha^1 c^n_\lambda, \gamma c^{n-1}_\mu)$ is obviously a basis for $C_n(P)$, where $\{c^n_\lambda\}$, $\{c^{n-1}_\mu\}$ are bases for $C_n(K)$, $C_{n-1}(K)$. The operator $\partial : C_n(P) \to C_{n-1}(P)$ is defined by

$$\begin{aligned} \partial \alpha^{\nu} c^{n}_{\lambda} &= \alpha^{\nu} \partial c^{n}_{\lambda} \qquad (\nu = 0, 1), \\ \partial \gamma c^{n-1}_{\mu} &= (\alpha^{1} - \alpha^{0} - \gamma\partial) c^{n-1}_{\mu}, \end{aligned} \tag{14.1}$$

which are the same as $\partial\alpha^\nu = \alpha^\nu\partial$, $\alpha^1 - \alpha^0 = \partial\gamma + \gamma\partial$.

Let $g^\nu : C(K) \to C(L)$ $(\nu = 0, 1)$ be chain mappings, which are related by

$$\bar{w} g^1 - g^0 = \partial\eta + \eta\partial, \tag{14.2}$$

where $\eta : C(K) \to C(L)$ is a chain deformation operator and $\bar{w} \in \pi_1(L)$. Then it follows from (14.1) and (14.2) that $g : C(P) \to C(L)$ is a chain mapping, where g is defined by

$$g\alpha^0 = g^0, \qquad g\alpha^1 = \bar{w} g^1, \qquad g\gamma = \eta. \tag{14.3}$$

Conversely (14.2) follows from (14.1), (14.3) and $g\partial = \partial g$.

Now let g^ν have a geometrical realization, $\phi^\nu : K \to L$, let g^0, g^1 be related by (14.2) and let

$$\psi : (K \times 0) \cup (e^0 \times I) \cup (K \times 1) \to L$$

be given by $\psi(q, \nu) = \phi^\nu q$ $(q \in K)$, $\psi(e^0, t) = \lambda(t)$, where $\lambda : I \to L^1$ represents $\bar{w}$. Then it is easily verified that ψ is a partial realization of $g : C(P) \to C(L)$. If dim $K \leq m$, whence dim $P \leq m + 1$, and if L is a

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[29] N.B. $\alpha^1 c^0 \neq c'^0$, where c^0, c'^0 are the elementary 0-chains associated with the base points $\tilde{e}^0 \in \tilde{K}^0$ and $(\tilde{e}^0, 0)$.

J_m-complex it follows from Theorem 16 that ψ can be extended throughout P, whence $\phi^0 \simeq \phi^1$. Thus $\phi^0 \simeq \phi^1$ if $g^0 \simeq g^1$.

Theorem 7 now follows from Lemmas 8, 9 and Theorem 10.

15. Examples. Let P and Q be 3-dimensional lens spaces (see [14, pp. 210, 279]) of types (m, p) and (m, q) where m is the order of $\pi_1(P)$ and $\pi_1(Q)$. It was proved (see also [15]) in [3] that, if there is an integer, k, such that $q \equiv \pm k^2 p$ (m), then $P \equiv Q$. We give an alternative proof of this by exhibiting a chain equivalence $u: C(P) \to C(Q)$.

As usual we suppose that P, Q are complexes such that $C_n(P)$, $C_n(Q)$ have bases consisting of single elementary n-chains,[30] a_n, b_n $(n=0, \cdots, 3)$, and that

$$(15.1)\qquad \begin{aligned} \partial a_1 &= (\xi - 1)a_0, & \partial a_2 &= \sigma_m(\xi)a_1, & \partial a_3 &= (\xi^p - 1)a_2,\\ \partial b_1 &= (\eta - 1)b_0, & \partial b_2 &= \sigma_m(\eta)b_1, & \partial b_3 &= (\eta^q - 1)b_2, \end{aligned}$$

where ξ, η generate $\pi_1(P)$, $\pi_1(Q)$ and

$$\sigma_t(\zeta^s) = 1 + \zeta^s + \cdots + \zeta^{(t-1)s} \qquad (\zeta = \xi \text{ or } \eta).$$

Let $q \equiv -k^2 p$ (m). Then we replace b_3 by $-b_3$ and b_2 by $\eta^q b_2$. Since $\eta^q \sigma_m(\eta) = \sigma_m(\eta)$ we have

$$\partial(-b_3) = (\eta^{-q} - 1)\eta^q b_2, \qquad \partial(\eta^q b_2) = \sigma_m(\eta) b_1.$$

Therefore this has the effect of replacing q by $-q$ and we assume that $q \equiv k^2 p$. Since $(p, m) = (q, m) = 1$ we have $(k, m) = 1$ and there are integers, h, l, such that $kl = 1 + hm$. Then $p \equiv l^2 q$ (m).

Let

$$u: C(P) \to C(Q), \qquad v: C(Q) \to C(P)$$

be the chain mappings, which are associated with the mutually inverse isomorphisms $\xi \to \eta^k$, $\eta \to \xi^l$, and are given by $ua_n = b_n$, $vb_n = a_n$ if $n = 0$ or 3 and

$$(15.2)\qquad \begin{aligned} ua_1 &= \sigma_k(\eta) b_1, & ua_2 &= \sigma_k(\eta^{kp}) b_2,\\ vb_1 &= \sigma_l(\xi) a_1, & vb_2 &= \sigma_l(\xi^{lq}) a_2. \end{aligned}$$

It is easily verified that $\partial u = u\partial$, $\partial v = v\partial$.

I say that, if s, t are integers such that $lt \equiv s$ (m), then

$$(15.3)\qquad \sigma_k(\xi^s)\sigma_l(\xi^t) = 1 + h\sigma_m(\xi).$$

For let $\omega_0 = 1, \omega_1, \cdots, \omega_{m-1}$ be the mth roots of unity and let χ_r be the character of $\pi_1(P)$, which is given by $\chi_r(\xi) = \omega_r$. It follows from the orthogonality relations,

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[30] In these examples the dimensionality of a chain is indicated by a subscript.

$$\sum_{r=0}^{m-1} \bar{\chi}_r(\xi^i)\chi_r(\xi^j) = m\delta^{ij} \qquad (i, j = 0, \cdots, m-1),$$

that $\lambda=0$ if $\chi_r(\lambda)=0$ for every $r=0, \cdots, m-1$, where λ is in the group ring of $\pi_1(P)$. But

$$\chi_0\{\sigma_k(\xi^s)\sigma_l(\xi^t)\} = kl = 1 + hm = \bar{\chi}_0\{1 + h\sigma_m(\xi)\}.$$

Also $ks \equiv klt \equiv t$ (m) and $\chi_r\{\sigma_m(\xi)\}=0$ if $r\neq 0$, whence

$$\begin{aligned}\chi_r\{\sigma_k(\xi^s)\sigma_l(\xi^t)\} &= \frac{\omega_r^{ks} - 1}{\omega_r^s - 1}\cdot\frac{\omega_r^{lt} - 1}{\omega_r^t - 1} = 1 \\ &= \bar{\chi}_r\{1 + h\sigma_m(\xi)\},\end{aligned}$$

which proves (15.3).

It follows from (15.2) and (15.3), first with $s=l$, $t=1$ and secondly with $s=klp\equiv p$, $t=lq$, that

$$\begin{aligned} vua_1 &= v\sigma_k(\eta)b_1 = \sigma_k(\xi^l)\sigma_l(\xi)a_1 \\ &= \{1 + h\sigma_m(\xi)\}a_1, \\ vua_2 &= v\sigma_k(\eta^{kp})b_2 = \sigma_k(\xi^{klp})\sigma_l(\xi^{lq})a_2 \\ &= \{1 + h\sigma_m(\xi)\}a_2. \end{aligned}$$

Therefore $vu-1=\partial\delta+\delta\partial$, where $\delta a_0=0$, $\delta a_1=ha_2$, $\delta a_2=0$, whence $vu\simeq 1$. Similarly $uv\simeq 1$ and it follows from Theorem 15 that u can be realized by a homotopy equivalence $P\rightarrow Q$.

Our next example is of a 5-dimensional homotopy system, $\rho=\{\rho_n\}$ which has no geometrical realization. It is defined as follows.

(a) ρ_1 has a single basis element x,

(b) ρ_2 has two basis elements, a_2, b_2 and $da_2=x^2$, $db_2=1$,

(c) ρ_n $(n=3, 4, 5)$ has a single basis element, a_n,

(d) $da_3=(\xi-1)b_2$, $da_4=(\xi+1)a_3$, $da_5=(\xi-1)a_4$, where $\xi=\bar{x}$.

Then $dd=0$ since $\xi^2=1$. We shall prove that ρ has no geometrical realization.

We can realize the system $(\rho_1, \cdots, \rho_4, 0, \cdots)$ by the topological product $Q^4=P^2\times S^2$, where $P^2=e_0^0\cup e_0^1\cup e_0^2$ and $S^2=e_1^0\cup e_1^2$ are complexes covering a real projective plane and a 2-sphere respectively. Assume that ρ has a realization K_0. Then Q^3 and K_0^3 are two realizations of the system $\rho^3=(\rho_1, \rho_2, \rho_3, 0, \cdots)$. By Theorem 4 there is a map $\phi: K_0^3\rightarrow Q^3$, which realizes the resultant, $\rho(K_0^3)\rightarrow\rho(Q^3)$, of given isomorphisms $\rho(K_0^3)\rightarrow\rho^3$ and $\rho^3\rightarrow\rho(Q^3)$. Let K be the complex formed by identifying each point $p\in K_0^3$ with $\phi p\in Q^3$. Then $K^3=Q^3$ and the map $\psi: K_0 \to K$, which is given by $\psi | K_0^3 = \phi$, $\psi | K_0 - K_0^3 = 1$, obviously induces an isomorphism $\rho(K_0) \to \rho(K)$. Therefore K is another realization of ρ. Let us take $\rho = \rho(K)$. It is easily verified that any basis for ρ_n ($n = 4, 5$) must consist of a single element, which is of the form $\pm \xi^r a_n$ ($r = 0$ or 1). Therefore K has but a single n-cell, e^n. Let a_n be represented by a characteristic map for e^n. We assume, as we obviously may, that a_2, b_2, a_3 are represented by characteristic maps for cells in Q^3. Since $(\xi - 1)a_4 = da_5 = j\beta a_5$, where j, β mean the same as in §5, and since $\beta j = 0$ we have

$$(\xi - 1)\beta a_4 = \beta(\xi - 1)a_4 = 0. \tag{15.4}$$

It follows from the condition (b) that $d_2^{-1}(1)$, in ρ_2, is a free Abelian group, which is freely generated by $(\xi - 1)a_2$, b_2, ξb_2. Moreover $j: \pi_2(Q^2) \to \rho_2$ is an isomorphism onto $d_2^{-1}(1)$ and we identify each $a \in \pi_2(Q^2)$ with $ja \in \rho_2$. Let $\tilde{P}^2$ be a universal covering complex of P^2. Then $\tilde{Q}^4 = \tilde{P}^2 \times S^2$, with a base point, $\tilde{e}^0 = (\tilde{e}_0^0, e_1^0)$ covering (e_0^0, e_1^0), is a universal covering complex of Q^4. We have

$$\tilde{Q}^2 = S_0^2 \times S_1^2 \times \tau(\xi) S_1^2,$$

where $\tau(\xi)$ means the same as in §12 and $S_0^2 = \tilde{P}^2 \times \tilde{e}_1^0$, $S_1^2 = \tilde{e}_0^0 \times S^2$ and S_1^2 and $\tau(\xi) S_1^2$ are attached to S_0^2 at the points $\tilde{e}^0$ and $\tau(\xi)\tilde{e}^0$. The elements $(\xi - 1)a_2$, b_2, ξb_2 are represented by homomorphisms

$$\partial I^3 \to S_0^2, \qquad \lambda: \partial I^3 \to S_1^2, \qquad \tau(\xi)\lambda: \partial I^3 \to \tau(\xi) S_1^2,$$

projected into Q^2 by the covering map $p: \tilde{Q}^4 \to Q^4$. Let γ be a generator of $\pi_3(\partial I^3)$ and, if $a \in \pi_2(X)$, where X is any space, let $a \cdot \gamma \in \pi_3(X)$ be the element respresented by the resultant of a map $\partial I^4 \to \partial I^3$, representing γ, followed by a map $\partial I^3 \to X$, representing a. Then it follows from an easy extension of Theorem 2 in [2], applied to $\tilde{Q}^2$ and transferred to Q^2 by means of $p: \tilde{Q}^4 \to Q^4$, that $\pi_3(Q^2)$ is a free Abelian group, which is freely generated by

$$u = (\xi a_2 - a_2) \cdot \gamma, \qquad v = b_2 \cdot \gamma, \qquad (\xi b_2) \cdot \gamma,$$

together with

$$[(\xi - 1)a_2, b_2], \qquad [(\xi - 1)a_2, \xi b_2], \qquad [b_2, \xi b_2],$$

where $[a, b]$ means the same as $a \cdot b$ in [2]. Obviously $\xi(a \cdot \gamma) = (\xi a) \cdot \gamma$. Also[31] $(-\bar{a})\ \gamma = \bar{a}\ \gamma$ if $\bar{a} \in \pi_2(S_0^2)$. Therefore

$$\xi u = (a_2 - \xi a_2) \cdot \gamma = (\xi a_2 - a_2) \cdot \gamma = u.$$

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[31] See Theorem IIb′ on p. 639 of [16].

Let i denote both injection homomorphisms $i:\pi_n(Q^2)\rightarrow\pi_n(Q^3)$ $(n=2, 3)$. Since $j_2=1$ we have

$$\beta a_3 = da_3 = (\xi - 1)b_2.$$

Therefore $i\xi b_2=ib_2$, $i\xi v=(i\xi b_2)\cdot\gamma=iv$ and

$$i[a, \xi b_2] = [ia, i\xi b_2] = [ia, ib_2] = i[a, b_2],$$

for any $a\in\pi_2(Q^2)$. Also a calculation of the Hopf invariant shows that $[b_2, b_2]= \pm 2v$. Therefore $i\pi_3(Q^2)$ is generated by $\bar{u}$, $\bar{v}$, $\bar{w}$, where

$$\bar{u} = iu, \qquad \bar{v} = iv, \qquad \bar{w} = iw, \qquad w = [(\xi - 1)a_2, b_2].$$

Notice that $\xi\bar{u}=i\xi u=\bar{u}$, $\xi\bar{v}=\bar{v}$.

Let $b_4\in\rho_4(Q^4)$ be the element which is represented by a characteristic map for $e_0^2\times e_1^2$, in the sense class which makes $db_4=da_4=(\xi+1)a_3$. Let $e'^2=\tilde{P}^2-\tilde{e}_0^0$. Then

$$S'^2 = \tilde{e}_0^0\cup e'^2, \qquad Q'^4 = S'^2\times S^2$$

are complexes of which $\tilde{P}^2$, $\tilde{Q}^4$ are sub-divisions. Let a_2', $b_2'\subset\rho_2(Q'^2)$ $b_4'\in\rho_4(Q'^4)$ be the elements represented by characteristic maps for $e'^2\times e_1^0$, $\tilde{e}_0^0\times e_1^2$ and $e'^2\times e_1^2$. Then it follows from the definition of $[a, b]$ that $\beta b_4' = \pm [a_2', b_2']$. Also the homomorphism $\rho(Q'^4)\rightarrow\rho(Q^4)$, which is induced by the identical map $Q'^4\rightarrow\tilde{Q}^4$, followed by $p:\tilde{Q}^4\rightarrow Q^4$, carries a_2', b_2', b_4' into $\pm(\xi-1)a_2$, $\pm b_2$, $\pm(\xi-1)b_4$. Therefore $\beta(\xi-1)b_4 =\pm\bar{w}$.

I say that $\bar{w}$ is not of finite order. For

$$\pi_4(Q^4) \approx \pi_4(S'^2) + \pi_4(S^2),$$

whence $\pi_4(Q^4)$ is of order 4. Since $\rho_4(Q^4)$ is a free Abelian group it follows that $j\pi_4(Q^4)=0$. Therefore it follows from the exactness of the sequence

$$\pi_4(Q^4) \xrightarrow{j} \rho_4(Q^4) \xrightarrow{\beta} \pi_3(Q^3)$$

that β is an isomorphism into $\pi_3(Q^3)$. Since $(\xi-1)b_4$ is not of finite order neither is $\bar{w}$.

We now contradict (15.4). Since $j(\beta a_4-\beta b_4)=da_4-db_4=0$ it follows that $\beta a_4-\beta b_4\in i\pi_3(Q^2)$. Therefore

$$\begin{aligned}\beta a_4 &= \beta b_4 + l\bar{u} + m\bar{v} + n\bar{w}\\ &= \beta b_4 + l\bar{u} + m\bar{v} \pm n\beta(\xi - 1)b_4\\ &= \{1 \pm n(\xi - 1)\}\beta b_4 + l\bar{u} + m\bar{v},\end{aligned}$$

where l, m, n are integers. Since $\xi^2=1$ we have $(\xi-1)^2=2(\xi-1)$ and

$$(\xi - 1)\{1 \pm n(\xi - 1)\} = (1 \pm 2n)(\xi - 1).$$

Also $(\xi - 1)\bar{u} = (\xi - 1)\bar{v} = 0$. Therefore

$$(\xi - 1)\beta a_4 = (1 \pm 2n)(\xi - 1)\beta b_4 = (1 \pm 2n)\bar{w} \neq 0.$$

Therefore it is absurd to suppose that ρ has a geometrical realization.

The above argument serves as an example of the use of the groups $\pi_n(K^n)$ and $j^{-1}(0) = i\pi_n(K^{n-1})$, which is discussed in §17 below.

16. Note on $\rho_n(K)$. Let $\rho = \rho(K)$, where K is a given complex. According to (D) in §5 of (I) any compact sub-set of K is contained in a finite sub-complex of K. Therefore the fact that ρ_n is a free $\bar{\rho}_1$-module if $n > 2$ follows from arguments on p. 417 of [2], restated in terms of relative homotopy groups. Alternatively one may prove the result for $H_n(\tilde{K}^n, \tilde{K}^{n-1})$, using the Eilenberg-Steenrod "excision" Theorem, where $\tilde{K}$ is a universal covering complex of K.

We now show how the arguments on pp. 422–425 of [2] can be used to prove that ρ_2 is a free crossed (ρ_1, d_2)-module. More generally let

$$X = X_0 \cup \{e_\lambda^2\}$$

where the cells $\{e_\lambda^2\}$ are attached to X_0 as in [2]. We allow this set of cells to be infinite provided X is a Hausdorff space such that

(a) X_0 is an arcwise connected, closed sub-set of X, and

(b) $Y \subset X$ is closed if $Y \cap X_0$ and all the sets $Y \cap \bar{e}_\lambda^2$ are closed.

Let this be so. Then it follows from the proof of (D) in §5 of (I) that any compact sub-set of X lies in the union of X_0 and a finite sub-set of the cells $\{e_\lambda^2\}$.

Let $\rho_2 = \pi_2(X, X_0)$, $\rho_1 = \pi_1(X_0)$, with a common base point $x_0 \in X_0$, and let $d: \rho_2 \to \rho_1$ be the boundary homomorphism. Let $a'_\lambda \in \rho_2$ be the element which is defined by a characteristic map[32] for e_λ^2, joined to x_0 by a path in X_0. We define a free crossed (ρ_1, ϕ)-module h_π $(= h_{\pi_1}$ in [2]), with a basis, $\{a_\lambda\}$, and with $\phi a_\lambda = da'_\lambda$. It follows from Lemma 2, in §2 above, that the correspondence $a_\lambda \to a'_\lambda$ determines an operator homomorphism, $\theta: h_\pi \to \rho_2$, such that $\phi = d\theta$. Obviously θ is onto ρ_2 and we shall prove that it is an isomorphism.

Let $\sigma_\lambda^2 \subset e_\lambda^2$ be a 2-simplex. Then we may absorb the closure of $e_\lambda^2 - \sigma_\lambda^2$ into X_0, for each λ. In order to simplify the notation we start again with $e_\lambda^2 = \sigma_\lambda^2 - \partial\sigma_\lambda^2$ and $\sigma_\lambda^2 \subset e_\lambda'^2$, where $e_\lambda'^2$ is a 2-cell, which is an open sub-set of X. Let p_λ be a vertex of σ_λ^2. Any element of ρ_2 may be represented by a map,

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[32] I.e. a map, $\psi: (I^2, \partial I^2) \to (X_0 \cup e_\lambda^2, X_0)$, such that $\psi | I^2 - \partial I^2$ is a homeomorphism onto e_λ^2.

$$f:(I^2, \partial I^2, I^0) \rightarrow (X, X_0, x_0),$$

which is *normal* in the following sense. The closure of $f^{-1}e^2_\lambda$ is a (finite) set of rectilinear simplexes, $\sigma^2_{1\lambda}, \sigma^2_{2\lambda}, \cdots \subset I^2 - \partial I^2$, such that each $f|\sigma^2_{i\lambda}$ is a barycentric map on σ^2_λ. Let $\{s_{i\lambda}\}$ be a set of polygonal segments in I^2, which join I^0 to the vertices $p_{i\lambda} = f^{-1}p_\lambda \cap \sigma^2_{i\lambda}$ and do not meet each other or the simplexes $\sigma^2_{j\mu}$, except in the end points I^0, $p_{i\lambda}$. For reasons given on p. 422 of [2], a normal map, f, together with such a set $\{s_{i\lambda}\}$, determines a unique element, $\psi f \in h_\pi$, such that $\theta\psi f \in \rho_2$ is represented by f. Moreover, given $a \in h_\pi$, there is a normal map, $f_a: I^2 \rightarrow X$, such that $a = \psi f_a$.

Let $a \in \theta^{-1}(0)$ and let $f_a: I^2 \rightarrow X$ be a normal map such that $a = \psi f_a$. Let $A^3 = p^* I^2$ be the join of I^2 with a new point p^* and let $E^2 = p^* \partial I^2 \subset \partial A^3$. Since f_n represents $\theta a = 0$ it may be extended to a map

$$f:(A^3, E^2) \rightarrow (X, X_0).$$

We assume that $f^{-1}\sigma^2_\lambda$, for each λ, is a simplicial complex, which is mapped by f simplicially on σ^2_λ. Then

$$L = f^{-1}\{q_\lambda\}$$

is a polygonal linkage, where $q_\lambda \in e^2_\lambda$. We assume that the vertices of L are in general position, relative to the projection of L from p^* on I^2. Finally we assume that the segment $p^* I^0$ is mapped by f on x_0. All these assumptions can be justified by geometrical arguments of standard type.

Let $G = \pi_1(A^3 - L)$ with p^* as base point. Then we define the homomorphism $F: G \rightarrow h_\pi$ as on pp. 423, 424 of [2]. Let $g \in G$ be the element which is represented by ∂I^2, joined to p^* by the segment $p^* I^0$. Assume that $a = Fg$. Since $\partial I^2 = \partial E^2$ and $E^2 \subset A^3 - L$ it follows that $g = 1$, whence $a = 0$. Therefore θ is an isomorphism. The proof that $Fg = \psi f_a = a$ is the final stage in the proof of Theorem 4 in [2].

17. The groups $\pi_n(K^n)$. The methods and results in this paper can be carried further, in certain special cases at least, by introducing the system of groups, $\alpha_n(K) = \pi_n(K^n)$, associated with a given complex K. We then have the combined system $(\rho, \alpha) = \{\rho_n, \alpha_n\}$, where $\rho_n = \rho_n(K)$, $\alpha_n = \alpha_n(K)$ $(\rho_1 = \alpha_1)$, with the homomorphisms

$$\rho_{n+1} \xrightarrow{\beta} \alpha_n \xrightarrow{j} \rho_n.$$

We define a homomorphism, $(f, g):(\rho, \alpha) \rightarrow (\rho', \alpha')$, into another such system, as a family of operator homomorphisms, $f_n: \rho_n \rightarrow \rho'_n$, $g_n: \alpha_n$

$\to\alpha'_n$, associated with f_1, such that $j_n g_n = f_n j_n$, $g_n \beta_{n+1} = \beta_{n+1} f_{n+1}$. We define two such homomorphisms, (f, g) and (f^*, g^*), as *equivalent*, and write $(f, g) \simeq (f^*, g^*)$, if, and only if, there is a deformation operator, $\xi: \rho \to \rho'$, associated with f_1, and an element $w' \in \alpha'_1$, such that

$$(17.1) \qquad \begin{aligned} &\text{(a)} \quad w' f_n^* - f_n = d_{n+1}\xi_{n+1} + \xi_n d_n, \\ &\text{(b)} \quad w' \alpha_n^* - \alpha_n = \beta_{n+1}\xi_{n+1} j_n. \end{aligned}$$

It is easily verified that a (cellular) map, $K \to K'$ in a complex L, induces a homomorphism $(\rho, \alpha) \to (\rho', \alpha')$, where $\rho' = \rho(K')$, $\alpha' = \alpha(K')$. Let $\phi_t: K \to L$ be a (cellular) homotopy, such that $\phi_0 e^0 = \phi_1 e^0 = e'^0$, where $e^0 \in K^0$, $e'^0 \in K'^0$ are the base points. Let $w' \in \alpha'_1$ be the element which is represented by the map $\theta: I \to K'^1$, where $\theta(t) = \phi_t e^0$. Let $\xi: \rho \to \rho'$ be the deformation operator, which is determined by ϕ_t. That is to say, if $a \in \rho_n$ is represented by

$$(17.2) \qquad \lambda: (I^n, \partial I^n, I^0) \to (K^n, K^{n-1}, e^0),$$

then $\xi a \in \rho'_{n+1}$ is represented by $\mu: I^{n+1} \to K'$, where

$$(17.3) \qquad \mu(t, t_1, \cdots, t_n) = \phi_t \lambda(t_1, \cdots, t_n).$$

Let (f, g) and (f^*, g^*) in (17.1) be the homomorphisms induced by ϕ_0, ϕ_1. Then (16.1a) may be verified directly, or deduced from the proof of Theorem 13 in §13, and the concluding remarks in §10. Let $\lambda(\partial I^n) = e^0$ in (17.2), and let λ be interpreted as a map representing a given element $b \in \alpha_n$. Then μ, given by (17.3), represents $\xi_{n+1} j_n b$ and (17.1b) follows without difficulty from (12.1) and the fact that $\mu(t \times \partial I^n) = \theta(t)$.

The value of this scheme is restricted by the fact that, in general, we know practically nothing about $\pi_n(K^n)$. But suppose that the groups $\alpha_1, \cdots, \alpha_k$ have been calculated, for some $k > 2$. Then we can ignore the groups α_n for $n > k$ and, in some cases, the results in §§6, 7 above can be improved with the help of the system $(\rho, \alpha_1, \cdots, \alpha_k)$. For example, $j_n: \alpha_n \approx d_n^{-1}(0)$ for $n = 1, \cdots, m$ if K is a J_m-complex. In this case the groups $\alpha_1, \cdots, \alpha_m$ do not give us anything new and the interest begins with α_{m+1}. Let K be a finite complex and let $\pi_n(K) = 0$ for $n = 1, \cdots, m-1$. Further assume that K^{m+2} is a reduced complex, as defined in [7]. Then we can calculate[33] α_{m+1}, and also α_{m+2} if $m \geq 4$.

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[33] See §11 in [6], Theorem 8 on p. 265 of [5], Lemma 4 on p. 418 of [2] and §12 in [12].

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NUMBERS OF SOLUTIONS OF EQUATIONS IN FINITE FIELDS

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The equations to be considered here are those of the type

$$a_0x_0^{n_0} + a_1x_1^{n_1} + \cdots + a_rx_r^{n_r} = b. \tag{1}$$

Such equations have an interesting history. In art. 358 of the *Disquisitiones* [1 a],[1] Gauss determines the Gaussian sums (the so-called cyclotomic "periods") of order 3, for a prime of the form $p=3n+1$, and at the same time obtains the numbers of solutions for all congruences $ax^3-by^3\equiv 1 (\bmod p)$. He draws attention himself to the elegance of his method, as well as to its wide scope; it is only much later, however, viz. in his first memoir on biquadratic residues [1b], that he gave in print another application of the same method; there he treats the next higher case, finds the number of solutions of any congruence $ax^4-by^4\equiv 1 \pmod p$, for a prime of the form $p=4n+1$, and derives from this the biquadratic character of 2 mod p, this being the ostensible purpose of the whole highly ingenious and intricate investigation. As an incidental consequence ("*coronidis loco,*" p. 89), he also gives in substance the number of solutions of any congruence $y^2\equiv ax^4-b \pmod p$; this result includes as a special case the theorem stated as a conjecture ("*observatio per inductionem facta gravissima*") in the last entry of his *Tagebuch* [1c];[2] and it implies the truth of what has lately become known as the Riemann hypothesis, for the function-field defined by that equation over the prime field of p elements.

Gauss' procedure is wholly elementary, and makes no use of the Gaussian sums, since it is rather his purpose to apply it to the determination of such sums. If one tries to apply it to more general cases, however, calculations soon become unwieldy, and one realizes the necessity of inverting it by taking Gaussian sums as a starting point. The means for doing so were supplied, as early as 1827, by Jacobi, in a letter to Gauss [2a] (cf. [2b]). But Lebesgue, who in 1837 devoted two papers [3a, b] to the case $n_0= \cdots =n_r$ of equation (1), did not

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] It is surprising that this should have been overlooked by Dedekind and other authors who have discussed that conjecture (cf. M. Deuring, Abh. Math. Sem. Hamburgischen Univ. vol. 14 (1941) pp. 197–198).

succeed in bringing out any striking result. The whole problem seems then to have been forgotten until Hardy and Littlewood found it necessary to obtain formulas for the number of solutions of the congruence $\sum_i x_i^n \equiv b \pmod{p}$ in their work on the singular series for Waring's problem [4]; they did so by means of Gaussian sums. More recently, Davenport and Hasse [5] have applied the same method to the case $r=2$, $b=0$ of equation (1) as well as to other similar equations; however, as they were chiefly concerned with other aspects of the problem, and in particular with its relation to the Riemann hypothesis in function-fields,[3] the really elementary character of their treatment does not appear clearly.

As equations of type (1) have again recently been the subject of some discussion (cf. e.g. [6]), it may therefore serve a useful purpose to give here a brief but complete exposition of the topic. This will contain nothing new, except perhaps in the mode of presentation of the final results, which will lead to the statement of some conjectures concerning the numbers of solutions of equations over finite fields, and their relation to the topological properties of the varieties defined by the corresponding equations over the field of complex numbers.

We consider equation (1) over a finite field k with q elements; the a_i are in k, and not 0; the n_i are integers, which we assume to be >0 (only trifling modifications would be required if some were <0). We shall first discuss the case $b=0$.

Let therefore N be the number of solutions in k of the equation

$$a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r} = 0.$$

For each i, let $d_i = (n_i, q-1)$ be the g.c.d. of n_i and $q-1$; for each i and for each u in k, let $N_i(u)$ be the number of solutions of the equation $x^{n_i} = u$; $N_i(u)$ is 1 for $u=0$, and is otherwise equal to d_i or to 0 according as u is or is not a d_ith power in k. Put $L(u) = \sum_{i=0}^{r} a_i u_i$; we have

$$N = \sum_{L(u)=0} N_0(u_0) \cdots N_r(u_r), \tag{2}$$

where the sum is taken over all sets of values for the u_i satisfying $L(u)=0$, or, as we may say, over all points $(u) = (u_0, \cdots, u_r)$ in the linear variety defined by $L(u)=0$ in the vector-space of dimension $r+1$ over k.

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[3] As to this, cf. H. Hasse, J. Reine Angew. Math. vol. 172 (1935) pp. 37–54. I regret that I did not quote either of these papers, where the connection between various kinds of exponential sums and the Riemann hypothesis is quite clearly expressed, in my recent note on the same subject, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 204–207.

If k^* is the multiplicative group of all non-zero elements in k, we shall denote by the letter χ any character of k^*; as k^* is cyclic of order $q-1$, such a character is fully determined if one assigns its value at a generating element w of k^* (a "primitive root"), and this value may be any $(q-1)$th root of unity. Selecting such an element w once for all, we shall denote by χ_α the character of k^* determined by $\chi_\alpha(w)=e^{2\pi i\alpha}$, where α is a rational number satisfying $(q-1)\alpha\equiv 0 \pmod 1$. We also put $\chi_\alpha(0)=0$ for $\alpha\not\equiv 0 \pmod 1$ and $\chi_\alpha(0)=1$ for $\alpha\equiv 0 \pmod 1$. Then we have

$$N_i(u) = \sum_\alpha \chi_\alpha(u) \qquad (d_i\alpha \equiv 0 \ (\mathrm{mod}\ 1),\ 0 \leqq \alpha < 1).$$

In fact, for $u=0$, both sides have the value 1; for $u\neq 0$, the right-hand side can be written as $\sum_{\nu=0}^{d_i-1}\zeta^\nu$, with $\zeta=\chi_{1/d_i}(u)$; and ζ is then a d_ith root of unity, equal to 1 if and only if u is a d_ith power in k^*.

Using this in (2), we get:

$$N = \sum_{u,\alpha} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r)$$
$$(L(u) = 0;\ d_i\alpha_i \equiv 0 \ (\mathrm{mod}\ 1),\ 0 \leqq \alpha_i < 1).$$

As there are q^r points in $L(u)=0$, the terms in the above sum which correspond to $\alpha_0=\cdots=\alpha_r=0$, being all equal to 1, give a sum q^r. We now show that those terms for which some, but not all, of the α_i are 0, give a sum 0. In fact, consider e.g. those for which $\alpha_0, \cdots, \alpha_{s-1}$ have given values, other than 0, and $\alpha_s=\cdots=\alpha_r=0$, with $s\leqq r$; as there are q^{r-s} points (u) in the variety $L(u)=0$ for which $u_0, \cdots, u_{s-1}$ have arbitrarily assigned values, the sum of those terms is

$$q^{r-s}\prod_{i=0}^{s-1}\Big(\sum_{u_i}\chi_{\alpha_i}(u_i)\Big),$$

and this is 0 since each factor is 0. This gives

$$N = q^r + \sum_{u,\alpha} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r)$$
$$(L(u) = 0;\ d_i\alpha_i \equiv 0 \ (\mathrm{mod}\ 1),\ 0 < \alpha_i < 1).$$

In this, we replace the u_i, respectively, by u_i/a_i, and get

$$N = q^r + \sum_\alpha \chi_{\alpha_0}(a_0^{-1})\cdots\chi_{\alpha_r}(a_r^{-1})\cdot S(\alpha)$$
$$(d_i\alpha_i \equiv 0 \ (\mathrm{mod}\ 1),\ 0 < \alpha_i < 1),$$

if we put, for any values of α_i satisfying $(q-1)\alpha_i\equiv 0 \pmod 1$, $\alpha_i\not\equiv 0$

(mod 1):

$$S(\alpha) = S(\alpha_0, \cdots, \alpha_r) = \sum_{\Sigma u_i = 0} \chi_{\alpha_0}(u_0) \cdots \chi_{\alpha_r}(u_r).$$

As to the latter sum, the terms for which $u_0=0$ are 0, and we may exclude them; we may then put $u_i=u_0v_i$ $(1 \leqq i \leqq r)$; the terms, in our sum, corresponding to given values of the v_i (satisfying $1+\sum_{i=1}^{r} v_i=0$) give

$$\chi_{\alpha_1}(v_1) \cdots \chi_{\alpha_r}(v_r) \sum_{u_0 \neq 0} \chi_\beta(u_0),$$

with $\beta=\sum_{i=0}^{r}\alpha_i$, and this last sum is $q-1$ for $\beta \equiv 0$ (mod 1), and 0 otherwise, so that in the latter case $S(\alpha)$ is 0.

Let us therefore define, for any set of α_i satisfying the conditions

$$(q-1)\alpha_i \equiv 0, \qquad \alpha_i \not\equiv 0, \qquad \sum_{i=0}^{r} \alpha_i \equiv 0 \pmod 1,$$

a number $j(\alpha)$ by the relation

$$\begin{aligned} j(\alpha) &= \sum_{1+v_1+\cdots+v_r=0} \chi_{\alpha_1}(v_1) \cdots \chi_{\alpha_r}(v_r) \\ &= \frac{1}{q-1} \sum_{u_0+\cdots+u_r=0} \chi_{\alpha_0}(u_0) \cdots \chi_{\alpha_r}(u_r). \end{aligned}$$

In terms of the $j(\alpha)$, the number N of solutions of $\sum_{i=0}^{r} a_i x_i^{n_i}=0$ is now seen to be given by

$$N = q^r + (q-1)\sum_{\alpha} \chi_{\alpha_0}(a_0^{-1}) \cdots \chi_{\alpha_r}(a_r^{-1}) \cdot j(\alpha) \tag{3}$$

$$(d_i\alpha_i \equiv 0;\ \sum \alpha_i \equiv 0 \pmod 1;\ 0 < \alpha_i < 1).$$

The $j(\alpha)$ may be called the Jacobi sums for the field k; they were first introduced and studied, for the case of a prime field, by Jacobi [**2a**, **b**], later by Stickelberger [**7**], and more recently by Davenport and Hasse [**5**]. They are closely related to the Gaussian sums for k:

$$g(\chi) = \sum_{x \in k} \chi(x)\psi(x),$$

where ψ is a character of the additive group of k, chosen once for all, and not everywhere equal to 1, and where χ is any one of the above defined multiplicative characters, other than χ_0. For the convenience of the reader, we shall briefly recall some of the known properties of these sums. In the first place, in the sum which defines $g(\chi)$, we may, as χ is not χ_0, restrict x to be $\neq 0$. Then we get

$$g(\chi)\bar{g}(\chi) = \sum_{y\neq 0}\sum_{x\neq 0}\chi(xy^{-1})\psi(x - y),$$

where we may substitute xy for x in the sum for x, and then interchange the order of summations:

$$g(\chi)\bar{g}(\chi) = \sum_{x\neq 0}\chi(x)\sum_{y\neq 0}\psi[(x - 1)y].$$

As the sum of all values of ψ on k is 0, the second sum has the value $q-1$ for $x=1$, and -1 for $x\neq 1$; as the sum of all values of χ on k^* is 0, this gives

$$g(\chi)\bar{g}(\chi) = q. \tag{4}$$

Now, in the definition of $g(\chi)$, write tx for x with any $t\neq 0$ in k; this gives

$$g(\chi) = \chi(t)\sum_{x}\chi(x)\psi(tx),$$

hence, using (4), and interchanging x and t:

$$\chi(x) = \frac{g(\chi)}{q}\sum_{t}\bar{\chi}(t)\bar{\psi}(tx),$$

which is also true for $x=0$; this is the Fourier expansion of $\chi(x)$ on k according to the additive characters of k. Using this in the definition of $j(\alpha)$, we get

$$(q - 1)j(\alpha) = q^{-r-1}\cdot g(\chi_{\alpha_0}) \cdots g(\chi_{\alpha_r})\sum_{t}\bar{\chi}_{\alpha_0}(t_0)\cdots\bar{\chi}_{\alpha_r}(t_r)$$
$$\cdot\sum_{\sum u_i=0}\bar{\psi}\Big(\sum_{i} t_i u_i\Big).$$

But, in the additive group of all vectors $(u)=(u_0, \cdots, u_r)$, the vectors satisfying $\sum u_i=0$ form a subgroup of q^r elements, on which $\bar{\psi}(\sum t_i u_i)$ is a character; the sum of the values of this character on the subgroup must therefore be either q^r, if the character has the constant value 1, or 0 otherwise. The former case occurs if and only if all the t_i are equal, since otherwise we can solve the equations $\sum u_i=0$, $\sum t_i u_i=z$, where z is any element of k, e.g. one such that $\psi(z)\neq 1$. As we have $\sum \alpha_i\equiv 0 \pmod 1$ by the definition of $j(\alpha)$, this gives

$$j(\alpha) = \frac{1}{q}\, g(\chi_{\alpha_0}) \cdots g(\chi_{\alpha_r}).$$

As a consequence, we have

$$i(\alpha)j(\alpha) = q^{r-1},$$

and therefore

$$| N - q^r | \leqq M(q-1)q^{(r-1)/2},$$

where M is the number of systems of rational numbers α_i satisfying

$$n_i\alpha_i \equiv 0, \sum \alpha_i \equiv 0 \ (\text{mod } 1), 0 < \alpha_i < 1,$$

and is therefore an integer depending only upon the n_i.

From the above results, we can easily derive the number N_1 of solutions of the equation $\sum_{i=0}^{r} a_i x_i^{n_i} + 1 = 0$. In fact, let N, as before, be the number of solutions of $\sum_{i=0}^{r} a_i x_i^{n_i} = 0$, and let N' be the number of solutions of $\sum_{i=0}^{r} a_i x_i^{n_i} + x_{r+0}^{q-1} = 0$. The previous results apply to the latter equation, with $d_{r+1} = n_{r+1} = q-1$. But, since x_{r+1}^{q-1} has the value 1, except for $x_{r+1} = 0$, we have

$$N' = (q-1)N_1 + N.$$

This gives at once an expression for N_1; in order to write it more conveniently, we shall define the symbol $j(\alpha)$ even in the case when some, but not all, of the α_i are 0. Let the β_j be numbers, satisfying $(q-1)\beta_j \equiv 0$, $\sum_j \beta_j \equiv 0$ (mod 1), and not all $\equiv 0$ (mod 1); assume that s of them are $\equiv 0$ (mod 1), and let $\alpha_0, \cdots, \alpha_r$ be the others, in any order; then we put $j(\beta) = (-1)^s j(\alpha)$. This being so, the formula for N_1 can be written as

$$N_1 = q^r + \sum_{\alpha} \chi_{\alpha_0}(a_0^{-1}) \cdots \chi_{\alpha_r}(a_r^{-1}) j\left(\alpha_0, \cdots, \alpha_r, -\sum_{i=0}^{r} \alpha_i\right)$$

$$(d_i\alpha_i \equiv 0 \ (\text{mod } 1),\ 0 < \alpha_i < 1),$$

and we get, as before:

$$| N_1 - q^r | \leqq M_1 q^{r/2},$$

where M_1 is now given by

$$M_1 = (d_0 - 1) \cdots (d_r - 1) < n_0 n_1 \cdots n_r.$$

It is a matter of considerable interest to be able to compare the number of solutions of an equation (or, more generally, the number of rational points on an algebraic variety) in a given finite field and in all the extensions of finite degree of that field. This can easily be done, for the type of equations under consideration in this note, if we use a relation, due to Davenport and Hasse [5], between Gaussian sums in a finite field and in its extensions. We shall first give a brief account, in elementary language, of the proof of Davenport and Hasse for this relation.

Let k' be an extension of k, of degree ν; for y in k', let $N(y)$ and $T(y)$ denote the norm and the trace of y, respectively, over k. If w denotes, as before, a generator of the multiplicative group k^*, there is a generator z of k'^*, such that $N(z)=w$; then, if we denote, as before, by $\chi'_\alpha(y)$ the multiplicative character on k' determined by $\chi'_\alpha(z)=e^{2\pi i\alpha}$, we have, for $(q-1)\alpha\equiv 0 \pmod 1$, $\chi'_\alpha(y)=\chi_\alpha[N(y)]$. We also put $\psi'(y)=\psi[T(y)]$; this is an additive character of k', not everywhere equal to 1 since it is known that $T(y)$ maps k' *on* k. Let now $g'(\chi'_\alpha)$ be the Gaussian sum in k':

$$g'(\chi'_\alpha) = \sum_{y\in k'} \chi'_\alpha(y)\psi'(y).$$

The theorem of Davenport and Hasse is as follows:

$$-\,g'(\chi'_\alpha) = [-\,g(\chi_\alpha)]^\nu. \tag{5}$$

In order to prove this, consider the polynomials with coefficients in k, and highest coefficient 1; to every such polynomial

$$F(X) = X^n + c_1X^{n-1} + \cdots + c_n,$$

of degree $n\geqq 1$, we attach the number

$$\lambda(F) = \chi_\alpha(c_n)\psi(c_1).$$

For two such polynomials F_1, F_2, we have $\lambda(F_1F_2)=\lambda(F_1)\lambda(F_2)$. If we also denote by $n(F)$ the degree of such a polynomial F, and by U an indeterminate, this gives the formal identity

$$1 + \sum_F \lambda(F)\cdot U^{n(F)} = \prod_P\, [1 - \lambda(P)\cdot U^{n(P)}]^{-1},$$

where the sum in the left-hand side is taken over *all* polynomials F over k, of degree $\geqq 1$, with highest coefficient 1, and the product in the right-hand side is taken over all *irreducible* polynomials P over k, with highest coefficient 1. As usual, this follows at once from the fact that every F can be expressed in a unique manner as product of powers of irreducible polynomials.

In the sum in the left-hand side, consider first the terms which correspond to polynomials $F(X)=X+c$ of degree 1; the sum of these terms is equal to $g(\chi_\alpha)U$. As to the sum of the terms corresponding to any given degree $n>1$, it is 0, since, with the above notations, it is equal to

$$q^{n-2}\sum_{c_n}\chi_\alpha(c_n)\sum_{c_1}\psi(c_1)\cdot U^n,$$

where both sums are taken over k and are therefore 0. This gives

$$(6)\qquad 1+g(\chi_\alpha)U=\prod_P[1-\lambda(P)\cdot U^{n(P)}]^{-1}.$$

Similarly, if $F'(X)=X^n+d_1X^{n-1}+\cdots+d_n$ is a polynomial over k', we write

$$\lambda'(F')=\chi'_\alpha(d_n)\psi'(d_1),$$

and, taking another indeterminate U', get the formal identity

$$(6')\qquad 1+g'(\chi'_\alpha)U'=\prod_{P'}[1-\lambda'(P')\cdot U'^{n(P')}]^{-1}$$

where the product is taken over all irreducible polynomials P' over k', with highest coefficient 1.

Now let P be as above; let P' be one of the irreducible factors of P over k'; let $-\xi$ be one of the roots of P'. Then ξ generates over k an extension $k(\xi)$ of degree $n=n(P)$, and over k' an extension $k'(\xi)$ of degree $n'=n(P')$; as $k'(\xi)$ is the composite of $k(\xi)$ and k', its degree over k must be the l.c.m. of the degree n of $k(\xi)$ over k, and of the degree ν of k' over k, i.e. equal to $n\nu/d$ if we write $d=(n,\nu)$. This gives $n'=n/d$; hence P has over k' exactly d irreducible factors, all of degree n/d. Moreover, if a and b are respectively the norm and the trace of ξ, taken in $k(\xi)$ relatively to k, we have

$$P(X)=X^n+bX^{n-1}+\cdots+a,$$

hence

$$\lambda(P)=\chi_\alpha(a)\psi(b).$$

Similarly, if a' and b' are the norm and the trace of ξ, taken in $k'(\xi)$ relatively to k', we have

$$\lambda'(P')=\chi'_\alpha(a')\psi'(b')=\chi_\alpha(Na')\psi(Tb'),$$

where Na' and Tb' are the norm of a' and the trace of b', taken in k' relatively to k; hence Na' and Tb' are respectively equal to the norm and to the trace of ξ, taken in $k'(\xi)$ relatively to k. We can therefore also obtain Na' by taking the norm of ξ in $k'(\xi)$ relatively to $k(\xi)$, this being equal to $\xi^{\nu/d}$, and then the norm of this in $k(\xi)$ relatively to k, which is $a^{\nu/d}$. Hence we have $Na'=a^{\nu/d}$, and similarly $Tb'=(\nu/d)b$, and therefore

$$\lambda'(P')=\lambda(P)^{\nu/d}.$$

Now, in the right-hand side of (6′), we can put together the d factors corresponding to all the irreducible factors of P over k'; if, moreover, we replace U' by U^ν, we get

$$[1 - \lambda(P)^{\nu/d} U^{\nu n/d}]^{-d},$$

which can also be written as

$$\prod_{\rho=0}^{\nu-1} [1 - \lambda(P)\cdot(\zeta^\rho U)^n]^{-1}$$

where ζ is any primitive νth root of unity. This gives

$$\begin{aligned} 1 + g'(\chi_\alpha')U^\nu &= \prod_{\rho=0}^{\nu-1} \prod_P [1 - \lambda(P)\cdot(\zeta^\rho U)^{n(P)}]^{-1} \\ &= \prod_{\rho=0}^{\nu-1} (1 + g(\chi_\alpha)\zeta^\rho U) \\ &= 1 + (-1)^{\nu+1} g(\chi_\alpha)^\nu U^\nu, \end{aligned}$$

which proves (5).

Now, N_ν being the number of solutions of an equation of type (1), with or without constant term, over the extension of degree ν of the ground-field k, it is easy, using the above results, to give a simple expression for the "generating power-series" for N_ν, i.e. for the formal power-series $\sum_1^\infty N_\nu U^\nu$; this turns out to be the expansion of a certain rational function in U. We shall, however, illustrate this idea by considering the case of the homogeneous equation

$$a_0 x_0^n + \cdots + a_r x_r^n = 0, \tag{7}$$

considered as the equation of a variety (without singular points) in the projective space P^r of dimension r over k. The number $\overline{N}$ of rational points over k, on that variety, is related to the number N of solutions of the same equation in affine space by $N = 1 + (q-1)\overline{N}$, so that, putting $d = (n, q-1)$, we get, from our earlier results:

$$\overline{N} = 1 + q + \cdots + q^{r-1} + \sum_\alpha \overline{\chi}_{\alpha_0}(a_0) \cdots \overline{\chi}_{\alpha_r}(a_r)\cdot j(\alpha)$$
$$(d\alpha_i \equiv 0,\ \sum \alpha_i \equiv 0 \pmod 1;\ 0 < \alpha_i < 1).$$

Now call $\overline{N}_\nu$ the number of rational points, on the variety defined by (7), over the extension k_ν of k of degree ν; we shall calculate the series $\sum_1^\infty \overline{N}_\nu U^{\nu-1}$.

In order to do this, consider any set of rational numbers $\alpha_0, \cdots, \alpha_r$ satisfying $n\alpha_i \equiv 0$, $\sum \alpha_i \equiv 0 \pmod 1$, $0 < \alpha_i < 1$. For this set, let

$\mu = \mu(\alpha)$ be the smallest integer such that $(q^\mu - 1)\alpha_i \equiv 0 \pmod 1$ for $0 \leqq i \leqq r$; then the extensions k_ν of k such that $(q^\nu - 1)\alpha_i \equiv 0 \pmod 1$ are those for which ν is a multiple of μ, and those only. Choosing a primitive root in k_μ, we can now, as before, define in k_μ the characters χ_{α_i}, the Gaussian sums $g(\chi_{\alpha_i})$, and the Jacobi sum

$$j(\alpha) = \frac{1}{q} g(\chi_{\alpha_0}) \cdots g(\chi_{\alpha_r}).$$

Furthermore, if we denote by χ'_{α_i}, $g'(\chi'_{\alpha_i})$ and $j'(\alpha)$ the corresponding characters and sums for the extension $k' = k_{\lambda\mu}$ of k of degree $\lambda\mu$, where λ is any integer, we get from our earlier results:

$$\chi'_{\alpha_i}(a_i) = \chi_{\alpha_i}(a_i)^\lambda,\ g'(\chi'_{\alpha_i}) = (-1)^{\lambda-1} g(\chi_{\alpha_i})^\lambda,\ j'(\alpha) = (-1)^{(\lambda-1)(r-1)} j(\alpha)^\lambda.$$

Then we get:

$$\begin{aligned}\sum_1^\infty \overline{N}_\nu U^{\nu-1} = &- \sum_{h=0}^{r-1} \frac{d}{dU} \log (1 - q^h U) \\ (8) \qquad &+ (-1)^r \sum_\alpha \frac{1}{\mu(\alpha)} \frac{d}{dU} \log \left[1 - C(\alpha) \cdot U^{\mu(\alpha)}\right] \\ &(n\alpha_i \equiv 0,\ \textstyle\sum \alpha_i \equiv 0 \pmod 1;\ 0 < \alpha_i < 1)\end{aligned}$$

where we have put

$$C(\alpha) = (-1)^{r-1} \overline{\chi}_{\alpha_0}(a_0) \cdots \overline{\chi}_{\alpha_r}(a_r) \cdot j(\alpha).$$

Furthermore, it is easily seen that $C(q\alpha) = C(\alpha)$, since $x \to x^q$ is an automorphism of k_μ which leaves the a_i invariant. Therefore, in the last sum in (8), the $\mu(\alpha)$ terms corresponding to the set $(\alpha) = (\alpha_0, \cdots, \alpha_r)$ and to the sets $(q^\rho \alpha)$ for $1 \leqq \rho \leqq \mu - 1$ are all equal, so that, putting them together, we can make the denominator $\mu(\alpha)$ disappear.

Let A be the number of solutions, in rational numbers α_i, of the system $n\alpha_i \equiv 0$, $\sum \alpha_i \equiv 0 \pmod 1$, $0 < \alpha_i < 1$. Then one finds[4] that the Poincaré polynomial (in the sense of combinatorial topology) of the variety defined, in the projective space P^r over complex numbers, by an equation of the form

$$c_0 x_0^n + \cdots + c_r x_r^n = 0$$

is equal to

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[4] As obligingly communicated to me by P. Dolbeault in Paris.

$$\sum_{h=0}^{r-1} X^{2h} + A\cdot X^{r-1}.$$

This, and other examples which we cannot discuss here, seem to lend some support to the following conjectural statements, which are known to be true for curves, but which I have not so far been able to prove for varieties of higher dimension.

Let V be a variety without singular points, of dimension n, defined over a finite field k with q elements. Let N_ν be the number of rational points on V over the extension k_ν of k of degree ν. Then we have

$$\sum_{1}^{\infty} N_\nu U^{\nu-1} = \frac{d}{dU}\log Z(U),$$

where $Z(U)$ is a rational function in U, satisfying a functional equation

$$Z\left(\frac{1}{q^n U}\right) = \pm\, q^{n\chi/2}U^{\chi}Z(U),$$

with χ equal to the Euler-Poincaré characteristic of V (intersection-number of the diagonal with itself on the product $V\times V$).

Furthermore, we have:

$$Z(U) = \frac{P_1(U)P_3(U)\cdots P_{2n-1}(U)}{P_0(U)P_2(U)\cdots P_{2n}(U)},$$

with $P_0(U)=1-U$, $P_{2n}(U)=1-q^nU$, and, for $1\leqq h\leqq 2n-1$:

$$P_h(U) = \prod_{i=1}^{B_h}(1-\alpha_{hi}U)$$

where the α_{hi} are algebraic integers of absolute value $q^{h/2}$.

Finally, let us call the degrees B_h of the polynomials $P_h(U)$ the *Betti numbers* of the variety V; the Euler-Poincaré characteristic χ is then expressed by the usual formula $\chi=\sum_h(-1)^hB_h$. The evidence at hand seems to suggest that, if $\overline{V}$ is a variety without singular points, defined over a field K of algebraic numbers, the Betti numbers of the varieties $V_{\mathfrak{p}}$, derived from $\overline{V}$ by reduction modulo a prime ideal $\mathfrak{p}$ in K, are equal to the Betti numbers of $\overline{V}$ (considered as a variety over complex numbers) in the sense of combinatorial topology, for all except at most a finite number of prime ideals $\mathfrak{p}$. For instance, consider the Grassmann variety $G_{m,r}$, the points of which are the r-dimensional linear varieties in a projective m-dimensional space, over a field with q elements. The number of rational points on the variety is easily seen to be $F(q)$, where F is the polynomial defined by

$$F(X) = \frac{(X^{m+1}-1)(X^{m+1}-X)\cdots(X^{m+1}-X^r)}{(X^{r+1}-1)(X^{r+1}-X)\cdots(X^{r+1}-X^r)}.$$

Then, if the above conjectures are true, the Poincaré polynomial of the Grassmann variety $G_{m,r}$ over complex numbers must be $F(X^2)$. This is indeed so, as can easily be verified from the well-known results of Ehresmann [8].[5]

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[5] *Added in proof.* Results, substantially identical to our formula (3), have just been published by L. K. Hua and H. S. Vandiver, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 94–99.

THE FEBRUARY MEETING IN NEW YORK

The four hundred forty-third meeting of the American Mathematical Society was held at Columbia University on Saturday, February 26, 1948. The attendance was about two hundred seventy, including the following two hundred fifty-nine members of the Society:

Milton Abramowitz, C. R. Adams, E. J. Akutowicz, C. B. Allendoerfer, D. B. Ames, R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, Helmut Aulbach, F. E. Baker, S. F. Barber, Joshua Barlaz, F. D. Bateman, P. T. Bateman, F. P. Beer, E. G. Begle, A. H. Berger, Stefan Bergman, Lipman Bers, Garrett Birkhoff, J. H. Blau, E. K. Blum, R. P. Boas, G. L. Bolton, Samuel Borofsky, D. G. Bourgin, S. G. Bourne, Paul Brock, A. B. Brown, J. H. Bushey, W. E. Byrne, W. R. Callahan, E. W. Cannon, P. G. Carlson, Jeremiah Certaine, K. Chandrasekharan, J. O. Chellevold, K. T. Chen, Y. W. Chen, S. S. Chern, P. L. Chessin, Randolph Church, J. A. Clarkson, G. R. Clements, E. A. Coddington, Carl Cohen, I. S. Cohen, R. M. Cohn, T. F. Cope, L. M. Court, W. H. H. Cowles, V. F. Cowling, J. E. Crawford, J. S. Cronin, C. W. Curtis, M. D. Darkow, D. A. Darling, Martin Davis, Philip Davis, A. S. Day, C. H. Dowker, T. C. Doyle, Jacques Dutka, Aryeh Dvoretzky, Samuel Eilenberg, Bernard Epstein, M. P. Epstein, R. M. Exner, W. H. Fagerstrom, A. B. Farnell, O. J. Farrell, Herbert Federer, J. M. Feld, F. G. Fender, R. S. Finn, Edward Fleisher, W. W. Flexner, R. H. Fox, Gerald Freilich, K. O. Friedrichs, David Gale, G. N. Garrison, L. L. Gavurin, Hilda Geiringer, Abe Gelbart, J. C. Gibson, B. P. Gill, Harriet Griffin, E. J. Gumbel, C. F. Hall, Philip Hartman, K. E. Hazard, C. M. Hebbert, Alex Heller, G. C. Helme, L. A. Henkin, A. P. Hillman, I. I. Hirschman, A. J. Hoffman, Banesh Hoffmann, T. R. Hollcroft, E. M. Hull, T. R. Humphreys, L. C. Hutchinson, Nathan Jacobson, A. R. Jacoby, R. A. Johnson, R. E. Johnson, Bjarni Jónsson, M. L. Juncosa, Robert Kahal, Shizuo Kakutani, Aida Kalish, Irving Kaplansky, Edward Kasner, M. E. Kellar, J. F. Kiefer, P. J. Kiernan, H. S. Kieval, S. A. Kiss, J. R. Kline, Morris Kline, E. G. Kogbetliantz, E. R. Kolchin, N. H. Kuiper, Wouter van der Kulk, Jack Laderman, A. W. Landers, R. E. Langer, J. R. Lee, Joseph Lehner, R. A. Leibler, R. B. Leipnik, Benjamin Lepson, W. J. LeVeque, Howard Levi, J. H. Lewis, Julius Lieblein, M. A. Lipschutz, Marie Litzinger, Charles Loewner, A. J. Lohwater, L. H. Loomis, E. R. Lorch, Lee Lorch, R. C. Lyndon, L. A. MacColl, H. M. MacNeille, Kurt Mahler, M. M. Maloney, Dis Maly, Irwin Mann, A. J. Maria, M. H. Maria, W. T. Martin, D. G. Mead, R. C. Meacham, A. E. Meder, K. S. Miller, W. H. Mills, G. C. Miloslavsky, Don Mittleman, Deane Montgomery, A. H. Moore, G. D. Mostow, Andrzej Mostowski, T. S. Motzkin, F. J. Murray, D. S. Nathan, C. A. Nelson, D. J. Newman, Morris Newman, P. B. Norman, I. L. Novak, C. O. Oakley, J. C. Oxtoby, F. W. Perkins, R. M. Peters, C. G. Plithides, E. L. Post, M. H. Protter, A. L. Putnam, Hans Rademacher, H. V. Rådström, L. R. Raines, G. N. Raney, John Rausen, G. E. Raynor, C. J. Rees, Helene Reschovsky, Moses Richardson, C. E. Rickart, R. F. Rinehart, J. F. Ritt, M. S. Robertson, Robin Robinson, S. L. Robinson, I. H. Rose, N. J. Rose, P. C. Rosenbloom, H. D. Ruderman, H. J. Ryser, C. W. Saalfrank, H. E. Salzer, Arthur Sard, Henry Scheffé, M. M. Schiffer, Pincus Schub, Abraham Schwartz, Abraham Seidenberg, Seymour Sherman, Edward Silverman, James Singer, M. H. Slud, P. A. Smith, J. J. Sopka, A. H. Sprague, George Springer, E. P. Starke, J. R. K. Stauffer, Fritz Steinhardt, F. M. Stewart, R. C. Stewart, R. C. Strodt, Walter Strodt, M. M. Sullivan, Fred Supnick,

R. T. Tear, Feodor Theilheimer, D. L. Thomsen, C. B. Tompkins, A. W. Tucker, J. W. Tukey, Annita Tuller, S. M. Ulam, D. H. Wagner, Sylvan Wallach, J. L. Walsh, M. A. Weber, J. H. Weiner, Alexander Weinstein, Louis Weisner, M. J. Weiss, J. G. Wendel, John Wermer, M. E. White, G. T. Whyburn, Albert Wilansky, A. B. Willcox, J. N. Williams, W. L. G. Williams, Jacob Wolfowitz, Y. K. Wong, M. Y. Woodbridge, Daniel Zelinsky, P. W. Zettler-Seidel, H. J. Zimmerberg, Leo Zippin.

On Saturday there were two general sessions for contributed papers, one at 10:00 A.M. in which Professor J. H. Bushey presided, and one at 3:30 P.M. in which Professor C. A. Nelson presided.

At 2:00 P.M. Dr. Stefan Bergman of Harvard University gave an address on *Some recent developments in the theory of analytic functions.* Professor W. T. Martin presided.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 230 was read by Professor Tucker and paper number 254 was read by Mr. Kaplan. Dr. Rauch, Mr. Kaplan, and Mr. O'Brien were introduced by Professor T. R. Hollcroft.

ALGEBRA AND THEORY OF NUMBERS

221*t*. Richard Bellman: *On a generalization of an identity due to Wilton.*

Wilton, Proc. London Math. Soc. vol. 31 (1930) pp. 11–12, gave an identity relating the product $\zeta(s)\zeta(s')$, $\zeta(s+s'-1)$, and an infinite series of Dirichlet type, which, apart from its general interest, is of value in the estimation of the mean value of the square of the zeta-function. In the present paper a corresponding formula involving the product $\zeta^2(s)\zeta^2(s')$, $\zeta^2(s+s'-1)$, and a similar infinite series of Dirichlet type is obtained. Wilton's identity is derived using the Poisson summation formula. The present identity requires the Voronoi summation formula (which can be used similarly to obtain the classical approximate functional equation for $\zeta^2(s)$ due to Hardy and Littlewood) and some difficult results due to Atkinson concerning trigonometric sums with divisor functions as coefficients. It may also be applied to obtain mean value results, for $\sigma > 1/2$. (Received February 11, 1949.)

222*t*. Richard Bellman: *On form-preserving linear transformations.* Preliminary report.

It is shown that only transformations of the form $x' = Ax$, where x has the components x_k, $1 \leq k \leq N$, and A is an $N \times N$ matrix, which preserve the sums of kth powers, $k \geq 3$, are the trivial ones. From this it follows easily that the only identities of the form (sum of kth powers) × (sum of kth powers) equals (sum of kth powers), where the variables on the right are bilinear forms in the variables on the left, are again the trivial ones. (Received December 21, 1948.)

223*t*. Richard Bellman: *On some divisor sums associated with Diophantine equations.*

Asymptotic results for the sums $S_3 = \sum_{n \leq N} d(n)d_3(n+l)$, $S_4 = \sum_{n \leq N} d(n)d_4(n+l)$

are obtained. The number-theoretic functions $d_3(n)$, $d_4(n)$ are the coefficients in the Dirichlet series for $\zeta(s)^3$ and $\zeta(s)^4$, respectively, $d(n)$ is the divisor function. These sums arise when considering the number of positive, integral rational solutions of the diophantine equations $x_1x_2x_3-y_1y_2=l$, $x_1x_2x_3x_4-y_1y_2=l$, $y_1y_2\leqq N$, respectively. The problem reduces to that of finding estimates for the sums $\sum_{n\leqq N}d_3(kn+l)$, $\sum_{n\leqq N}d_4(kn+l)$ with sufficiently accurate error terms, which now depend upon k as well as N. These error terms are obtained, to the desired order of accuracy, followin a method of Hardy and Littlewood, using the mean value of the fourth power of the corresponding L-series mod k. (Received February 11, 1949.)

224t. Richard Bellman: *The average value of arithmetical functions.*

Consider the number-theoretical function, $\sigma_s(n)$, equal to the sum of the sth powers of the divisors of n. It is shown that $\sigma_s(f(n))$ has a mean value for any $s<0$, where $f(x)$ is any polynomial in x which is integer-valued for integer values of x. The mean value is a constant if n runs through the integers and equals $c/\log n$ if n runs through the primes. (Received December 21, 1948.)

225t. Sarvadaman Chowla and H. J. Ryser: *Combinatorial problems.* I.

Let v elements be arranged in v sets, and list the elements in a row and the sets in a column. Form an incidence matrix by inserting a one in row i and column j if the jth element belongs to the ith set, and a zero otherwise. Problem I: Arrange v elements into v sets such that (I_1) every set contains exactly k distinct elements, (I_2) every pair of sets has exactly $\lambda=k(k-1)/(v-1)$ elements in common $(0<\lambda<k<v)$. Problem II: Arrange v elements into v sets such that (II_1) each element occurs in exactly k distinct sets, (II_2) every pair of elements occurs in the v sets exactly $\lambda=k(k-1)/(v-1)$ times $(0<\lambda<k<v)$. Problem III: Arrange v elements into v sets fulfilling (I_1) and (I_2), (II_1), and (II_2). Problem (IV): Arrange v elements into v sets fulfilling (I_1) and (I_2) so that the incidence matrix is cyclic. A solution of any one of the problems I, II, and III is necessarily a solution of the remaining two. However, values for v and k exist for which Problem III has a solution and Problem IV has no solution. The theory is applied to Hadamard matrices, projective planes, block designs, and difference sets. (Received January 3, 1949.)

226t. Sarvadaman Chowla and H. J. Ryser: *Combinatorial problems.* II.

The theorems which follow concern the impossibility of Problem I described in the preceding abstract. Clearly the impossibility of Problem I for a given v and k implies the impossibility of Problems II, III, and IV. (1) if v is even and if $k-\lambda$ is not a square, then Problem I has no solution. (2) If $v\equiv 1 \pmod 4$ and if there exists an odd prime p such that p divides the squarefree part of $k-\lambda$ and if $(\lambda|p)=-1$, where $(\lambda|p)$ is the Legendre symbol, then Problem I has no solution. (3) If $v\equiv 3 \bmod 4$ and if there exists an odd prime p such that p divides the squarefree part of $k-\lambda$ and if $(-\lambda|p)=-1$, then Problem I has no solution. Theorems 2 and 3 are a rather straightforward generalization of a theorem of Bruck and Ryser on the nonexistence of certain finite projective planes, and for projective planes they give no new information (Bull. Amer. Math. Soc. Abstract 54-7-291). However, the proofs are now independent of the Minkowski-Hasse theory on the invariants of a quadratic form under rational transformations. (Received January 3, 1949.)

227t. Carl Cohen: *The Laplace transform of the generalized dilogarithm.*

Let $dl^{(k)}t$ be the integral from 0 to t of $x^{-1}dl^{(k-1)}x$, $dl^{(0)}t=-\log(1-t)$, then the Laplace transforms of this family of functions satisfy the law $(d/ds)[s\cdot L\{dl^{(k)}t\}]=-L\{dl^{(k-1)}t\}$, while $L\{dl^{(0)}t\}=s^{-1}\exp(-s)\,EI(s)$ is convergent if $s>0$. Take $\exp(t)$ instead of t as variable, then $(d/dt)[dl^{(k)}(\exp t)]=dl^{(k-1)}(\exp t)$ and $s\cdot L\{dl^{(k)}(\exp t)\}=\zeta(k+1)+L\{dl^{(k-1)}(\exp t)\}=\sum_{\mu=0}^{k-1}s^{-(\mu+1)}\zeta(k-\mu+1)+s^{-k}L\{dl^{(0)}\cdot(\exp t)\}$. $L\{dl^{(0)}(\exp t)\}=s^{-1}[\psi(1)\pm in\pi-\psi(s)]$, $s>0$, where $\psi(s)=(d/ds)[\log\Gamma(s)]$. The Laplace transform has been used to study the identities of the generalized dilogarithm. (Received November 26, 1948.)

228. R. H. Fox: *Free differential calculus and free groups.*

Iteration of a previously given formula (Bull. Amer. Math. Soc. Abstract 55-3-115) gives a "Taylor series expansion" $u=u^0+\sum_j(\partial u/\partial x_j)^0(x_j-1)+\sum_{jk}(\partial^2u/\partial x_j\partial x_k)^0\cdot(x_j-1)(x_k-1)+\cdots$ about the point $(1, 1, \cdots)$ of any element u of the integral group ring P of a free group F. This expansion differs only by a translation $x_j=1+s_j$ from an expansion considered by Magnus (Math. Ann. vol. 111 (1935) pp. 259–280) for the case $u\in F$. The element u of P is determined uniquely by the coefficients of this series. Several known theorems follow from this: (1) The ring P has no divisors of zero (Higman, Proc. London Math. Soc. vol. 46 (1939) pp. 231–248); (2) the intersection of the subgroups of the lower central series of F consists of the identity element alone (Magnus, loc. cit.). (Received January 13, 1949.)

229. David Gale: *Convex cones and solutions of two-person games.*

A subset C of a finite-dimensional vector space V is called a *convex cone* if it is closed under the operations of addition and multiplication by non-negative scalars. The set of (convex) cones in V is closed under the operations of intersection ($\cap$) and algebraic sum ($+$). In addition there is an operation which attaches to each cone C its polar cone C^*, defined to be the set of all vectors y in V such that $y\cdot x\geqq 0$ for all x in C. For cones which are topologically closed the important property $(C^*)^*=C$ holds, and the set of cones becomes an orthocomplemented lattice under $+$, $\cap$ and *. If the unit vectors of V are taken to be the pure strategies of a zero-value, two-person game, the set of good mixed strategies will be a closed cone in the above sense. By making strong use of the lattice properties a complete characterization of the possible sets of good strategies is derived for any two-person game, a result obtained independently by S. Sherman and by H. F. Bohnenblust, S. Karlin and L. S. Shapley. (Received January 10, 1949.)

230. David Gale, H. W. Kuhn, and A. W. Tucker: *A game problem equivalent to a maximum problem.*

Let B be an m by n matrix, c a p-vector (p by 1 matrix), and D a p by n matrix (all real). Let X be the convex set of m-vectors x got by the linear mapping $x=By$ from the convex set of n-vectors y determined by the $n+p$ linear inequalities $y\geqq 0$ and $c\geqq Dy$ (where such vector inequalities hold for each component separately). A vector of a of X is called maximal if $x\geqq a$ for x in X implies $x=a$. The problem of finding a maximal a when B, c, D are given is closely related to econometric problems considered by G. B. Dantzig and T. C. Koopmans. Let G be the 0-sum 2-person game (see von Neumann and Morgenstern, *The theory of games and economic behavior*) associated with the $m+p$ by $n+1$ composite matrix formed from a, $-B$, $-c$, and D. A submatrix of a game matrix is called essential if both players have good mixed strategies in which the pure strategies corresponding to the rows and columns of the submatrix are used with positive probabilities. Theorem: A vector a in X is maximal if and only if the game G has value zero and the submatrix a is essential to G. (Received March 3, 1949.)

231*t*. Florence D. Jacobson and Nathan Jacobson: *Classification and representation of semi-simple Jordan algebras.*

The authors give a determination of the semi-simple special Jordan algebras over any field of characteristic 0. An *imbedding* of a Jordan algebra K is defined to be a linear mapping $a \rightarrow a^R$ of K into an associative algebra such that $(a \cdot b)^R = 2^{-1}(a^R b^R + b^R a^R)$. An imbedding in an algebra of linear transformations is called a *representation*. They show that the imbedding problem can be reduced to the problem of homomorphism of a certain universal associative algebra, and determine the structure of these universal associative algebras for semi-simple Jordan algebras of characteristic 0. These universal algebras are semi-simple. It follows that any representation of a semi-simple Jordan algebra is completely reducible. Moreover, the irreducible representations can be given by using these results. (Received January 28, 1949.)

232. Irving Kaplansky: *Primary ideals in group algebras.*

Let A be the L_1-algebra of any locally compact abelian group. It is shown that any closed primary ideal in A is maximal. The method is an extension of that of Segal and Ditkin, who proved the theorem for the group of integers or of real numbers. Some applications are noted. (Received January 13, 1949.)

233*t*. Jakob Levitzki: *A theorem on polynomial identities.*

Let S be a ring satisfying a polynomial identity $f(x_1, x_2, \cdots, x_n)=0$ of degree $d \geqq 1$. If d is of minimal value we say in short: S is a PI-ring of degree $d=d(S)$. Denote by $N(S)$ the radical of S (that is, the sum of all nilpotent ideals). The following theorem is proved: If S is a PI-ring of degree d, then the nilpotent elements of S satisfy the relation $x^{[d/2]} \in N(S)$. This implies, for example, that if S is the ring of all k by k matrices over a field F, then $k \leqq [d/2]$. Scalar extension of the underlying field of an algebra yield further results. Other applications concern PI-nil-rings. Starting with $N_1(S)=N(S)$, R. Baer (*Radical ideals*, Amer. J. Math. vol. 65 (1943)) defined a transfinite ascending chain of ideals $N_\alpha(S)$ terminating in the lower radical $U(S)$. The author proves that each PI-nil-ring is an L-ring, that is, such that $S=U(S)$. Since each L-ring is semi-nilpotent (examples show that the converse is false) this includes a recent result of I. Kaplansky (*Rings with a polynomial identity*, Bull. Amer. Math. Soc. vol. 54 (1948)). If $\lambda=\lambda(S)$ denotes the length of an L-ring (that is, the smallest ordinal such that $N_\lambda(S)=N_{\lambda+}(S)=S$) we prove that if S is a PI-nil-ring—and hence an L-ring—then $\lambda(S)$ is finite. Moreover: $\lambda(S) \leqq \log d(S)/\log 2$ ($S \neq 0$; if $S=0$ then $\lambda=d=1$). In particular: $\lambda(S)<d(S)$ for $S \neq 0$. (Received January 3, 1949.)

234. R. C. Lyndon: *The representation of algebras of relations.*

A complete set of axioms is given which characterizes the class of all abstract relational algebras (cf. Jónsson and Tarski, Bull. Amer. Math. Soc. Abstract 54-1-89) that are isomorphic to algebras of concrete binary relations. This set is infinite, and the axioms of the set contain bound variables, in addition to the operations union, intersection, complement, converse, and relational product. It is shown that this class of algebras is not characterizable by any set of axioms containing only free variables. This provides a negative solution to the representation problem for relational algebras posed by Jónsson and Tarski (loc. cit. problem one). (Received January 10, 1949.)

235. Seymour Sherman: *Size, shape and situation of solutions in strategy simplices.*

Let $M_i = df.$ closed convex polyhedral subset of Sm_i; the unit simplex of real n_i-dimensional euclidean space U_i; $E_i = df.$ minimal face of Sm_i containing M_i; $\epsilon_i = df.$ number of vertices in E_i; $\sigma_i = df.\ n_i - \epsilon_i$; $\mu_i = df.$ the dimension of M_i; $m_i = df.$ number of maximal faces of M_i; $e_i = df.$ number of maximal faces of M_i contained in proper faces of E_i. Necessary and sufficient conditions that Sm_i and M_i, $i = 1, 2$, represent the strategy simplices and solution polyhedra of some zero-sum, two-person, $n_1 \times n_2$ game are (1) $\epsilon_1 - \mu_1 = \epsilon_2 - \mu_2$, (2) $\sigma_2 \geqq m_1 - e_1$, and (3) $\sigma_1 \geqq m_2 - e_2$. The necessity of (1) has been shown by Bohnenblust, Karlen, and Shapley, *Solutions of discrete, two-person games* (unpublished), and Gale, Kuhn, and Tucker, *Multiplicity of solutions of two-person zero-sum games* (unpublished). Applications of the theorems are given. (Received December 20, 1948.)

236t. Robert Steinberg: *A geometric approach to the representations of the full linear group over a Galois field.*

Through a consideration of the permutation of the geometric entities of $PG(n-1, q)$ effected by the group G, which is essentially the collineation group of $PG(n-1, q)$ of all n-ary linear homogeneous substitutions of nonzero determinant with marks in $GF(q)$, a set $p(n)$ of irreducible representations of G is determined. This set immediately leads to a large number of irreducible representations of G, and a correspondence is seen to exist between sets of the representations so determined and sets of classes of conjugate elements of G. Through a favorable comparison of G with the symmetric group of degree n, much use is made throughout the paper of the methods of Frobenius in his determination of the irreducible representations of Sn (Berliner Berichte (1900) pp. 516–534). (Received December 20, 1948.)

ANALYSIS

237. D. B. Ames: *Certain inversion formulas for the Laplace transform.*

The case of a periodic determining function $F(t)$, real or complex, is first discussed. Inversion formulas are derived which have the forms of the Fourier series of the functions $\int_0^t F(q)e^{-xq}dq$ and $F(t)e^{-xt}$. The coefficients involve the transform and its real and imaginary parts. Similar formulas hold when $F(t)$ is non-periodic and zero for $t > t_0 > 0$. From these results and extensions of them, several theorems are derived about the form of $F(t)$ when the transform vanishes at points on a vertical line, yielding a contrast with Lerch's theorem. Necessary and sufficient conditions are obtained for $F(t)$ to be null. By a natural generalization of results in the periodic case an inversion formula, of similar form, is derived for the general case where $F(t)$ is non-periodic and has a finite abscissa of absolute convergence. This holds when $F(t)$ is real and integrable but not necessarily bounded. It is shown, with no implication that the transform is analytic, that $F(t)$ is null if its transform vanishes at all points of any vertical line in the half-plane of absolute convergence. (Received January 6, 1949.)

238*t*. Stefan Bergman and M. M. Schiffer: *Kernel functions and conformal mapping*. I.

Let B be a domain in the complex plane bounded by analytic curves; let $g(z, \zeta)$ be Green's function of B. Define the functions $K(z, \bar{\zeta}) = -(2/\pi)g_{z\bar{\zeta}}$ and $l(z, \zeta) = (1/\pi)(z-\zeta)^{-2} - (2/\pi)g_{z\zeta}$ which are analytic in their arguments. For each analytic $f(z)$ with $\iint |f|^2 d\tau_B < \infty$ we have $\iint K(z, \bar{\zeta})f(\zeta)d\tau_B = f(z)$, $\iint \overline{l(z, \zeta)}f(\zeta)d\tau_B = (1/\pi) \cdot \iint \overline{(z-\zeta)^{-2}} \cdot f(\zeta)d\tau_B$. $K(z, \bar{\zeta})$ becomes infinite if z and ζ converge to the same boundary point of B: $l(z, \zeta)$ is regular in the closed region $B+C$. The identity (a) $\iint l(z,\zeta)\overline{l(\zeta, w)}d\tau_B = K(z,\bar{w}) - \Gamma(z, \bar{w})$, with $\Gamma(z, \bar{w}) = \iint (\zeta - z)^{-2}\overline{(\zeta - w)^{-2}}d\tau_{\tilde{B}}$, $\tilde{B}$ = complement of B, shows that $K - \Gamma$ is regular in the closed region $B+C$. One has the inequality (b) $\left| \sum l(z_i, z_k)x_i x_k \right| \leq \sum K(z_i, \bar{z}_k)x_i \bar{x}_k$ for arbitrary constants x_i and points $z_i \in B$ $(i=1, \cdots, N)$. The transform $Tf(z) = (1/\pi)\iint (z-\zeta)^{-2}\overline{f(\zeta)}d\tau_B$ is defined in $B+\tilde{B}$; l is TK in B. The inversion formula $f(z) = (1/\pi)\iint (z-\zeta)^{-2}\overline{Tf(\zeta)}d\tau_{B+\tilde{B}}$ is proved. The integral equation $\phi_\nu = \lambda_\nu \cdot T\phi_\nu$ is studied by means of (a). (Received February 24, 1949.)

239*t*. Stefan Bergman and M. M. Schiffer: *Kernel functions and conformal mapping*. II.

If $w = \phi(z)$ maps the domain B upon a domain B_1 with corresponding kernels $K_1(w, \bar{\omega})$ and $l_1(w, \omega)$, one has the transformation formulas: $K_1(w, \bar{\omega})\phi'(z)\overline{\phi'(\zeta)} = K(z, \bar{\zeta})$, $l_1(w, \omega)\phi'(z)\phi'(\zeta) = l(z, \zeta) + U(z, \zeta)$ with $U(z, \zeta) = (1/\pi)[\log (\phi(z) - \overline{\phi'(\xi)} - \log (z-\zeta)]_{z\zeta}$. Since the inequality (b) has to hold also in B_1, $\left| \sum [l(z_i, z_k) + U(z_i, z_k)]x_i x_k \right| \leq \sum K(z_i, \bar{z}_k)x_i \bar{x}_k$ which gives, for K and l known and fixed, an inequality for all functions $\phi(z)$ univalent in B. One obtains easily by specialization Grunsky's necessary and sufficient conditions of univalence. The following lemma is used: Let $V(z, \zeta) = \sum d_{mn}z^m\zeta^n$ be symmetric in z and ζ and analytic around the origin; let $K(z, \bar{\zeta}) = \sum k_{mn} z^m \cdot \bar{\zeta}^n$ be the development of the K-kernel of B around the origin. If $\left| \sum d_{mn} x_m x_n \right| \leq \sum k_{mn} x_m \bar{x}_n$ for every choice of x_n, then $V(z, \zeta)$ is regular analytic for both variables in B. (Received February 24, 1949.)

240*t*. Stefan Bergman and M. M. Schiffer: *Kernel functions and pseudo-conformal mapping*. I.

Let D^4 be a domain in the z_1, z_2-space bounded by two analytic hypersurfaces $\mathfrak{h}_1^3 = [z_2 = e^{i\lambda_1}]$ and $\mathfrak{h}_2^3 = [z_1 = h(z_2, \lambda), 0 \leq \lambda \leq 2\pi]$. $S^2 = \mathfrak{h}_1^3 \cap \mathfrak{h}_2^3$ is the distinguished boundary surface of D^4. The authors consider functions of the "extended class" $E(D^4)$, that is, real functions $H(x_1, y_1, x_2, y_2)$ which are harmonic in x_1, y_1 in every intersection $D^4 \cap [z_2 = z_2^0] = D^2(z_2^0)$ and such that $H(\mathrm{Re}[h(z_2, \lambda)], \mathrm{Im}\,[h(z_2, \lambda)], x_2, y_2)$ is harmonic in $|z_2| < 1$ for every λ. Every function which is the real part of an analytic function in z_1, z_2 belongs to $E(D^4)$. The generalized Green's formula $\{\phi, \psi\} = \iint \phi\psi_{n_1n_2}ds_1ds_2 = \iint \phi_{n_1}\psi_{n_2}ds_1ds_2 = \iint \phi_{n_2}\psi_{n_1}ds_1ds_2 = \iint \phi_{n_1n_2}\psi ds_1ds_2$ is proved for ϕ, $\psi \in E(D^4)$ where the integration is extended over S^2 and $\{\phi, \psi\} = \iiiint_{D^4}[\phi_{x_1x_2}\psi_{x_1x_2} + \phi_{y_1x_2}\psi_{y_1x_2} + \phi_{x_1y_2}\psi_{x_1y_2} + \phi_{y_1y_2}\psi_{y_1y_2}]dx_1dy_1dx_2dy_2$. Orthonormal functions of the class $E(D^4)$ in the $\{\phi, \psi\}$-metric are defined, the convergence of their kernel K is proved. (Received February 24, 1949.)

241*t*. Stefan Bergman and M. M. Schiffer: *Kernel functions and pseudo-conformal mapping*. II.

Fundamental solutions in the class $E(D^4)$ of the form $\log |z_1 - \zeta_1| \log |z_2 - \zeta_2| + \delta_1(z_2, \zeta_2) \log |z_1 - \zeta_1| + \delta_2(z_1, \zeta_1; \zeta_2) \log |z_2 - \zeta_2| + s(z_1, z_2; \zeta_1, \zeta_2)$, $s \in E(D^4)$, are introduced. δ_1, δ_2 are harmonic in $|z_2|<1$ and $z_1 \in D^2(\zeta_2)$, respectively. Four Green's functions $g^{(i)}(z_1, z_2; \zeta_1, \zeta_2)$ are defined as follows: $g^{(i)}$ is a fundamental solution with properly chosen δ_1 and δ_2 and satisfies on S^3 the conditions: $g^{(1)}=0$, $g^{(2)}_{n_1 n_2}=c_1(z_1)+d_1(z_2)$, $g^{(3)}_{n_1}=c_2(z_2)$, $g^{(4)}_{n_2}=c_3(z_1)$. They lead to the solution of boundary values problems for biharmonic functions. $L=g^{(1)}+g^{(2)}-g^{(3)}-g^{(4)}$ is an element of $E(D^4)$. Let K be the kernel of this class, then $K-L$ is orthogonal to all biharmonic functions in the $\{\phi, \psi\}$-metric. In the case of special domains, for example, product domains, one has $K=L$. This result represents a generalization of a previous theorem for harmonic functions of two real variables (see Bergman-Schiffer, Duke Math. J. vol. 14 (1947) p. 623). (Received February 24, 1949.)

242. Lipman Bers: *Isolated singularities of minimal surfaces.* II.

The results announced previously (Bull. Amer. Soc. Abstract 54-7-239) are generalized to the case of a solution of the minimal surface equation (1) $[\phi_x(1+\phi_x^2+\phi_y^2)^{-1/2}]_x + [\phi_y(1+\phi_x^2+\phi_y^2)^{-1/2}]_y=0$ with finitely many-valued gradients. If $\phi(x, y)$ has an isolated singularity at a finite point, say at $z=x+iy=0$, then $w=\phi_x-i\phi_y$ has a limit w_0 at 0. If $\phi(x, y)$ has an isolated singularity at ∞, then w has a *finite* limit at ∞. If the gradient at the singularity is finite, then $\phi(x, y)=\phi_0(x, y)+R(x, y)$ where $\phi_0(x, y)$ considered on the Riemann surface of $z^{1/m}$ (m a positive integer) is harmonic and $R(x, y)$ is small compared to $\phi_0(x, y)$. If the gradient w is infinite at the singular point, a new kind of singularity arises which is in general topologically different from any singularity of harmonic functions. Explicit parametric and asymptotic formulas for all possible singularities are given. (Received February 9, 1949.)

243. T. M. Cherry: *Uniform asymptotic expansions of Bessel, hypergeometric, and other functions with transition points.*

The functions concerned involve a large parameter ν, and satisfy a differential equation of the form (1) $d^2y/dz^2+y\{\nu^2 f(z)+g(z, \nu^{-2})\}=0$, where $f(z)$ is regular at $z=0$ and vanishes to the first order at this point, and g is regular at $z=0$, $\nu^{-2}=0$. Thus in the real case the solutions are monotonic on one side of $z=0$, oscillating on the other. A-method is given for approximating to the solutions of any such equation in terms of those of another such equation, for example, the Airy equation (2) $d^2y/dz^2-\nu^2 zy=0$. *The approximations are uniform to any desired order $2n$ in ν^{-1}, for z in a domain having the point $z=0$ in its interior*, and are found by transforming the equations (1) and (2) to forms which differ only by terms of order $\nu^{-2n}y$. They are made by means of asymptotic series proceeding in integral powers of ν^{-2}. In the case of the Bessel functions $J_\nu(\nu z)$, $Y_\nu(\nu z)$, the approximations by means of Airy functions are uniform for all values of z, and the $O(\nu^{-4})$ approximation gives accuracy to 7 significant figures for $\nu \geqq 5$, or to 5 figures for $\nu \geqq 2$. (Received December 10, 1948.)

244t. I. I. Hirschman and J. A. Jenkins: *Note on a result of Levine and Lifschitz.*

The connection between the integral orders of the zeros of a function represented by a Fourier series and the degree of lacunarity of the series has been studied by S. Mandelbrojt and after him by B. Levine and M. Lifschitz, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 693–711. This note contains certain refinements of the results previously obtained. (Received December 20, 1948.)

245t. Meyer Karlin: *Systems of extremals for the simplest isoperimetric problem.*

The system of extremals of the simplest isoperimetric problem $\int F(x, y, y')dx = \min$ while $\int G(x, y, y')dx = \text{constant}$ is shown to be a triply infinite system of curves of the form $y''' = Ay''^3 + By''^2 + Cy'' + D$ where A, B, C, D are functions of x, y, y'. For a given lineal element there corresponds ∞^1 curves. The locus of centers of curvatures of the evolutes of these curves is a cubic. Some special and degenerate cases are discussed. (Received December 9, 1948.)

246*t*. Norman Levinson: *Criteria for the limit-point case for second order linear differential operators.*

If $f(x) \leqq Kx^2$ for large x then it is shown that at most one solution of $u'' + f(x)u = 0$ is integrable squared over (x_0, ∞). This is a special case of the following: The equation $(pu')' + qu = 0$ is in the limit-point case if there exists a positive monotone increasing function $M(x)$ such that for large x, (a) $q(x) \leqq M(x)$, (b) $\int^\infty dx/(p(x)M(x))^{1/2} = \infty$, and (c) $\lim\sup p^{1/2}(x)M'(x)/M^{3/2}(x) < \infty$. (Received January 3, 1949.)

247*t*. Norman Levinson: *On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase.*

The solution $y(x, \lambda)$ of $y'' + (\lambda^2 - P(x))y = 0$ with initial values $y(0, \lambda) = 0$, $y'(0, \lambda) = 1$ will, in case $P(x)$ is small for large x, satisfy $y(x, \lambda) \rightarrow B(\lambda) \sin(\lambda x - \phi(\lambda))$ as $x \rightarrow \infty$ where $B(\lambda)$ and $\phi(\lambda)$ are continuous for $0 < \lambda < \infty$. In physics the problem of the determination of $P(x)$ for a given phase function $\phi(\lambda)$ arises. It is shown that if $\int_0^1 x|P(x)|dx + \int_1^\infty x^2|P(x)|dx < \infty$ and if $\phi(\infty) - \phi(+0) < \pi$ then $\phi(\lambda)$ determines $P(x)$ uniquely. It is also shown that under more general conditions $\phi(\lambda)$ determines $B(\lambda)$ and conversely. In case $P(x) \geqq 0$ then the condition at ∞ becomes $\int_1^\infty xP(x)dx < \infty$. The condition $\phi(\infty) - \phi(+0) < \pi$ is equivalent to the condition that the boundary value problem should have no discrete spectrum. (Received January 3, 1949.)

248. L. H. Loomis: *Note on a theorem of Mackey.*

The author gives a simpler proof, by new methods, of the following slight generalization of a theorem of Mackey. Let $\sigma \rightarrow U_\sigma$ be a weakly continuous representation of a locally compact group G by unitary operators on a Hilbert space H, and let $A \rightarrow \overline{A}$ be a nontrivial Boolean homomorphism from the open subsets of G with compact closures onto projection operators on H, such that $U_\sigma \overline{A} = \overline{(\sigma A)} U_\sigma$. Then H is the direct sum of subspaces H_i each of which is unitary-equivalent to $L^2(G)$ by a mapping T_i such that U_σ becomes left translation through $\sigma(T_i U_\sigma T_i^{-1}(f(x)) = f(\sigma^{-1}x))$ and $\overline{A}$ becomes multiplication by the characteristic function $A(x)$ of $A(T_i \overline{A} T_i^{-1}(f(x)) = A(x)f(x))$. (Received February 26, 1949.)

249*t*. S. H. Min: *On the order of $\zeta(1/2+it)$.*

The problem of finding an upper bound for θ such that $\zeta(1/2+it) = O(t^\theta)$ has been attacked by van der Corput and Koksma, Walfisz, Titchmarsh, Phillips and Titchmarsh. Their results obtained are, neglecting a factor involving $\log t$, $\theta \leqq 1/6$, 163/988, 27/164, 229/1392 and 19/116 respectively. The object in the present paper is to prove that $\zeta(1/2+it) = O(t^{15/92+\epsilon})$ for $\epsilon > 0$. The proof depends essentially on the estimation of a double exponential sum of the form $\sum\sum \exp(2\pi i\phi(x, y))$ where $\phi(x, y)$ is a function whose Hessian $H(x, y)$ vanishes along certain lines. The method used in this paper can be applied to problems in the analytic theory of numbers, whose solutions depend upon the estimation of similar exponential sums. (Received November 17, 1948.)

250. H. E. Rauch: *A "Poisson" formula and the Hardy-Littlewood theorem for matrix spaces.* Preliminary report.

An important type of "matrix space" of Siegel and Hua (cf. Hua, Amer. J. Math. vol. 66 (1944) pp. 470–488) is the space H consisting of the $n\times n$ complex matrices, Z, for which $I-ZZ^*>0$ (positive definite). This space, viewed as a subset of the space of n^2 complex variables, presents a remarkable analogy to the unit circle, $1-z\bar{z}>0$ (of which it is a direct generalization), the characteristic feature being the existence of a transitive group of analytic homeomorphisms which can be written as $W=(AZ+B)(CZ+D)^{-1}$ where A, B, C, D are $n\times n$ complex matrices satisfying the matrix analogues of the usual conditions. Clearly, the boundary of H contains $U(n)$, the unitary group, as the analogue of $z\bar{z}=1$. For the above and other matrix spaces the author has proved the following exact generalization of a well known theorem of Hardy and Littlewood on analytic functions in the unit circle: Let $f(Z)$ be an analytic function of n^2 complex variables, $Z\in H$, such that $\int_{U\in U(n)} |f(rU)|^{\lambda}dv\leqq M^{\lambda}$ where dv is the volume element on $U(n)$, $0\leqq r<1$, $\lambda>0$. Then $\int_{U\in U(n)} \{\sup_{0<r<1} |f(rU)|\}^{\lambda}dv \leqq \alpha M^{\lambda}$ where α is independent of f. The proof combines previous methods of the author (C. R. Acad. Sci. Paris vol. 227 (1948) pp. 887–889) and a new formula directly generalizing the classical Poisson integral for real parts of analytic functions: $f(Z)=\int_{U\in U(n)} f(U)(\det|I-\bar{Z}Z'|)^{n}(\det |I-[U^*Z'+\bar{Z}U]+\bar{Z}Z'|)^{-n}dv$. (Received January 13, 1949.)

251*t*. H. M. Schwartz: *Contributions to existence theory of ordinary differential equations in the real domain.* I.

Carathéodory (*Vorlesungen über reele Funktionen*, 2d ed., pp. 665–688) has given far-reaching existence theorems for a system of ordinary differential equations in the real domain of the form (1) $y_i'(x)=f_i(x, y_1, \cdots, y_n)$ $(i=1, 2, \cdots, n)$. His assumptions on f_i are as general as can be desired as far as their dependence on x is concerned, but they are unduly restrictive in what concerns their dependence upon the $y|f_i|<$summable function of x (Theorem 2, p. 672). The author considers here some of the possibilities for relaxing the latter restriction. No attempt is made here at attaining at the same time full generality in the x-dependence of the f_i, but rather to single out some results which can be derived by simple means. A typical result is as follows. If the f_i are continuous for x in an interval I and all finite y, except possibly for a subset of I whose kth derived set vanishes where k is any positive integer, then the system (1) has a solution continuous in I and satisfying there any given initial values, provided that the f_i satisfy in I and for all finite y the following inequality: $|f_i|\leqq g(x)+h(x)\phi(\sum|y_i|)$, $\phi(t)$ monotonically increasing for $t>0$, $\limsup_{t\to\infty}\phi(t)/t<\infty$, and g, h summable in I. (Received January 17, 1949.)

252. Edward Silverman: *A definition of Lebesgue area for surfaces in arbitrary metric spaces.* Preliminary report.

An area is defined for surfaces in arbitrary metric spaces which agrees with Lebesgue area for surfaces in Euclidean space. The procedure is to define the area of a triangle in a Banach space B in such a way as to permit the use of the original definition of Lebesgue area. Let this area be denoted by L_B. Surfaces are then studied in m, the space of bounded sequences. In particular, it is shown that isometric surfaces in m have the same L_m area. Since any surface can be mapped isometrically into m, its Lebesgue area is defined to be the area of an isometric image in m. Let this area be denoted by L. If B is finite-dimensional, then it is shown that $L=L_B$. (Received February 24, 1949.)

APPLIED MATHEMATICS

253. T. C. Doyle: *Invariant theory of the general, ordinary, linear, homogeneous, second order, differential boundary problem.*

There is presented an invariantive formulation of the boundary problem treated in R. E. Langer's first Herbert Ellsworth Slaught memorial paper, *Fourier's series, the genesis and evolution of a theory*, Amer. Math. Monthly vol. 54 (1947). The solution $y(x, \lambda)$ of the differential equation, $y''+p(x)y'+[q(x)\lambda+r(x)]y=0$, is subjected to the boundary conditions $\sum_{\rho=1}^{2} [(\beta_{j\rho}^{-1}\lambda+\gamma_{j\rho}^{-1})y\cdot(a^{\rho}, \lambda)+(\beta_{j\rho}^{0}\lambda+\gamma_{j\rho}^{0})y(a^{\rho}, \lambda)]=0$, where the β's and γ's are constants and $y\cdot(x, \lambda)$ is the covariant derivative of $y(x, \lambda)$ under the most general group of transformations, $y=e^{\sigma(x)}\bar{y}$, $\bar{x}=\xi(x)$, preserving the differential equation. The Green's function is obtained in invariant form and its residues are computed by reduction of the system to canonical form. (Received December 7, 1948).

254. Sidney Kaplan and George O'Brien: *A note on Richardson's solution of the heat equation by finite difference techniques.*

Richardson's finite difference formula for the solution of the heat equation: $\partial^2\phi/\partial x^2=\partial\phi/\partial t$, namely: $(\phi_{k+1,j}-\phi_{k-1,j})/2\Delta t=(\phi_{k,j+1}-2\phi_{k,j}+\phi_{k,j-1})/(\Delta X)^2$ (L. F. Richardson, *The approximate arithmetical solution by finite differences of physical problems involving differential equations*, Philos. Trans. Roy. Soc. London, Ser. A vol. 210 (1910) p. 313) is proved to be unstable notwithstanding the fact that his formula possesses a remarkably small truncation error. The authors prove both theoretically and by examples that at a very early stage the solutions begin to oscillate, and at $X=0.4$, $t=0.017$, the values diverge beyond the boundary. The authors then exhibit a stable finite difference formula: $(\phi_{k+1,j}-\phi_{k,j})/\Delta t=(\phi_{k,j+1}-2\phi_{k,j}+\phi_{k,j-1})/(\Delta X)^2$ which can be used to solve this classical problem. (Received January 10, 1949.)

255t. Eric Reissner: *On bending of curved thin-walled tubes.*

The problem under consideration may be thought of as a problem of determining an axi-symmetrical stress distribution in a shell of revolution. Previous work on shells of revolution by H. Reissner, E. Meissner and others presupposed rotationally symmetric univalued displacements. It is shown that by admitting suitable multivalued expressions for displacements the known results may be generalized in such a way that the results for the problems of tube bending, previously studied by von Kármán, Lorenz and others by energy methods, are included. (Received January 13, 1949.)

256t. I. F. Ritter: *Noise abatement in numerical inversion of a matrix.*

"Noise" being defined as the accumulated influence of the rounding off errors, its occurrence in inverting a matrix by the elimination method is described and estimates for its magnitude are derived in a rigorous manner by J. von Neumann and H. H. Goldstine (Bull. Amer. Math. Soc. vol. 53 (1947)). In a manner quite trivial, compared with this elaborate solution of the involved estimation problem, an iteration process can be carried out by which, from a noisy approximation to the inverse, an unlimited number of correct digits of the true inverse can be computed. The effort required to remove the noise from the first solution does not exceed the work involved in obtaining the latter. The process is a matrix iteration, hence lends itself to automatic operation. If carried out on a desk calculator it is not impeded by limited width of the keyboard. It is particularly efficient in finding the numerical solution of a set of many linear equations to any desired precision. (Received November 19, 1948.)

257. Charles Tompkins: *Projection methods in calculation of some linear problems.*

The author deals with methods of calculation. In problem I he seeks a solution of equations $\sum A_{ik}x_k = b_i$; in problem II, to minimize the linear function $J = \sum c_k x_k$ subject to the restriction $\sum A_{ik}x_k \geqq b_i$, where $A_{ik} \geqq 0$. Problem II is of interest in connection with various linear supply problems concerning cost; in these the restrictive inequalities frequently include $x_i \geqq 0$. Successive projection from a point to the various hyperplanes represented by the equations of problem I (in terms of a multidimensional coordinate geometry with rectangular cartesian coordinates) is studied. Each projection is computationally short, and each produces a point nearer a solution (if one exists) by a factor definitely less than one. These two statements imply stability in the calculation and convergence. A possible helpful utilization of the fact of slow convergence in particular problems is noted. A similar attack is made on problem II; this attack involves motion from a point within the allowed region along a line whose direction numbers are the gradient of the linear function; when the boundary of the region is reached, reflection into the region is taken along another direction—one perpendicular to a bounding hyperplane reached. Slow convergence is again interpreted. (Received January 14, 1949.)

258. Alexander Weinstein: *On generalized potential theory and the equation of Darboux-Tricomi.*

It is shown that the two fundamental integrals of the equation $(^*)\phi_{xx}+\phi_{yy}+2py^{-1}\phi_y=0$ investigated in a previous paper (A. Weinstein, *Discontinuous integrals and generalized potential theory*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 342–354) are given by the formulas $\phi_1 = \pi^{-1}b^{-1/2}y^{-p}Q_{p-1}(1+2\epsilon^2)$ and $\phi_2 = \pi^{-1}b^{-1/2}y^{-p}Q_{-p}(1+2\epsilon^2)$ where $0<2p<2$, $y \geqq 0$, $b>0$, $\epsilon^2 = [x^2+(y-b)^2](4by)^{-1}$, and P and Q are Legendre functions of the first and second kind. It is shown that $\phi_1 = -\pi^{-1}b^{-1/2}y^{-p}P_{p-1}(1+2\epsilon^2)\log\epsilon + R_1(x, y-b)$ and $\phi_2 = -\pi^{-1}b^{-1/2}y^{-p}P_{-p}(1+2\epsilon^2)\log\epsilon + R_2(x, y-b)$ where R_1 and R_2 denote power series. The coefficient of $\log\epsilon$ in both cases is $u(x, y) = \pi^{-1}b^{-1/2}y^{-p}F(p, 1-p, 1, -\epsilon^2)$ where F denotes the hypergeometric series. The function $u(x, y)$ is a solution of $(^*)$ and coincides, for $p=1/6$, with the solution given by F. Frankl (Bull. Acad. Sci. URSS. vol. 10 (1946) pp. 135–166) in his investigations on Darboux-Tricomi's equation and its applications to transonic flows. (Received January 7, 1949.)

STATISTICS AND PROBABILITY

259t. B. O. Koopman: *The law of small numbers in Markoff processes.*

The author considers the probability of just s successes in a sequence of n trials forming a simple Markoff chain. The chain may be stationary (constant probabilities and correlations), semi-stationary (varying probabilities, constant correlations), or non-stationary (varying probabilities and correlations). In the limiting case of increasing n, probabilities of order $1/n$, and fixed correlations, a generating function is obtained which is of exponential form and reduces to the Poisson distribution generating function when the correlations are all zero. With its aid, the probabilities of s successes are given as simple expressions in terms of Laguerre polynomials, both in the stationary and semi-stationary cases. In the non-stationary case, the generating function is a simple exponential expression involving the generating function of the auto correlation. Various extension are considered to trials having many different outcomes, and to many-step processes. (Received December 13, 1948.)

260*t*. R. H. Fox: *On the asphericity of regions in a 3-sphere.*

If $X_1, \cdots, X_n$ are the 1-cells and $U_1, \cdots, U_m$ the 2-cells of a cell complex K normalized to contain only one vertex, then the fundamental group G of K is generated by $x_1, \cdots, x_n$ with defining relations $u_1 = 1, \cdots, u_m = 1$, where x_j and u_i are carried by X_j and U_i respectively. In $\tilde{K}$, the covering space of K determined by a normal subgroup G_1 of G, the boundary of a 2-cell rU_i, $r \in R(G/G_1)$, the integral ring of G/G_1, is $r \cdot \sum_{j=1}^{n} (\partial u/\partial x_j)^{\psi\phi} X_j$ (cf. Bull. Amer. Math. Soc. Abstract 55-3-115), where ϕ is the natural homomorphism of the ring of the free group $[x_1, \cdots, x_n]$ onto the ring $R(G)$ of G and ψ the natural homomorphism of $R(G)$ onto $R(G/G_1)$. Using this, Whitehead's condition (Fund. Math. vol. 32 (1939), pp. 149–166) for the asphericity of the complement of a knot may be replaced by the condition: $\sum_i \xi_i (\partial u_i/\partial x_j)^{\phi} = 0$, $\xi_i \in R(G)$ implies $\xi_i = 0$ in the ring of the knot group in one of its common presentations. (Received January 13, 1949.)

TOPOLOGY

261*t*. G. W. Whitehead: *On the characteristic cohomology class of a fibre bundle.*

Let X be a fibre bundle over a topological space B, with $\pi: X \rightarrow B$ as projection, and suppose that the fibre F is arcwise connected and that $\pi_i(F) = 0$ for $i < n$. For each $b \in B$, let $F_b = \pi^{-1}(b)$. The characteristic cohomology class c^{n+1} of the bundle may be regarded as an element of the $(n+1)$st cohomology group of B modulo a fixed point b_0 with local coefficients in the system of local groups $\{\pi_n(F_*)\}$. Let d^n be the obstacle to contracting F_{b_0} over itself to a point; d^n belongs to the nth cohomology group of F_b with coefficients in $\pi_n(F_{b_0})$. The cohomology classes c^{n+1} and d^n are connected by the formula $\pi^* c^{n+1} = \delta d^n$, where π^* is the homomorphism induced by the mapping $\pi: (X, F_{b_0}) \rightarrow (B, b_0)$, and δ is the coboundary operator. (Received December 7, 1948.)

262*t*. Leo Zippin: *Two-ended topological groups.*

Freudenthal has proved (see his *Neuaufbau der Endentheorie*, Ann. of Math. vol. 43 (1942) pp. 261–279) that a locally compact, connected, topological group G has at most two "ends"; these ends are ideal points which serve to make compact the group manifold, and groups with no ends are compact. In this note it is shown that the two-ended group manifolds are the topological product of a real axis by a compact connected set. The compact set is homeomorphic to the coset-space of G by a one-parameter subgroup. (Received February 26, 1949.)

T. R. HOLLCROFT,
Associate Secretary

THE FEBRUARY MEETING IN CHICAGO

The four hundred forty-fourth meeting of the American Mathematical Society was held at the University of Chicago, on Saturday February 26, 1947. The attendance was eighty-one, including the following seventy-seven members of the Society:

A. A. Albert, W. R. Allen, Reinhold Baer, E. W. Banhagel, S. F. Bibb, W. M. Boothby, A. J. Brandt, C. R. Cassity, Herman Chernoff, E. W. Chittenden, A. M. Christiansen, H. M. Clark, Byron Cosby, D. M. DeWitt, Flora Dinkines, John Dyer-Bennet, W. F. Eberlein, Chester Feldman, J. V. Finch, Harley Flanders, L. R. Ford, Evelyn Frank, C. G. Fry, M. P. Gaffney, David Gilbarg, S. H. Gould, L. M. Graves, L. W. Griffiths, William Gustin, M. M. Gutterman, P. R. Halmos, H. L. Harter, I. N. Herstein, L. A. Hostinsky, Ralph Hull, M. M. Johnsen, Louis Joseph, D. E. Kibbey, D. D. Kosambi, M. S. Kramer, W. C. Krathwohl, M. Z. Krzywoblocki, E. P. Lane, J. S. Leech, A. W. McGaughey, Saunders MacLane, Morris Marden, J. M. Mitchell, E. J. Moulton, S. B. Myers, Leopoldo Nachbin, Zeev Nehari, E. H. Ostrow, D. W. Pounder, W. T. Reid, Haim Reingold, Herman Rubin, L. J. Savage, A. C. Schaeffer, Lowell Schoenfeld, W. T. Scott, I. E. Segal, M. A. Seybold, H. A. Simmons, Annette Sinclair, A. H. Smith, E. H. Spanier, M. P. Steele, A. F. Strehler, E. A. Trabant, E. F. Trombley, Henry Van Engen, L. R. Wilcox, R. E. Wright, L. C. Young, J. W. T. Youngs, Antoni Zygmund.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor S. B. Myers delivered an address entitled *Normed linear spaces of continuous functions*. Professor Myers was introduced by Professor L. M. Graves of the University of Chicago. Contributed papers were presented to the Society at a session on Saturday afternoon, presided over by Professor L. R. Ford of the Illinois Institute of Technology.

Visiting mathematicians and their guests were entertained by Mrs. Saunders MacLane on Saturday afternoon following the conclusion of the session for contributed papers.

Abstracts of all papers presented at the meeting are given below. Papers read by title are indicated by the letter "*t*."

ALGEBRA AND THEORY OF NUMBERS

263*t*. Harvey Cohn: *Minkowski's conjectures on critical lattices in the metric* $(|\xi|^p+|\eta|^p)^{1/p}$.

The conjecture deals with the minimum constant c_p for which each lattice in the $\xi\eta$-plane will necessarily have two points no further apart than $c_pA^{1/2}$ in the sense of the metric $(|\xi|^p+|\eta|^p)^{1/p}$, $p>1$, where A is the eculidean area of the fundamental parallelogram of the lattice. The main result is that for p large enough, $c_p=2^{1/p}(1-\tau_p)^{1/2}/(1+\tau_p)^{1/2}$, where τ_p is the root of $\tau_p^p+1=2(1-\tau_p)^p$. Its significance lies in the fact that it verifies, in part, some conjectures of Minkowski, other parts of which have been recently disproved. The solution is determined, in principle, by a theorem of Minkowski which in this case asserts that $c_p=\delta_p^{-1/2}$, where δ_p is the minimum area of the simple infinity of parallelograms one vertex of which lies on the origin and the other three vertices of which lie of the curve $|\xi|^p+|\eta|^p=1$. The conditions defining minimum are treated asymptotically as p approaches infinity, and it is shown that for p large enough, only the obvious symmetry positions enter into consideration as optima. (Received January 10, 1949.)

264. I. N. Herstein: *A proof of a conjecture of Vandiver.*

Vandiver, in Ann. of Math. vol. 48 (1946) p. 28, conjectured that the Wedderburn theorem, that every finite skew-field is commutative, could be generalized as follows: Every finite, non-commutative ring has an element which is a divisor of zero and is not in the centrum of the ring. Using elementary ring theory, this conjecture is proved. (Received January 21, 1949.)

265*t*. C. M. Price: *Jordan division algebras.* Preliminary report.

It is shown that the only central Jordan division algebras $S^{(+)}$ over an algebraic number field F which are not fields are obtained by replacing ordinary multiplication by the quasi-multiplication $a \bigcirc b=(ab+ba)/2$ in S, where S is either (1) an associative central division algebra D of odd degree over F or (2) the set of all J-symmetric elements of D of odd degree over F, where J is of the second kind. More generally if S is an associative central simple crossed product of degree n over any field F, then $S^{(+)}$ is a Jordan division algebra if and only if S is a division algebra and n is odd. If S is the set of all J-symmetric elements of an associative central simple algebra A of degree n over its center F, then $S^{(+)}$ or $A^{(+)}$ is a Jordan division algebra not a field only if A is a division algebra. In this case if J is of the first kind so that $n=2^m$, and A contains an imprimitive quartic field over F when $n>2$, then $S^{(+)}$ is a Jordan division algebra if and only if $S^{(+)}$ is a field and $n=1, 2$. Also if n is odd and B is any subset of A (division algebra), then $B^{(+)}$ can have no (Jordan) divisors of zero. (Received January 19, 1949.)

266*t*. K. G. Ramanathan: *On the Fourier coefficients of certain modular forms.*

The present paper is devoted to the study of the congruence properties of the arithmetic function $P_\nu(n)$ defined by $\nu>0$, $\sum_0^\infty P_\nu(n)x^n=[(1-x)(1-x^2)\cdots]^{-\nu}$ for the moduli 5^α and 7^β. One of the many results proved is: if $\nu>0$, $\nu\equiv-24 \pmod{420}$ and $24m\equiv\nu \pmod{5^\alpha 7^\beta}$ then $P_\nu(m)\equiv 0 \pmod{5^\alpha 7^\beta}$. The Ramanujan-Watson congruences for the partition function are contained as particular cases. (Received January 13, 1949.)

267. H. A. Simmons: *Interpretation of belonging-to exponents in terms of ever-repeating decimals.*

In this paper, some number scales used have positive bases; others, negative bases. When the base of a scale that is used is negative, the digits in the associated decimal may not all be taken with the same sign. When they are not, the decimal is called a "pseudo-decimal." In a particular situation, our procedure leads naturally either to a customary decimal or to a "pseudo-decimal." Our basic result is this: *if* $(a', m)=1$, *where* $m>1$ *and* a' *is numerically larger than* 1, *then a necessary and sufficient condition that* a' *belong to the exponent* e *modulo* m *is that* $1/m$ *be expressible in the scale of* a' *by an ever-repeating decimal of shortest period* e. One corollary of this theorem is about the expression of b/m as an ever-repeating decimal when $(b, m)=1$,

$m>1$, and $1\leqq|b|<m$; another corollary relates to the replacement of a base a' by bases of the form $a'+km$; and finally five additional theorems are stated which are analogous to known theorems of elementary number theory. (Received March 2, 1949.)

268*t*. M. F. Smiley: *A remark on a theorem of Marshall Hall.*

The identity $x(yz-zy)^2=(yz-zy)^2x$ characterizes Cayley-Dickson algebras among alternative but not associative division rings. (Cf. Marshall Hall, *Projective planes*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 229–277.) This follows from Hall's proof and a result of A. A. Albert (*Absolute valued algebraic rings*, Bull. Amer. Math. Soc. Abstract 54-11-416). A geometric consequence is that the stated identity and Hall's Theorem L ensure the uniqueness of the coordinate ring in a projective plane. (Received January 8, 1949.)

ANALYSIS

269. William Gustin: *Areal mean value families.*

A family $\mathfrak{S}$ of plane surfaces is called an areal mean value family if every function f regular in a simply-connected domain D has the same areal mean value over any two surfaces of $\mathfrak{S}$ covered by D. Many areal mean value families besides the concentric circles (Gauss) and the confocal ellipses (Asgeirsson) are shown to exist. (Received January 12, 1949.)

270*t*. Samuel Karlin: *Orthogonal properties of independent functions.*

This paper presents a systematic study of the analytical orthogonal properties of independent functions. The first part treats of convergence of expansions of functions in series of independent functions. It is shown under mild conditions that almost all modes of convergence are equivalent. In the second part, the relationship of convergence and summability questions are investigated. The Lebesgue kernel of an independent system is shown to be nonsummable. Finally, the connection of independent systems and lacunary orthogonal systems is discussed. (Received February 6, 1949.)

271. Josephine M. Mitchell: *On convergence and* $(C, 1, 1)$ *summability of double orthogonal series.*

Let (i) $\sum_{m,n=1}^{\infty}a_{mn}\phi_{mn}(P)$, $a_{mn}=\int_E f(P)\phi_{mn}(P)dA_P$, be the orthogonal development of an arbitrary function $f(P)$ of class L^2 defined on a measurable set E of r-dimensional Euclidean space, with respect to the complete orthonormal system $\{\phi_{mn}(P)\}$ $(m, n=1, 2, \cdots)$ of functions of class L^2. The rôle of the Lebesgue functions, $L_{mn}(P)=\int_E|K_{mn}(P, Q)|dA_Q$, where $K_{mn}(P, Q)$ is the mnth kernel function, $\sum_{i,j=1}^{m,n}\phi_{ij}(P)\phi_{ij}(Q)$, in the convergence of series (i) is investigated. It is proved that if $L^*_{mn}(P)=\int_E \max_{1\leqq j\leqq n}|K_{mj}(P, Q)|dA_Q$ is $O(u^2(m))$ $(u(m)\leqq u(m+1))$, then the sequence $s_{mn}(P)/u(m)$, where $s_{mn}(P)$ is the mnth partial sum of (i), approaches a limit amost everywhere in E. In particular, if E is the Cartesian product of two measurable sets E_1 and E_2 and $\{\phi_{mn}\}=\{\phi_m^{(1)}\phi_n^{(2)}\}$, where $\{\phi_n^{(\kappa)}\}$ is a complete orthonormal set defined on E_κ $(\kappa=1, 2)$, then the result corresponding to the above for simple orthogonal series may be exactly generalized (S. Kaczmarz, Studia Math. vol. 1 (1929) p. 101) in the case of positive kernel functions. Similar results are obtained for $(C, 1, 1)$ summability. (Received January 13, 1949.)

272t. Zeev Nehari: *A class of domain functions and some allied extremal problems.*

Let D be a smoothly bounded finite domain of connectivity n in the complex z-plane, s the length-parameter on the boundary Γ of D, and $\lambda(s)$ a continuous positive function on Γ. It is shown that there always exist two uniquely determined single-valued analytic functions $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ which are regular in D with the exception of a simple pole of $L_\lambda(z, \zeta)$ at $z=\zeta$ ($\zeta \in D$) and which, for $z \in \Gamma$, are connected by the identity $\lambda(s)\overline{K_\lambda(z, \zeta)}ds = -iL_\lambda(z, \zeta)dz$. This identity implies that $K_\lambda(z, \zeta)$ has the reproducing property $\int_\Gamma \lambda(s)\overline{K_\lambda(z, \zeta)}f(z)ds = f(\zeta)$ with respect to the class C of functions $f(z)$ which are regular and single-valued in D and whose boundary values are of class $L^2(\Gamma)$. The function $K_\lambda(z, \zeta)$ can be computed by means of the bilinear formula $K_\lambda(z, \zeta) = \sum_{\nu=1}^{\infty} \phi_\nu(z)\overline{\phi_\nu(\zeta)}$, where $\{\phi_\nu(z)\}$ is a complete set of functions of C orthonormalized by the conditions $\int_\Gamma \lambda(s)\phi_\nu(z)\overline{\phi_\mu(z)}ds = \delta_{\nu\mu}$. An algorithm of similar type yields the function $L_\lambda(z, \zeta)$. It is then shown that the functions $K_\lambda(z, \zeta)$, $L_\lambda(z, \zeta)$, and certain combinations of them provide the solutions for a number of extremal problems in the theory of conformal mapping. (Received January 12, 1949.)

273. Lowell Schoenfeld: *On second order linear partial differential equations with singular coefficients.*

Let $L(U) = U_{z\bar{z}} + AU_z + BU_{\bar{z}} + CU$ where A, B and C are functions of the two independent complex variables $z = x+iy$, $\bar{z} = x-iy$ and x and y can take complex values. Using the work of Bergman (Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 1169–1198 and Trans. Amer. Math. Soc. vol. 57 (1945) pp. 299–331), the author proves that if $L(U) = 0$ and certain auxiliary conditions are satisfied then $U(z, \bar{z})$ is singular in the plane $\bar{z} = \bar{z}_0$. The following is typical of the two principal results. If $k \geqq 0$ is an integer, $B_{\bar{z}z} - A_{z\bar{z}} = 0 = 2^{-1}(2k-1)A_z - 2^{-1}(2k+1)B_{\bar{z}} + C - AB$, $B(z, \bar{z})$ is singular at $\bar{z}_0 \neq 0$ and has no other singularity, $A(z, \bar{z})$ is entire, H_{mn}, defined by $U \exp \{\int_0^{\bar{z}} A(z, \zeta)d\zeta + \int_0^z B(\xi, 0)d\xi\} = \sum_{m,n=0}^{\infty} H_{mn} z^m \bar{z}^n$, is such that $H_{0n} = 0$ for $n \geqq 1$, $H_{m0} = 0$ for $m < k$, $H_{k0} \neq 0$, $H_{m+1,0}/H_{m,0} \to 0$ as $m \to \infty$ and $L(U) = 0$, then $U(z, \bar{z})$ is single-valued for $|z|$, $|\bar{z}| < \infty$, is singular in the plane $\bar{z} = \bar{z}_0$, and has no other singularities for $|z|$, $|\bar{z}| < \infty$. (Received January 12, 1949.)

APPLIED MATHEMATICS

274. David Gilbarg: *A characterization of non-isentropic irrotational flows.*

Steady irrotational flows are considered for which entropy is required to be constant only along streamlines. It is shown that all plane subsonic irrotational flows are in the large either isentropic or are vortex flows. If certain general restrictions are placed on the boundaries of the flow region, the same result holds for mixed and supersonic flows. Analogous theorems are proved for axially symmetric and general spatial flows. The proofs depend on uniqueness theorems for the Cauchy initial value problem applied to the equation for the velocity potential of isentropic irrotational flow. A new uniqueness theorem is proved for the case of parabolic data on the initial curve, that is, for data on the sonic line. (Received January 14, 1949.)

275. M. Z. Krzywoblocki: *On shock theory in viscous gases.*

In most existing shock theories the assumption is that the shock curve is a geometric surface (line) of discontinuity in pressure, velocity, and so on. The location of the shock and its shape is more or less unknown at the very beginning of the problem. The author, starting with the application of the methods known from the theory of the "limiting lines" in an inviscid fluid to a viscous fluid, has obtained new results. The first result is that the "limiting lines" cannot exist in a viscous fluid. The question of whether a limiting line is a characteristic feature of a transformation (hodograph transformation) may be solved by using other kinds of transformation. The second result is that the curve of the beginning of a shock is the locus of points in which the change of pressure with respect to the running coordinate along a streamline is of the same order as the change of the components of the stress tensor with respect to the same coordinate. Hence the criterion for the beginning of the shock in a viscous fluid is the limited deformation in a continuous medium. (Received January 14, 1949.)

GEOMETRY

276*t*. Edward Kasner and John DeCicco: *Physical families in conservative fields of force.*

A physical system S_k of ∞^3 curves in a conservative field of force is obtained as the extremals of the variation problem $\int(W+h)^{(k+1)/2}ds=\min$, where W is the work function and h is the energy constant. According as k is 0, 1, -2, the system S_k consists of dynamical trajectories, general catenaries, brachistochrones. A velocity system is defined by the variation problem $\int e^{W/v_0^2}ds=\min$. In a non-homothetic conformal representation of a surface Σ_1 upon a surface Σ, complete systems S_k do not, in general, correspond. There is only one conservative field of force, namely the one for which the work function is $w=(ds_1/ds)^{2/(k+1)}$, on Σ, for which the complete system S_k becomes a complete system S_k on Σ_1. The corresponding work function on Σ_1 is $w_1=1/W$. There is no complete system S_k on Σ, which by a non-homothetic conformal map on Σ_1 becomes a complete system S_{k_1} with $k_1\neq k$ on Σ_1. (Received December 30, 1949.)

277*t*. Edward Kasner and John DeCicco: *Physical systems of curves in space.*

Connected with an arbitrary positional field of force in space are the systems S_k of ∞^5 curves along any one of which a constrained motion is possible such that the osculating plane at each point contains the force vector F, and the pressure P is proportional to the normal component N of F. Thus $P=kN$. Special cases are (a) dynamical trajectories, (b) general catenaries, (c) generalized brachistochrones, (d) velocity curves, according as k is (a) 0, (b) 1, (c) -2, (d) ∞. The locus of the centers of the osculating spheres of the curves of a system S_k through a given lineal element is a straight line. By varying the direction through the point, the associated lines form a congruence Ω of order one and class three. This congruence Ω is composed of the secants of a twisted cubic curve Γ. The associated quadrics through Γ are studied, from which can be deduced a geometric interpretation of the curl of F. Many other related results are also discussed. (Received December 27, 1949.)

LOGIC AND FOUNDATIONS

278*t*. A. R. Schweitzer: *On the place of Kant's Critique of Pure Reason in a genetic logic of mathematics.*

In his article, *Les idées directrices de la logique génétique des mathématiques* (Rev. de Mét. et de Mor., Paris, 1914) the author quotes from an article by Felix Klein mediating, in effect, between intuition and logical deduction in evolutionary mathematics. In this paper Klein's remark is interpreted as a special case of a statement by Kant (*Critique of pure reason*, translated by F. Max Muller, New York, 1911, p. 41): "... it is equally necessary to make our concepts sensuous ... as to make our intuitions intelligible ...." This leads to finding a place for Kant's Critique in a genetic logic of mathematics by interpreting the Critique as a theory of knowledge elaborating a relation between intuition and logic. For intuition is associated by Kant with sensibility and logic is subdividied by him into analytic, associated with the understanding, and dialectic, associated with reason. Also Kant assumes (ibid., p. 242; compare p. 563) that "All our knowledge begins with the senses, proceeds thence to the understanding, and ends with reason." (Received January 13, 1949.)

279t. A. R. Schweitzer: *Outline of a theory of knowledge suggested by mathematics*. Preliminary report.

This paper consists of three parts: I. Systems of notation (representation); II. Terms and processes. III. Relations. Instruments of notation are: letters, words, signs, symbols. Systems of notation include: language, literature, philosophy, science and mathematics. Terms include: entities, things, objects, facts, thoughts, concepts, ideas, elements, points, units, monads, atoms. Processes include: classification, orientation, working hypothesis, assumption, definition, composition, inference, method, proof, existence, construction, approximation, limitation, replacement, transformation. Mathematics is held to be a science of relations. In part II reference is made to Kant's statement, "Whatever the process and the means may be by which knowledge reaches its objects, there is one that reaches them directly, and forms the ultimate material of all thought, viz. intuition ...." (*Critique of pure reason*, translated by F. Max Müller, New York, 1911, p. 15). (Received January 13, 1949.)

TOPOLOGY

280t. M. K. Fort: *Essential and non-essential fixed points.*

Let X be a compact metric space having the fixed point property. Let $f\in X^X$ and p be a fixed point under f. The point p is an essential fixed point under f if corresponding to each neighborhood U of p there exists $\epsilon>0$ such that every g for which $\rho(f, g)<\epsilon$ has a fixed point in U. An approximation theorem is proved which shows that every $f\in X^X$ can be approximated arbitrarily close by a transformation, all of whose fixed points are essential. It is proved that if f has a single fixed point, then this fixed point is essential. Another existence theorem is proved, to the effect that if X is a n-manifold and the set of fixed points of $f\in X^X$ is totally disconnected, then f has at least one essential fixed point. Various examples and special cases are discussed. (Received January 13, 1949.)

281t. E. E. Moise: *A note on the pseudo-arc.*

It has been shown by R. H. Bing (Duke Math. J. vol. 15 (1948) pp. 729–742) that a certain continuum described by the author (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 581–594) is homogeneous. The present note gives a proof of Bing's result, based on the methods of the author's original paper. (Received February 3, 1949.)

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BOOK REVIEW

Functional analysis and semi-groups. By Einar Hille. American Mathematical Society Colloquium Publications, vol. 31. New York, American Mathematical Society, 1948. 12+528 pp. $7.50.

This is a very interesting and important treatise, embodying several books as it were, the largest among the latter being the one actually devoted to semi-groups themselves. The treatise might be described as a "cours d'analyse" for the study of vector-valued functions, Banach norms and spectral theory; but it also contains such "classical" material as Laplace integrals, Fourier series, Hermite expansions, real functions in Euclidean spaces, and so on.

1. The treatise incorporates first of all a kind of monograph, very up-to-date but not entirely complete, on ordinary Banach spaces with a goodly emphasis on "categories" which the author avers arises from the influence of Max Zorn. This is a very valuable supplement to Banach's own manual. The author then rapidly proceeds to the development of the notions of Banach valued measurable and integrable functions with which he will afterwards operate incessantly. We bow to the author in gratitude for naming his principal integral after us. But we were surprised to find that he does not describe how the transition from the ordinary Riemann integral to Lebesque integral may in itself be derived by a process of Banach-completion of a normed vector-space at first incomplete.

2. In the second place, Chapters V and XX of the treatise constitute a most handy monograph on so-called Banach algebras. A Banach algebra (and who does not know it by now?) is a Banach space with a super-imposed ring multiplication satisfying the relation

$$\|x\cdot y\| \leqq \|x\|\cdot\|y\|.$$

The Abelian prototype is the space of complex-valued functions $f(t)$, $-\infty < t < \infty$, with the norm $\int_{-\infty}^{\infty} |f(t)| dt$ and the multiplication

$$h(t) = \int_{-\infty}^{\infty} f(t-u)g(u)du.$$

The author gives an account of maximal ideals, including Gelfand's fundamental theorem on numerically-valued functions of maximal ideals for commutative Banach algebras. However the author does not include Gelfand's application of his theorem to the proof of Wiener's theorem on ordinary Fourier series. This is a most regrettable omission, seeing that this application is still probably the most telling link between the theory of Banach algebras and analysis of the homespun variety. The author has theorems on "structure" and "representation" (but no positive-definite functions); he greatly stresses the efforts of the contemporary school which originated technically with Sam Perlis and N. Jacobson to avoid the addition of a ring-unit whenever one is not present at the start. In the previously described prototype of a commutative Banach algebra the missing unit would be the "Dirac function" $e=dF(t)/dt$, $F(t)=0$, $-\infty<t<0$, $F(t)=1$, $0\leq t<\infty$. In this and all other cases the unit can be added very economically by forming the elements $\lambda e+a$. It is originally needed for the definition of "inverse" elements; but the basic relation $(a+e)(b+e)=e$ reduces to $ab+a+b=0$, and in the latter version the unit has been eliminated. It is claimed that the artificial addition of a unit would have distortive consequences, but this claim is not substantiated by any illustrative material.

3. Then there is a treatment of complex analytic functions from Banach space to Banach space. Assuming, for our discussion now, the values of the functions to be complex numbers, this is an extension of functions $f(z_1, \cdots, z_k)$ from a complex E_k to a Banach space B. In the latter case $f(a)$ is called analytic in D, $D\subset B$, if for a in D, b in B, and t complex, $f(a+bt)$ is analytic in t, "uniformly" in a, b. This analyticity is also described in terms of differentiability properties, mainly with a view to obtaining statements on the domain of convergence of what corresponds to the Taylor-expansion of $f(z_1, \cdots, z_k)$ around the origin in its diagonal arrangement $\sum_{n=0}^{\infty} P_n(z_1 \cdots z_k)$. In this entire context some very astute "category arguments" are used and the results are very interesting; but in connection with the theorem stated on page 90 it should be emphasized that it is a generalization of a classical theorem of Hartogs for functions in k complex variables, and that Hartogs' own version is presupposed known for the generalization and apparently has not been proved anew by the category argument.

4. If e is the unit and a any other element of a B-algebra, for example, an algebra of endomorphisms in some Banach space, the spectral analysis of a consists of a discussion of the expression

$$(\lambda e - a)^{-1} \tag{1}$$

considered as a function from the complex numbers λ to elements in the algebra; the resolvent set is the open λ-set for which the inverse exists, and the spectrum is its complementary set. The author indicates in what manner N. Dunford, E. R. Lorch, C. E. Rickart, A. E. Taylor and others have successfully extended and expanded previous results about operators in Hilbert space to general B-algebras. In this connection it seems that it would be worth while to investigate the nature of the expression.

$$(\lambda_1 e_1 + \cdots + \lambda_k e_k - a)^{-1}$$

as a function of k complex variables $\lambda_1, \cdots, \lambda_k$, for a set of "unit-like" elements $e_1, \cdots, e_k$. For instance, if $e = e_1 + \cdots + e_k$ with $e_i e_j = \delta_{ij} e_i$, then most likely the resolvent set is a topological product of open sets in the λ_j-planes separately, and is thus of rather special structure. But even in the most general set-up, the resolvent set could not be any given open set in λ-space (due to structural requirements on domains of analyticity for $k \geqq 2$), but our best guess is that an open set is a resolvent set provided it is the set of uniform convergence of a series

$$\sum_{n=0}^{\infty} (\alpha_{n1}\lambda_1 + \cdots + \alpha_{nk}\lambda_k + \alpha_{n0})^{-1}$$

with suitable numerical coefficients α_{rj}.

For fixed λ but variable a the expression (1) is an analytic function from a Banach algebra to itself. For such functions in general, E. R. Lorch has given a definition of analyticity and of the Cauchy integral $\int_C f(x)dx$, for x and $y = f(x)$ both in a B-algebra. From the author's standpoint the most important instances of such functions are either the resolvent (1), or an exponential e^{-tx} and its inverse, $\log x \equiv \int (1/x)dx$, both of which are fundamental to the very definition of a semi-group. They are special cases of general numerical functions of x, x in B, the latter being then roughly speaking a power series $\sum \alpha_n x^n$ with *numerical* coefficients α_n. For a non-commutative B-algebra, more general functions $f(x)$ have hardly been approached, even in the finite-dimensional case, and research in this direction would be most desirable.

5. Two of the most appealing sections of the treatise are Chapters VI and VII, insofar as they are devoted to an analysis of subadditive functions and to a purely geometric description of the boundary of Euclidean semimodules (that is, additive Abelian semi-groups), in particular to those having the origin of the space on its boundary (angular semi-modules). The author's digest of the existing material is most useful. We are taking the liberty of replenishing his list of references by quoting two items of our own in which angular semi-modules have been explicitly introduced for the purposes of Fourier analysis and Cauchy formulas, namely: *Group invariance of Cauchy's formula in several variables*, Ann. of Math. vol. 45 (1944) p. 686, and *Bound-*

ary values of analytic functions in several variables, Ann. of Math. vol. 45 (1944) p. 708.

6. A semi-group is a solution $f(\xi)$ of the functional equation

$$f(\xi_1 + \xi_2) = f(\xi_1)f(\xi_2) \tag{2}$$

which is defined from the half-line $0<\xi<\infty$, or a larger semi-module, to a Banach algebra, preferably an algebra of bounded linear transformations of a Banach space into itself; in the latter case it is better to write the above in the form:

$$T(\xi_1 + \xi_2)x = T(\xi_1)[T(\xi_2)x]. \tag{3}$$

With regard to the numerical solutions of (2) one may make two statements: (i) if a solution has any kind of "approximate" continuity, for example if it is Lebesgue measurable, then it automatically is "strictly" continuous; and (ii) if it is so continuous, it has the trivial form

$$f(\xi) = e^{a\xi}. \tag{4}$$

Now, in the non-numerical case, statement (i) still remains in force on the whole, and we shall not discuss it any further. But statement (ii) is subject to serious qualifications. First of all, (4) must be immediately amended to read

$$f(\xi) = je^{a\xi} \tag{5}$$

where $j=j^2$ is any idempotent element. Secondly, the existence of an element a which satisfies (5), or of an *un*bounded transformation A which satisfies

$$T(\xi)x = j[e^{A\xi}x] \tag{6}$$

is predicated on a foreknowledge that the limits

$$\lim_{\xi\to+0} f(\xi) = f(+0), \qquad \lim_{\eta\to 0} \frac{f(\eta) - f(+0)}{\eta}$$

do exist in some appropriate manner, and the manner in which the latter exist greatly conditions the way in which the equations (5) and (6) are to be interpreted.

Also, due to the possible generality of the spectrum of a (or A), the actual complex-analytic behaviour of $f(\xi)$ is not that of a simple exponential monomial as the simplicity of the symbol (5) might deceptively suggest. Rather it is that of an analytic function defined only in an angular sector, and exhibiting there the behaviour of general functions of exponential type, complete with indicator diagram à la Pincherle-Pólya, Laplace-transforms, and such like. The most notable features by which to distinguish it from a function of exponential type in general are as follows. If $f(\xi)$ can be appropriately continued analytically into a complex neighborhood of some finite real interval $\alpha \leqq \xi \leqq \beta$ then it can be so continued into an entire semi-module, and for $\alpha=0$ this is even an angular semi-module. Furthermore, depending on the set-up, the function $f(\xi)$ has usually some kind of limit behaviour at the termini of the sector, for both $\xi \rightarrow +0$ and $\xi \rightarrow +\infty$. The body of statements concerning such limit behaviour is what is commonly called "ergodic theory"; and with some adaptation of approach a certain part of the "Tauberian theory" may also be thus included. In discussing the latter "theories" the author does not draw on the concepts of "partial ordering" except for a brief description of a paper by the reviewer on p. 381. This leads to a tangible curtailment of viewpoint when discussing such topics as the spectral resolution of a Hermitian operator in Hilbert space, or G. D. Birkhoff's version of the ergodic theorem. But the author has knowingly chosen his viewpoints, as undoubtedly was his privilege.

7. Finally, under the designation of "special semi-groups" the author has assembled several topics in "concrete" analysis each involving a semi-group and falling under the following general scheme.

Take a system of functions $\{\phi_n(x)\}$ on some point set S, the index n being discrete or continuous. Next take a vector space of functions on S each admitting an expansion

$$f(x) \sim \sum_n a_n \phi_n(x)$$

or

$$f(x) \sim \int a_n \phi_n(x) dn$$

with suitable constant coefficients $\{a_n\}$. Now, take a system of nonnegative exponents $\{\lambda_n\}$ and for $\xi > 0$ introduce the transformations

$$T(\xi)f \sim \sum e^{-\xi\lambda_n} a_n \phi_n(x).$$

They obviously form a semi-group. Furthermore, if the system $\{\phi_n(x)\}$ is orthonormal with respect to a suitable measure on S, we have the kernel representation

$$T(\xi)f \sim \int_S K(\xi, x, y) f(y) d\delta y \tag{7}$$

where

$$K(\xi, x, y) \sim \sum_{n} e^{-\xi\lambda_n}\phi_n(x)\phi_n(y). \tag{8}$$

Now, one may envisage admitting an "arbitrary" sequence of exponents $\{\lambda_n\}$ and attempt to describe the corresponding arbitrary semigroup $T(\xi)$ and its associated kernel (8). This is an interesting enough problem, but a somewhat artificial one. A much more natural set-up arises if the exponents $\{\lambda_n\}$ and the functions $\{\phi_n(x)\}$ are in some specific relationship to each other, to wit, if they are the eigenfunctions and eigenvalues of some suitable differential equation

$$A_x\phi = \lambda\phi.$$

In such a case the transformed function (7) is the solution of the associated "heat equation"

$$A_x f(x, \xi) = -\lambda\frac{\partial f(x, \xi)}{\partial \xi}$$

with the initial value $f(x, 0) = f(x)$. Or, if we have

$$A_\lambda\phi = \lambda^2\phi,$$

then the function is a solution of the "Laplace equation"

$$A_x f(x, \xi) = \frac{\partial^2 f(x, \xi)}{\partial \xi^2}$$

again with the same initial value, and under the condition of remaining bounded for $0<\xi<\infty$. The nature of the kernel (Gauss-kernel, Abel-kernel, and so on) is then pushed sharply into the foreground and its behaviour closely reflects the nature of the operator $A_x\phi$ under discussion. For instance, roughly speaking, whenever the operator is a positive-definite one, the kernel will be a non-negative function and only then. In broaching these topics the author has by no means exhausted the existing fund of information, whether actual or potential, or delineated all existing interconnections. But he has performed the important service of assembling and sifting some prerequisite material from an absorbing field of study of ever widening interest.

S. Bochner

NOTES

An Institute for Teachers of Mathematics will be held on the Wellesley College Campus August 23–30, 1949. This Institute is sponsored by the Association of Teachers of Mathematics in New England and is planned to serve teachers of mathematics in elementary schools, secondary schools, and colleges.

Dr. Harish-Chandra of the Institute for Advanced Study and Dr. J. A. Jenkins of Harvard University have been awarded Frank B. Jewett Fellowships by the American Telephone and Telegraph Company.

Professor P. H. Linehan of City College, New York City, has retired with the title emeritus.

Professor Enrico Bompiani of the University of Rome has been appointed to a professorship at the University of Pittsburgh. He will spend half of each academic year at the University of Rome.

Professor Lamberto Cesari of the University of Bologna has been appointed to a visiting professorship at the University of California.

Professor S. S. Chern of the Institute of Mathematics of the Chinese Academy is presently a member of the Institute for Advanced Study.

Professor D. D. Kosambi of Tata Institute of Fundamental Research, Bombay, India, has been appointed to a visiting professorship at the University of Chicago.

Professor T. G. Room of the University of Sydney has been appointed to a visiting professorship at the University of Tennessee.

Professor Enrico Volterra of the University of Rome has been appointed to an associate professorship of mechanics at Illinois Institute of Technology.

Dr. F. L. Alt of the Ballistic Research Laboratories, Aberdeen Proving Ground, has accepted a position as assistant and acting chief of the Computation Laboratory, National Bureau of Standards.

Mr. C. W. Cassel of the University of Dayton has accepted a position as chief of the Branch of Measurement and Analysis, Engineering Division, Clinton County Air Force Base, Wilmington, Ohio.

Dr. H. E. Ellingson of Rosemount Research Center, University of Minnesota, has accepted a position as Mathematician with the Naval Ordnance Laboratory, Washington, D. C.

Mr. L. N. Enequist of the Bendix Aviation Corporation has accepted a position as mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground.

Dr. H. G. Landau of the Ballistic Research Laboratories, Aberdeen

Proving Ground, has accepted a position as research associate with the Committee on Mathematical Biology, University of Chicago.

Mr. George Lausch of Cornell University has been appointed to an assistant professorship at the University of Pittsburgh.

Dr. R. K. Luneburg of the Institute for Mathematics and Mechanics of New York University has been appointed to an associate professorship at the University of Southern California.

Mr. E. W. Marchand of the University of Rochester has accepted a position as physicist with the Eastman Kodak Company, Rochester, New York.

Professor W. E. Milne of Oregon State College is on leave at the Institute for Numerical Analysis, National Bureau of Standards.

Associate Professor E. D. Rainville of the University of Michigan is on leave at the California Institute of Technology.

Mr. L. L. Rauch of Princeton University has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor W. P. Reid of Purdue University has been appointed to an assistant professorship at the Air Forces Institute of Technology, Wright Field, Ohio.

Associate Professor E. K. Ritter of the U. S. Naval Postgraduate School has accepted a position as supervisor, Performance Analysis Group, Aeronautical Research Center, University of Michigan.

Mr. J. H. Rosenbloom of Frankford Arsenal, Philadelphia, has accepted a position as physicist with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Mr. J. W. Wray of the University of Illinois has been appointed to an acting assistant professorship at State Teachers College, New Paltz, New York.

The following promotions are announced:

J. G. Herriot, Stanford University, to an associate professorship.

Samuel Karlin, California Institute of Technology, to an assistant professorship.

Z. I. Mosesson, Prudential Insurance Company of America, to senior actuarial assistant.

The following appointments to instructorships are announced:

Illinois Institute of Technology, Mr. B. A. Galler; University of California, Mr. J. L. Hodges; Iowa State College of Agriculture and Mechanic Arts, Mr. W. C. Hoffman.

Professor Torsten Carleman of the University of Stockholm died on January 11, 1949 at the age of fifty-six years.

Dean Emeritus T. M. Focke of Case Institute of Technology died

March 2, 1949 at the age of seventy-eight years. He had been a member of the Society for thirty-five years.

Dr. H. V. Gummere died on February 9, 1949 at the age of eighty years. He had been a member of the Society for fifty-five years.

Professor Emeritus W. S. Hall of Lafayette College died December 17, 1948 at the age of eighty-seven years.

Professor Emeritus W. D. MacMillan of the University of Chicago died November 14, 1948 at the age of seventy-seven years. He had been a member of the Society for thirty-four years.

Mr. J. S. Mikesh of the Lawrenceville School, Lawrenceville, New Jersey, died January 29, 1949 at the age of sixty-seven years.

Professor Maximilian Philip of City College, New York City, died January 17, 1949 at the age of seventy-one years.

Mr. H. I. Treiber of Watson Laboratories, Columbia University, died on February 27, 1949 at the age of twenty-seven years.

Professor John Williamson of Queens College died on February 9, 1949 at the age of forty-seven years. He had been a member of the Society for twenty-three years.

The following one hundred and seventeen doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1948 in universities in the United States and Canada. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

Milton Abramowitz, New York, February, minor in aeronautical engineering, *On the backflow of a viscous fluid in a diverging channel.*

N. B. Allison, Kentucky, June, *On ideals in generalized quaternion algebras and representation of integers by ternary quadratic forms.*

R. A. Alpher, George Washington, May, *On the origin and relative abundance of the elements.*

R. D. Anderson, Texas, June, *Concerning upper semi-continuous collections of continua.*

M. M. Andrew, Massachusetts Institute of Technology, June, minor in physics, *Application of Lagrangian procedures to an electromagnetic system consisting of a wave guide, an orifice, and a cavity.*

H. I. Ansoff, Brown, October, *The forming of a plastic sheet between fixed cylindrical guides with coulomb friction.*

T. M. Apostol, California, September, *A study of Dedekind sums and their generalization.*

R. W. Ball, Illinois, June, minor in philosophy, *Dualities of finite projective planes.*

J. W. Beach, Iowa State, June, minors in physics and chemical engineering, *Flow of viscous fluid between slowly rotating cylinders.*

T. A. Bickerstaff, Michigan, February, *Certain order probabilities in non-parametric sampling.*

B. P. Bogert, Massachusetts Institute of Technology, June, minor in electrical engineering, *On the separability of one variable in certain linear partial differential equations.*

Frances R. Brand, Brown, October, *An extension of the Prandtl-Reuss theory of plastic flow.*

N. A. Brigham, Pennsylvania, June, *On a certain weighted partition function.*

F. E. Browder, Princeton, June, *The topological fixed point theory and its application in functional analysis.*

L. J. Burton, Harvard, June, *On the complex elliptic geometry of Hilbert space.*

R. G. Calloway, Catholic University, October, minors in physics and chemistry, *The reality of the triangles in- and circumscribed to the plane rational quartic with one cusp and two nodes.*

Herman Chernoff, Brown, June, *Studentization in testing of hypotheses.*

Louise H. Chin, California, June, *Distributive and modular laws in relation algebras.*

F. E. Clark, Duke, June, minors in philosophy and physics, *Muirhead's theorem and its extension.*

E. A. Coddington, Johns Hopkins, June, *On the equations governing the stability of the laminar boundary layer in a compressible fluid.*

Harvey Cohn, Harvard, June, *Diophantine aspects of Poincaré theta functions.*

L. A. Colquitt, Ohio State, December, *On paths of minimum flight time.*

Damian Connelly, Catholic University, June, minors in physics and education, *An investigation of the conditions for reality of certain characteristics of the bicircular cuspidal quartic curves.*

J. C. Currie, Louisiana, *The field of values of a matrix.*

D. B. Dekker, California, September, *Hypergeodesic curvature and torsion.*

Yael N. Dowker, Radcliffe, March, *The ergodic theorems and invariant measure.*

James Dugundji, Massachusetts Institute of Technology, February, minor in electrical engineering, *Fundamental groups for spaces which are not* LC^1.

Helen M. Elliott, Radcliffe, June, *On the degree of approximation to harmonic functions by harmonic polynomials.*

H. W. Eves, Oregon State, June, minor in physics, *A class of projective space curves.*

Walter Federer, Iowa State, June, minor in plant breeding, *Evaluation of variance components from a group of experiments with multiple correlation.*

Evelyn Fix, California, June, *Distributions which lead to linear regressions.*

E. E. Floyd, Virginia, June, *The extension of homeomorphisms.*

M. K. Fort, Virginia, June, *A unification of the theory of semi-continuous functions.*

E. H. Gallagher, Cincinnati, June, *Borel summability of Fourier series of continuous functions.*

David Gans, New York, June, *An approach to elliptic geometry using translations.*

P. R. Garabedian, Harvard, June, *Schwarz's Lemma and the Szegö kernel function.*

B. R. Gelbaum, Princeton, June, *Expansions in Banach spaces.*

J. B. Giever, Massachusetts Institute of Technology, June, minor in physics, *On the equivalence of two continuous homology theories.*

E. N. Gilbert, Massachusetts Institute of Technology, June, minor in physics, *Asymptotic solution of relaxation oscillation problems.*

Oscar Goldman, Princeton, June, *On the theory of algebraic surfaces.*

R. D. Gordon, Indiana, October, minor in physics, *On the differential equation of the streamfunction for plane compressible flows.*

Arthur Grad, Stanford, June, *The region of values of the derivative of a schlicht function.*

Harold Grad, New York, June, minor in physics, *Approximation to the Boltzmann equation by moments.*

R. E. Graves, Minnesota, August, *Integral representations of linear and weak linear functionals defined over the Wiener space C.*

N. G. Gunderson, Cornell, September, minor in physics, *Derivation of criteria for the first case of Fermat's last theorem and the combination of these criteria to produce a new lower bound for the exponent.*

John Gurland, California, September, I. *Asymptotically normal estimates.* II. *Inversion formulae for the distribution of ratios.*

S. W. Hahn, Duke, June, minor in philosophy, *Universal spaces under strong homeomorphisms.*

Frank Harary, California, September, *The structure of Boolean-like rings.*

I. N. Herstein, Indiana, June, minor in physics, *Divisor algebras.*

M. O. Hizon, Michigan, February, *Actuarial study of Philippines government service insurance system.*

J. L. Hodges, California, September, *Studies in optimum statistical procedures. I. Initial sample size in the Stein procedure. II. Stringency in acceptance sampling.*

N. C. Hunsaker, Rice, June, *Extremal curves for the modulus of an analytic function.*

G. A. Hunt, Princeton, April, *On stationary stochastic processes.*

Richard Ingram, Johns Hopkins, June, *The characters of certain classes of the symmetric group.*

A. V. Jacobson, Michigan, February, *A generalized convolution for the finite Fourier transform.*

S. J. Jasper, Kentucky, August, *The homogeneity problem in the calculus of variations for a parametric form.*

J. A. Jenkins, Harvard, June, *Some problems in conformal mapping.*

P. S. Jones, Michigan, February, *The development of the mathematical theory of linear perspective and its connections with projective and descriptive geometry with especial emphasis on the contributions of Brook Taylor.*

M. L. Juncosa, Cornell, September, minor in physics, *On the distribution of the minimum of a sequence of mutually independent random variables.*

S. N. Karp, Brown, June, *On some problems in compressible flutter theory.*

J. B. Keller, New York, June, minor in physics, *Reflection and transmission of electro-magnetic waves by thin curved shells.*

J. B. Kelly, Massachusetts Institute of Technology, June, minor in physics, *On certain classes of algebraic integers.*

L. M. Kelly, Missouri, June, *New metric-theoretic properties of elliptic space.*

W. J. Klimczak, Yale, June, *Contributions of the theory of differential operators of indefinite order.*

R. G. Langebartel, Illinois, October, minor in astronomy, *An investigation of the convolutions of kernels of the type $x^\lambda - t^\lambda$ by a transform method and by fractional integration by parts.*

C. E. Langenhop, Iowa State, June, minors in physics and physical chemistry, *Properties of kernels of integral equations whose iterates satisfy linear relations.*

J. H. Laning, Massachusetts Institute of Technology, February, minor in physics, *Mathematical theory of lubrication-type flow.*

Harry Lass, California Institute of Technology, June, minor in physics, *The differential geometry of a space with a two-point differential metric.*

Mary A. Lee, Cornell, June, minor in physics, *Summability of di-*

agonal series formed from the terms of double series.

M. E. Levenson, New York, February, minor in physics, *Harmonic and subharmonic response for the Duffing equations* $\ddot{x}+\alpha x+\beta x^3 = F \cos \omega t$.

F. W. Light, Johns Hopkins, June, *A mathematical treatment of the two-dimensional motion of a perfect fluid.*

C. T. Loo, Chicago, June, *A note on the properties of Fourier coefficients.*

C. J. McNamara, Catholic University, June, minors in physics and physical chemistry, *The reality of the double tangents of the rational bicircular symmetric quartic.*

Clifford Maloney, Iowa State, June, minor in genetics, *Stratification in survey sampling.*

C. G. Maple, Carnegie Institute of Technology, June, *The Dirichlet problem: bounds at a point for the solution and its derivative.*

L. T. Mathews, St. Louis, June, minor in physics, *The study of constructive methods in the theory of algebraic equations.*

F. I. Mautner, Princeton, June, *Unitary representations of infinite groups.*

T. F. Morris, Toronto, June, *Quantum electrodynamics.*

G. D. Mostow, Harvard, March, *The extensibility of local Lie groups of transformations and groups on surfaces.*

D. N. Nanda, North Carolina, June, *Some contributions to the theory of multi-variate analysis.*

S. F. Neustadter, California, September, *Multiple-valued harmonic functions with circle as branch-curve.*

J. D. Newburgh, California at Los Angeles, June, *The variation of operator spectra.*

G. E. Nicholson, North Carolina, June, *The application of a regression equation to a new samples*

Ilse L. Novak, Radcliffe June, *On the consistency of Goedel's axioms for class and set theory relative to a weaker set of axioms.*

T. E. Oberbeck, California Institute of Technology, June, minor in physics, *Trigonometric expansions of doubly periodic functions of the second kind in any number of variables.*

G. K. Overholtzer, Indiana, June, minor in astronomy, *Some functions in elementary p-adic analysis.*

C. R. Putnam, Johns Hopkins, June, *An application of spectral theory to a singular calculus of variations problem.*

R. M. Redheffer, Massachusetts Institute of Technology, June, minor in electrical engineering, *Separation of Laplace's equation.*

O. W. Rechard, Wisconsin, June, *Some topics in theory of measure* *spaces.*

T. D. Reynolds, Duke, June, minor in physics, *Analytic solutions of integral equations with non-analytic kernels.*

Julia B. Robinson, California, June, *Definability and decision problems in arithmetic.*

C. L. Roney, Catholic University, June, minors in physics and chemistry, *Triangles in- and circumscribed to a rational quartic with simple symmetry.*

Hermann Rubin, Chicago, March, minor in physics, *Systems of linear stochastic equations.*

Herbert Ryser, Wisconsin, June, *Rational vector spaces.*

C. W. Saalfrank, Pennsylvania, February, *Retraction properties for normal Hausdorff spaces.*

F. J. Scheid, Massachusetts Institute of Technology, June, minor in physics, *On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid.*

Harold Shniad, California at Los Angeles, June, *Convexity properties of means of moduli of analytic functions.*

Edward Silverman, California, June, *Definitions of Lebesgue area for surfaces in metric spaces.*

W. H. Spragens, Cincinnati, June, *On the absolute Cesaro summability of double Fourier series.*

R. A. Staal, Toronto, November, *Star diagrams and modular properties of irreducible representations of the symmetric group.*

R. G. Stanton, Toronto, June, *On the Mathieu groups* M_{24}.

Robert Steinberg, Toronto, June, *Representations of the linear fractional groups.*

Ernst Straus, Columbia, June, *Some results in the generalized theory of relativity.*

G. L. Tiller, Kentucky, June, *The existence of a two dimensional potential flow with finite wake past a strictly convex profile symmetric with respect to the flow at infinity.*

C. J. Titus, Syracuse, June, *A topological characterization of a class of affine transformations.*

Deonisie Trifan, Brown, October, *A comparison between two theories of plasticity.*

W. R. Utz, Virginia, June, *Almost periodic geodesics on n-dimensional manifolds.*

A. H. Van Tuyl, Stanford, June, minor in chemistry, *The distribution of electricity on two neighboring charged spheres in the presence of an outside point charge.*

M. L. Vest, Michigan, February, *Birational space transformation* *associated with congruence of lines.*

Sylvan Wallach, Johns Hopkins, June, *Location of spectra of wave equations.*

Jean B. Walton, Pennsylvania, June, *Theta series in the Gaussian field.*

Herschel Weil, Brown, October, *On the extrusion of a very viscous liquid.*

J. G. Wendel, California Institute of Technology, June, minor in physics *A degenerate van der Pol equation.*

D. R. Whitney, Ohio State, August, *A comparison cf the power of non-parametric tests and tests based on the normal distribution under non-normal alternatives.*

B. R. Wicker, Iowa, June, *A differential equation of the second order and first degree containing two parameters.*

J. H. Wolfe, Harvard, June, *Tensor fields associated with Lipschitz cochains.*

M. A. Woodbury, Michigan, June, *Probability and expected values.*

The following two doctorates were conferred in 1947, but were not included in the list in the preceding volume of this Bulletin (vol. 54, pp. 496–501).

M. Francis B. Stauder, Notre Dame, June, *Studies on projective generalizations of metric geometry.*

M. Petronia Van Straten, Notre Dame, June, *The topology of the configurations of Desargues and Pappus.*

NEW PUBLICATIONS

Andronow, A. A., and Chaikin, C. E. *Theory of oscillations.* Ed. by S. Lefschetz. Princeton University Press, 1949. 10+358 pp. $6.00.

Arley, N. *On the theory of stochastic processes and their application to the theory of cosmic radiation.* New York, Wiley, 1949. 240 pp. $5.00.

Bhagavantam, S., and Venkatarayudu, T. *Theory of groups and its application to physical problems.* Waltair, Andhra University, 1948. 11+234 pp. 20 R.

Bourbaki, N. *Algèbre multilinéaire.* Paris, Hermann, 1948. 157 pp.

Chaikin, C. E. See Androvow, A. A.

Courant, R., and Friedrichs, K. O. *Supersonic flow and shock waves.* New York, Interscience, 1948. 16+464 pp. $7.00.

Coxeter, H. S. M. *The real projective plane.* New York, McGraw-Hill, 1949. 10+196 pp. $3.00.

Dailey, C. L., and Wood, F. C. *Computation curves for compressible fluid problems.* New York, Wiley, 1949. 10+33 pp. plus tables. $2.00.

Dark, H. J. *The life and works of Herbert Ellsworth Slaught.* Nashville, George Peabody College for Teachers, 1948. 8+152 pp.

Denjoy, A. *Leçons sur le calcul des coefficients.* Part IV. Paris, Gauthier-Villars, 1948. 389 pp. 3500 Fr.

Faulkner, T. E. *Projective geometry.* New York, Interscience, 1948. 128 pp. $3.00.

Finzi, B., and Pastori, M. *Calcolo tensoriale e applicazioni.* Bologna, Zanichelli, 1949. 8+427 pp. 2000 L.

Friedrichs, K. O. See Courant, R.

Hasse, H. *Zahlentheorie.* Berlin, Akademie-Verlag, 1949. 59 DM.

Hood, S. S., ed. by. *Archibald Henderson, the new Crichton.* New York, Beechhurst, 1949. 18+252 pp. $5.00.

Kendall, M. G. *Rank correlation methods.* London, Griffin, 1948. 8+160 pp. 18s.

Ledermann, W. *Introduction into the theory of finite groups.* New York, Interscience, 1948. 152 pp. $3.00.

Lefschetz, S. *Introduction to topology.* (Princeton Mathematical Series, no. 11.) Princeton University Press, 1949. 8+218 pp. $4.00.

Lefschetz, S. See Andronow, A. A.

Milne, E. A. *Vectorial mechanics.* New York, Interscience, 1948. 400 pp. $7.50.

Milne, W. E. *Numerical calculus.* Princeton University Press, 1949. 10+393 pp. $3.75.

Neyman, J., ed. by. *Proceedings of the Berkeley symposium on mathematical statistics and probability.* Berkeley, University of California Press, 1949. 8+501 pp. $7.50.

Pastori, M. See Finzi, B.

Reissner anniversary volume. *Contributions to applied mechanics.* Ed. by the Staff of the Department of Aeronautical Engineering and Applied Mechanics of the Polytechnic Institute of Brooklyn. Ann Arbor, Edwards, 1949. 8+493 pp. $6.50.

Reutter, F. *Darstellende Geometrie.* Vol. I: *Orthogonale Zweitafelprojektion.* Karlsruhe, Braun, 1948. 140 pp., 144 diagrams. 7.50 DM.

Schelkunoff, S. A. *Applied mathematics for engineerings and scientists.* New York, Van Nostrand, 1948. 480 pp.

Tables of the Bessel functions of the first kind of orders forty through fifty-one. (Annals of the Computation Laboratory of Harvard University, vol. 11.) Cambridge, Harvard University Press, 1948. 14+620 pp. $10.00.

Tables of generalized sine- and cosine-integral functions, Parts I–II. (Annals of the Computation Laboratory of Harvard University, vols. 18–19.) Cambridge, Harvard University Press, 1949. Part I, 38+462 pp. Part II, 8+560 pp. $10.00 each.

TARSKI, A. *Cardinal algebras*. New York, Oxford University Press, 1949. 12+326 pp. $10.00.

VAN OS, C. H. *Number and the cosmos*. In Dutch. Amsterdam, Meulenhoff, 1948. 280 pp. 7.90 Guilders.

VENKATARAYUDU, T. See BHAGAVANTAM, S.

WADE, T. L. *An introduction to the algebra of vectors and matrices*. Tallahassee, Wade, 1948. 6+97 pp.

WEIL, A. *Variétés abéliennes et courbes algébriques*. Paris, Hermann, 1948. 165 pp.

WOOD, F. C. See DAILEY, C. L.

ZASSENHAUS, H. J. *The theory of groups*. New York, Chelsea. 1949. 160 pp. $2.95.

THE SCHWARZIAN DERIVATIVE AND SCHLICHT FUNCTIONS

ZEEV NEHARI

It is customary to formulate the inequalities of the "Verzerrungssatz" type for analytic functions $w=f(z)$, schlicht in the unit circle, with reference to a specific normalization. The two normalizations mainly used are: (a) $f(z)$ is finite in $|z|<1$, $f(0)=0$, $f'(0)=1$; (b) $f(z)$ has a pole at $z=0$ with the residue 1. If we want to obtain inequalities which are independent of any particular normalization, we have to use quantities which are invariant with regard to an arbitrary linear transformation of the z-plane. The simplest quantity of this type is the Schwarzian differential parameter

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2, \tag{1}$$

also called the Schwarzian derivative of w with regard to z.

It is easy to obtain an upper bound for $\{w, z\}$ by a simple transformation of the classical inequality $|a_1| \leqq 1$ valid for functions $w=f(z)=z^{-1}+a_0+a_1z+\cdots$ schlicht in the unit circle. Indeed, applying this inequality to the coefficient of z in the expansion of the schlicht function

$$\begin{aligned} g(z) = \frac{f'(x)(1-|x|^2)}{f((x+z)/(1+\bar{x}z))-f(x)} &= \frac{1}{z} + \bar{x} - \frac{1}{2}\frac{f''(x)}{f'(x)}(1-|x|^2) \\ &- \frac{1}{6}(1-|x|^2)^2\left[\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right]z + \cdots \\ &\qquad\qquad (|x|<1), \end{aligned}$$

we obtain $|\{w, z\}| \leqq 6(1-|z|^2)^{-2}$.

We shall now show that by replacing the number 6 in this inequality by 2, this necessary condition for the schlichtness of $f(z)$ in $|z|<1$ becomes sufficient.

THEOREM I. *In order that the analytic function $w=f(z)$ be schlicht in $|z|<1$, it is necessary that*

$$|\{w, z\}| \leqq \frac{6}{(1-|z|^2)^2}$$

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and sufficient that

$$|\{w, z\}| \leqq \frac{2}{(1-|z|^2)^2}. \tag{2}$$

PROOF. By the classical theory of differential equations, the general solution of the differential equation

$$\{w, z\}/2 = p(z) \tag{3}$$

is of the form $w=y_1/y_2$, where y_1 and y_2 are two linearly independent solutions of the linear differential equation

$$y'' + p(z)y = 0. \tag{4}$$

To prove that the condition (2) entails the schlichtness of $w=f(z)$ in $|z|<1$ is therefore equivalent to showing that the ratio y_1/y_2 of two solutions of (4) is schlicht in $|z|<1$ if $p(z)$ is subject to the inequality

$$|p(z)| \leqq \frac{1}{(1-|z|^2)^2}. \tag{5}$$

Now it is easily seen that the schlichtness of y_1/y_2 can be expressed in a different form which is much easier to handle. In order that the ratio of two linearly independent solutions of (4) be schlicht in a certain domain, it is necessary and sufficient that no solution of (4) vanish there more than once. Indeed, if

$$\frac{y_1(z_1)}{y_2(z_1)} = \frac{y_1(z_2)}{y_2(z_2)} = \alpha,$$

then

$$y_1(z_1) - \alpha y_2(z_1) = 0, \qquad y_1(z_2) - \alpha y_2(z_2) = 0,$$

that is, the solution $y_1(z)-\alpha y_2(z)$ of (4) vanishes at the two points z_1 and z_2. Our task is therefore reduced to showing that no solution of (4) can vanish in $|z|<1$ more than once if the condition (5) holds.

For this purpose, we multiply equation (4) by $\bar{y}dz$ and integrate from z_1 to z_2 ($|z_1|<1$, $|z_2|<1$). This leads to

$$\int_{z_1}^{z_2} \bar{y}d(y') + \int_{z_1}^{z_2} p|y|^2dz = 0,$$

whence we obtain by partial integration

$$[\bar{y}y']_{z_1}^{z_2} - \int_{z_1}^{z_2} |y'|^2\overline{dz} + \int_{z_1}^{z_2} p|y|^2dz = 0,$$

the "Green's transform" of (4).

If $y(z_1) = y(z_2) = 0$, we have

$$\int_{z_1}^{z_2} |y'|^2 \overline{dz} = \int_{z_1}^{z_2} p\,|y|^2 dz, \tag{6}$$

an identity fundamental in Hille's investigations on the zero-free regions of the solutions of (4) [1].[1]

In view of the foregoing, Theorem I will be proved if we can show that (6) cannot be true if $p = p(z)$ is subject to the inequality (5). We may obviously confine ourselves to the case in which $|z_1| = |z_2| = \rho < 1$ without restricting the generality of the proof. It is even sufficient to show the incompatibility of (5) and (6) for $z_2 = -z_1$, since the condition

$$|\{w, z\}| \leqq \frac{2}{(1 - |z|^2)^2} \tag{7}$$

remains unaltered if z is made subject to any linear transformation $z = k(\alpha - x)(1 - \bar{\alpha}x)^{-1}$($|\alpha| < 1$, $|k| = 1$) which leaves the unit circle invariant. Indeed, any such transformation can be built up from two rotations of the type $z = kx$($|k| = 1$) and a substitution $z = (\beta - x) \cdot (1 - \beta x)^{-1}$ with real β. Now it is easily confirmed that

$$\{w, x\} = k^2\{w, kx\} = k^2\{w, z\} \tag{8}$$

and

$$\{w, x\} = \frac{(1 - \beta^2)^2}{(1 - \beta x)^4}\left\{w, \frac{\beta - x}{1 - \beta x}\right\} = \frac{(1 - \beta^2)^2}{(1 - \beta x)^4}\{w, z\}. \tag{9}$$

(8) shows that the substitution $z = kx$ does not affect the inequality (7). The same is true of the substitution $z = (\beta - x)(1 - \beta x)^{-1}$ since, by (7) and (9),

$$\begin{aligned} |\{w, x\}| &= \frac{(1 - \beta^2)^2}{|1 - \beta x|^4}|\{w, z\}| < 2\left[\frac{1 - \beta^2}{|1 - \beta x|^2(1 - |z|^2)}\right]^2 \\ &= 2\left[\frac{1 - \beta^2}{|1 - \beta x|^2 - |\beta - x|^2}\right]^2 = \frac{2}{(1 - |x|^2)^2}. \end{aligned}$$

Since, by a substitution $z = k(\alpha - x)(1 - \bar{\alpha}x)^{-1}$, any two points z_1, z_2($|z_1|$, $|z_2| < 1$) may be transformed into two points symmetrical with regard to the origin and neither the schlicht properties of $f(z)$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

in $|z| < 1$ nor condition (7) are affected by this substitution, it is sufficient for our purposes to consider the case $z_1 = -z_2$, where z_1 may obviously be taken positive. (6) takes in this case the form

$$\int_{-\rho}^{\rho} |y'|^2 dr = \int_{-\rho}^{\rho} p|y|^2 dr \qquad (0 < \rho < 1). \tag{10}$$

We shall now use the integral inequality

$$\int_{-1}^{1} \frac{u^2 dx}{(1 - x^2)^2} < \int_{-1}^{1} u'^2 dx \qquad (u = u(x)), \tag{11}$$

which holds for continuous real functions $u(x)$ which vanish at $x = \pm 1$ of the first order. (11) follows easily from the positive-definiteness of the integral

$$\int_{-1}^{1} \left(u' + \frac{xu}{1 - x^2}\right)^2 dx. \tag{12}$$

Expanding and integrating by parts, we obtain

$$\int_{-1}^{1} u'^2 dx + \left[\frac{xu^2}{1 - x^2}\right]_{-1}^{1} - \int_{-1}^{1} \frac{(1 + x^2)u^2}{(1 - x^2)^2} dx + \int_{-1}^{1} \frac{x^2u^2}{(1 - x^2)^2} dx > 0.$$

u being $O(1-x)$ and $O(1+x)$ at $x=1$ and $x=-1$ respectively, the integrals exist and the integrated part vanishes. Hence (11) follows. (11) is an inequality in the strict sense. Equality in (11) would imply the vanishing of the integral (12), whence $u' = -xu(1-x^2)^{-1}$, $u = c(1-x^2)^{1/2}$. Since this function does not satisfy our hypotheses, equality in (11) is excluded.

Substituting ρx for x in (11) we obtain the inequality

$$\rho^2 \cdot \int_{-\rho}^{\rho} \frac{u^2 dx}{(\rho^2 - x^2)^2} < \int_{-\rho}^{\rho} u'^2 dx, \tag{13}$$

valid for functions $u(x)$ possessing zeros of first order at $x = \pm\rho$.

Writing now $y = u + iv$, we have $|y|^2 = u^2 + v^2$, $|y'|^2 = u_x^2 + v_x^2$, and (10) takes the form

$$\int_{-\rho}^{\rho} (u_x^2 + v_x^2) dx = \int_{-\rho}^{\rho} p(u^2 + v^2) dx. \tag{14}$$

If p is subject to the inequality (5), we have

$$\left|\int_{-\rho}^{\rho} p(u^2+v^2)dx\right| \leqq \int_{-\rho}^{\rho} \frac{u^2+v^2}{(1-x^2)^2}dx < \rho^2 \int_{-\rho}^{\rho} \frac{u^2+v^2}{(\rho^2-x^2)^2}dx.$$

Since $y=u+iv$ is supposed to have zeros of the first order at $z=\pm\rho$, the same is true of u and v separately. Using (13), we obtain therefore

$$\left|\int_{-\rho}^{\rho} p(u^2+v^2)dx\right| < \rho^2 \int_{-\rho}^{\rho} \frac{u^2+v^2}{(\rho^2-x^2)^2}dx < \int_{-\rho}^{\rho} (u_x^2+v_x^2)dx,$$

which contradicts (14). We have thus proved that no solution of (4) can vanish in $|z|<1$ more than once if $p(z)$ satisfies (5). As shown above, this is equivalent to the schlichtness in $|z|<1$ of the ratio $w=y_1/y_2$ of two independent solutions of (4). In view of the identity (3), this completes the proof of Theorem I.

Replacing (11) by similar integral inequalities, it is possible to obtain sufficient conditions for the schlichtness of $w=f(z)$ in $|z|<1$ which are different from (2). As an example, we prove the following theorem:

THEOREM II. *If*

$$|\{w, z\}| \leqq \pi^2/2 \tag{15}$$

in $|z|<1$, *then* $w=f(z)$ *is schlicht in the unit circle. The constant* $\pi^2/2$ *in* (15) *is the best possible.*

PROOF. As shown above, it is sufficient to prove that the differential equation $y''+py=0$, where $p=\{w, z\}/2$, cannot have a solution y which vanishes in $|z|<1$ more than once.

Suppose now that $y(z_1)=y(z_2)=0$. Using the identity (6) and taking as the integration path the linear segment connecting z_1 and z_2 ($z=(z_1+z_2)/2+e^{i\theta}r$), we obtain

$$\int_{-\rho}^{\rho} |y'|^2 dr = e^{-2i\theta}\int_{-\rho}^{\rho} p|y|^2 dr, \qquad 2\rho = |z_2-z_1|. \tag{16}$$

Writing $y=u+iv$, we have $y'=e^{-i\theta}(u_r+iv_r)$, whence $|y|^2=u^2+v^2$, $|y'|^2=u_r^2+v_r^2$. Using this, (16) takes the form

$$\int_{-\rho}^{\rho} (u_r^2+v_r^2)dr = e^{-2i\theta}\int_{-\rho}^{\rho} p(u^2+v^2)dr. \tag{17}$$

We now use the inequality

$$\int_{-\pi/2}^{\pi/2} y^2 dx < \int_{-\pi/2}^{\pi/2} y'^2 dx \tag{18}$$

[2], which holds for a function $y(x)$ which is continuous for $-\pi/2 \leqq x \leqq \pi/2$ and has zeros of first order at $x=\pm\pi/2$. (18) follows immediately from the identity

$$\int_{-\pi/2}^{\pi/2} (y' + ytgx)^2 dx = \int_{-\pi/2}^{\pi/2} y'^2 dx + [y^2 tgx]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} y^2 dx$$

$$= \int_{-\pi/2}^{\pi/2} y'^2 dx - \int_{-\pi/2}^{\pi/2} y^2 dx,$$

which also shows that equality in (18) can only occur for $y \equiv c \cdot \cos x$. Replacing the variable x in (18) by $(\pi/2\rho)r$, we obtain

(19) $$\int_{-\rho}^{\rho} y^2 dr < \frac{4\rho^2}{\pi^2} \int_{-\rho}^{\rho} y'^2 dr, \qquad \left(y' = \frac{dy}{dr}\right),$$

valid for functions y vanishing of first order at $r=\pm\rho$.

Both u and v satisfy these conditions. In view of (3), (15) and (19), we therefore obtain

$$\left| e^{-2i\theta} \int_{-\rho}^{\rho} p(u^2 + v^2) dr \right| \leqq \frac{\pi^2}{4} \int_{-\rho}^{\rho} (u^2 + v^2) dr$$

$$\leqq \rho^2 \int_{-\rho}^{\rho} (u_r^2 + v_r^2) dr < \int_{-\rho}^{\rho} (u_r^2 + v_r^2) d_r.$$

This contradicts (17) and thus proves Theorem II.

That the constant $\pi^2/2$ in (15) is the best possible is shown by the function $w=f_0(z)=e^{\pi i(1+\epsilon)z}$, $\epsilon>0$. It is easily confirmed that

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{\pi^2}{2}(1+\epsilon)^2 > \frac{\pi^2}{2}.$$

The period of $f_0(z)$ being $2(1+\epsilon)^{-1}<2$, it follows that $f_0(z)$ cannot be schlicht in the unit circle.

For the applications it is more convenient to formulate Theorem II in a slightly different way.

COROLLARY. *The radius of univalence of the function* $w=f(z)$ *is at least equal to the smallest positive root* ρ_0 *of the equation*

(20) $$\rho^2 M(\rho) = \pi^2/2,$$

where

(21) $$M(\rho) = \max_{|z|=\rho} \left| \{w, z\} \right|.$$

Indeed, by (3), (15), (19), (20) and (21) we have, for $\rho<\rho_0$,

$$\left| e^{-2i\theta}\int_{-\rho}^{\rho} p(u^2+v^2)dr \right| \leqq M(\rho)\int_{-\rho}^{\rho}(u^2+v^2)dr < \frac{\pi^2}{2\rho^2}\int_{-\rho}^{\rho}(u^2+v^2)dr$$

$$\leqq \int_{-\rho}^{\rho}(u_r^2+v_r^2)dr,$$

which again contradicts (17).

As an application of Theorem II, or rather the corollary, we show that the error-function

$$w=\phi(z)=\int_0^z e^{-z^2}dz$$

is schlicht in a circle with radius

(22) $$\rho_0=(((\pi^2+1)^{1/2}-1)/2)^{1/2}$$

around the origin. Indeed,

$$\{w,z\} = -2(1+z^2)$$

and, by the corollary, the radius of univalence of $\phi(z)$ is therefore at least equal to the positive root of

$$2\rho^2(1+\rho^2)=\pi^2/2,$$

that is, (22).

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HARVARD UNIVERSITY AND
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REMARKS ON A PAPER BY ZEEV NEHARI

EINAR HILLE

In the preceding paper Zeev Nehari has proved some interesting inequalities for the Schwarzian derivative of a univalent (=schlicht) function. Thus *if* $f(z)$ *is univalent in the unit circle, then*

$$(1) \qquad |\{f(z), z\}| \leqq 6[1 - |z|^2]^{-2}$$

while if

$$(2) \qquad |\{f(z), z\}| \leqq 2[1 - |z|^2]^{-2},$$

then $f(z)$ *is univalent for* $|z| < 1$. The object of the present note is to show that 2 *is the best possible constant* in (2) in the following sense:

For every $C > 2$ *there exists a function* $f(z)$ *such that for* $|z| < 1$ *we have* (i) $f(z)$ *is holomorphic,* (ii) $f(z)$ *takes on the value one infinitely often, and* (iii) $|\{f(z), z\}| \leqq C[1 - |z|^2]^{-2}$ *with equality for real values of* z.

An explicit example of such a function is given by

$$(3) \qquad f(z) = \left(\frac{1 - z}{1 + z}\right)^{\gamma i},$$

where γ is a real constant, $f(0) = 1$, and $C = 2(1 + \gamma^2)$.

In view of the background of the problem, the following approach is natural. Let F_a denote the family of fractional linear transforms with constant coefficients of the quotient of two linearly independent solutions of the differential equation,

$$(4) \qquad \frac{d^2y}{dz^2} + a(1 - z^2)^{-2}y = 0,$$

where a is an arbitrary parameter. If $f(z) \in F_a$, then

$$\{f(z), z\} = 2a(1 - z^2)^{-2}.$$

If one function $f(z)$ of F_a is univalent in the unit circle, then they all are. Let us determine the region U of the a-plane such that if $a \in U$, then the functions of F_a are univalent for $|z| < 1$. By Nehari's Theorem I the region $|a| \leqq 1$ belongs to U and (1) makes it plausible that U is contained in $|a| \leqq 3$.

Equation (4) has elementary solutions. The indicial equations at

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the singular points $z = \pm 1$ are identical,

$$\rho^2 - \rho + \tfrac{1}{4}a = 0,$$

and if its two roots are α and β, a simple calculation shows that

$$y(z) = (1 - z)^{\alpha}(1 + z)^{\beta}$$

is a solution. If $a \neq 1$, $\alpha \neq \beta$ and $y(-z)$ is a second linearly independently solution. The family F_a is generated by their quotient

$$f(z) = \left(\frac{1 - z}{1 + z}\right)^{\delta}, \qquad \delta = \beta - \alpha = (1 - a)^{1/2} = \mu + i\nu, \tag{5}$$

where $f(0) = 1$ and the square root is $+1$ for $a = 0$.

It remains to determine when $f(z)$ is univalent in $|z| < 1$ or, what is equivalent, when t^{δ} is univalent in the right half-plane. A necessary and sufficient condition that for every choice of v the equation $t^{\delta} = v$ shall have at most one solution with $|\arg t| < \frac{1}{2}\pi$ is given by $2|\mu| \geqq \mu^2 + \nu^2$. This implies that either $|\delta - 1| \leqq 1$ or $|\delta + 1| \leqq 1$. Since $a = 1 - \delta^2$, we see that the boundary of U is the cardioid

$$a = -2e^{i\phi} - e^{2i\phi}, \qquad -\pi < \phi \leqq \pi, \tag{6}$$

and U consists of the interior and boundary of the cardioid. We note that the largest circle with center at the origin contained in U is $|a| = 1$ and that $|a| \leqq 3$ in U with equality only for $a = -3$.

Every point a outside of U leads to a family of functions which are not univalent in the unit circle. In particular, for every real γ, $\gamma \neq 0$, the point $a = 1 + \gamma^2$ is outside of U. Thus the function (3) is not univalent for $|z| < 1$ and a simple calculation shows that it takes on the value one infinitely often in the unit circle.

In conclusion, we observe that the boundary point $a = -3$ of U is of considerable interest. One of the functions of F_{-3} is $4z(1+z)^{-2}$ which is the extremal function of Koebe's mapping problem.

YALE UNIVERSITY

SPECIAL PROPERTIES OF MEASURE PRESERVING TRANSFORMATIONS

STEPHEN P. DILIBERTO

1. **Summary.** In studying problems[1] concerned with the qualitative description of bounded trajectories, in a region free of singular points, associated with a flow

$$dx_i/dt = P_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n) \tag{1.1}$$

the P_i being holomorphic in the x_i, we considered the possibility of finding "point conditions" on the P_i which would insure a smooth behavior on the part of a trajectory—or more specifically, on any motion in its limit set. For example, the restriction that the transformation defined by equations (1.1) preserve n measure is expressible by the point condition

$$\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} = 0.$$

Being unable to ensure the behavior desired by this condition, we sought stronger conditions. In particular we asked, "What are the conditions on the P_i such that the transformations defined by (1.1) preserves p measure where p is restricted to values $1 \leqq p \leqq n-1$?" The answer to this question and also to the question, "Is this a restrictive condition?" is contained in the following theorem.

THEOREM I. *The condition that the flow defined by*

$$dx_i/dt = P_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

the P_i being holomorphic, preserve p measure, p any (fixed) integer between 1 *and $n-1$, is that*

$$\partial P_i/\partial x_j = - \partial P_j/\partial x_i$$

for all i and j. These conditions imply that the motion is rigid.

It is an open and apparently difficult question as to whether every point transformation (we are considering only homeomorphisms)—of a sufficiently differentiable class—of E^3 onto itself is obtainable from a flow, that is, from the solutions of a system of first order equations of the type (1.1). (In E^2 there are point transformations which are *not* embeddable in flows.) Thus it is natural to ask for how

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[1] *On systems of ordinary differential equations*, which will appear elsewhere.

large a class of transformations the above result holds. We have a partial sharpening of Theorem I to the following theorem.

THEOREM II. *The only point transformations, of class C^3, of Euclidean three space into itself which preserve area are rigid motions (and reflections).*

2. A determinant expansion. We state next, without proof, a well known property about the product of an $n\times p$ by $p\times n$ matrix:

LEMMA. *Let*

$$A = (a_{ij}) = \begin{pmatrix} a_{11} \cdots a_{1n} \\ a_{21} \cdots a_{2n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{p1} \cdots a_{pn} \end{pmatrix}, \qquad B = (b_{ij}) = \begin{pmatrix} b_{11} \cdots b_{1p} \\ b_{21} \cdots b_{2p} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n1} \cdots b_{np} \end{pmatrix},$$

$$C = A\cdot B = (c_{ij}) = \begin{pmatrix} c_{11} \cdots c_{1p} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_{p1} \cdots c_{pp} \end{pmatrix}, \qquad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Then

$$|C| = \sum_{(l_i)} \begin{vmatrix} a_{1l_1} \cdots a_{1l_p} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{pl_1} \cdots a_{pl_p} \end{vmatrix} \begin{vmatrix} b_{l_11} \cdots b_{l_1p} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{l_p1} \cdots b_{l_pp} \end{vmatrix}.$$

This lemma says that the determinant of the product of $p\times n$ matrix by an $n\times p$ matrix is expressible as the sum of products of p square matrices, the first factor being a p square determinant from A selected by choosing columns $l_1, \cdots, l_p$ and the second factor being a p square determinant from B selected by choosing rows $l_1, \cdots, l_p$ from B, the summation being over all ways, $C_{n,p}$, of choosing p numbers from a set of n (and we take $l_1<l_2< \cdots <l_p$).

'We make use of this lemma to generalize the well known product formula for Jacobians. Let X_i and x_i be functions of class C^2 where

$$X_i = X_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, p,$$

$$x_i = x_i(u_1, \cdots, u_p), \qquad i = 1, \cdots, n;$$

then, the Jacobian of the X_i with respect to the u_i is expressible as

$$\frac{\partial(X_1, \cdots, X_p)}{\partial(u_1, \cdots, u_p)} = \left|\frac{\partial X_i}{\partial u_j}\right| = \begin{vmatrix} \partial X_1/\partial u_1 \cdots \partial X_1/\partial u_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \partial X_n/\partial u_1 \cdots \partial X_n/\partial u_n \end{vmatrix}$$

but
$$\partial X_i/\partial u_j = \sum_{k=1}^{n} (\partial X_i/\partial x_k)\cdot(\partial x_k/\partial u_j)$$

and so $\partial(X_1, \cdots, X_p)/\partial(u_1, \cdots, u_p)$ has the same form as $|C|$ in the last lemma. Thus

$$\frac{\partial(X_1, \cdots, X_p)}{\partial(u_1, \cdots, u_p)} = \sum_{(l_i)} \frac{\partial(X_1, \cdots, X_p)}{\partial(u_{l_1}, \cdots, u_{l_p})}\cdot\frac{\partial(x_{l_1}, \cdots, x_{l_p})}{\partial(u_1, \cdots, u_p)}.$$

3. **Proof of Theorem I.** Beginning with equation (1.1), let

$$x_i = f_i(x_{10}, \cdots, x_{n0}, t) \tag{3.1}$$

be the solution of that equation which at time t_0 passes through the point $(x_{10}, \cdots, x_{n0})$. We take a regular surface to be given by

$$x_i = g_i(u_1, \cdots, u_p), \qquad i = 1, \cdots, n, \tag{3.2}$$

where the rank of the functional matrix

$$\begin{pmatrix} \partial g_1/\partial u_1 \cdots \partial g_1/\partial u_p \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \partial g_n/\partial u_1 \cdots \partial g_n/\partial u_p \end{pmatrix}$$

is exactly p and where the g_i are analytic in the u_i and defined for values of u_i in an open convex p-cell, P, in the E^p space of the u_i. Then the p area, A_p, of this surface is given by

$$A_p = \int \cdots \int \left\{ \sum_{(l_i)} \frac{\partial(g_{l_1}, \cdots, g_{l_p})^2}{\partial(u_1, \cdots, u_p)} \right\}^{1/2} du_1, \cdots, du_p. \tag{3.3}$$

If we use capital letters for the transform of the surface (3.2), the transformation being given by (3.1), its equation is, at "time t,"

$$X_i = f_i(g_1, \cdots, g_n, t), \qquad i = 1, \cdots, n. \tag{3.4}$$

Its area $A_p(t)$ is given by the integral

$$A_p(t) = \int \cdots \int \left\{ \sum_{(l_i)} \frac{\partial(X_{l_1}, \cdots, X_{l_p})^2}{\partial(u_1, \cdots, u_p)} \right\}^{1/2} du_1 \cdots du_p. \tag{3.5}$$

But by hypothesis $A_p(t) \equiv A_p$, and since this must be true for each piece of the surfaces we must have that the integrands in (3.5) and (3.3) are equal. Using the lemma of the last section this equality becomes

$$(3.6)\quad \sum_{(l_\nu)} \frac{\partial(g_{l_1},\cdots,g_{l_p})^2}{\partial(u_1,\cdots,u_p)} = \sum_{(m_\nu)} \left\{ \sum_{(n_\nu)} \frac{\partial(X_{m_1},\cdots,X_{m_p})}{\partial(x_{n_1},\cdots,x_{n_p})}\cdot\frac{\partial(g_{n_1},\cdots,g_{n_p})}{\partial(u_1,\cdots,u_p)}\right\}^2.$$

We now proceed to draw consequences of (3.6) by making special choices of the g_i of (3.2). For a particular choice we do two things: (a) verify that this reduction is true for $t=t_0$; (b) impose the condition that this be an identity in t_0, that is, see that these equations, after differentiation with respect to t_0, are verified at $t_0=t$.

(A) *First choice of the* g_ν. Let

$$(3.7)\quad \begin{aligned} x_i &= u_i, & i &= 1,\cdots,p,\\ x_i &= 0, & i &> p. \end{aligned}$$

We now verify (a) for (3.6) for this choice of the g_i. First,

$$\frac{\partial(x_{l_1},\cdots,x_{l_p})}{\partial(u_1,\cdots,u_p)} = \begin{cases} 1, & l_i = i,\ i = 1, 2, \cdots, p,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, (3.6) reduces to

$$(3.8)\quad 1 = \sum_{(l_\nu)} \frac{\partial(X_{l_1},\cdots,X_{l_p})}{\partial(x_1,\cdots,x_p)}.$$

Now at $t_0=t$

$$\frac{\partial(X_{l_1},\cdots,X_{l_p})}{\partial(x_1,\cdots,x_p)} = \begin{cases} 1 & \text{if } l_i = i,\ i = 1,\cdots,p,\\ 0 & \text{otherwise.}\end{cases}$$

This last fact may be seen as follows: the expression $\partial X_k/\partial x_j$ is the derivative of the kth component X_k, of a solution of (1.1) with respect to the jth component of its initial point. According to (3.1)

$$X_i = f_i(x_{10},\cdots,x_{n0},t)$$

and when $t=t_0$, $X_\nu(t_0)=f_\nu(x_1,\cdots,x_n,t_0)=x_\nu$. Thus $\partial X_k/\partial x_j|_{t_0=t}=\delta_{kj}$, explaining the value of the last Jacobian. From this it follows that at $t_0=t$ (3.8), and hence (3.6), is verified.

We now verify (3.8) (that is, (3.6) reduced by this choice of the g_ν) for condition (b). To do this we must differentiate (3.8) with respect to t_0 and then put $t_0=t$. The differentiation will produce a sum of products, and when t_0 is put equal to t all products containing a factor of the form

$$\frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_p)}, \qquad \text{some } l_j \neq j,$$

will be zero. Hence, *eliminating all such terms before differentiation*, (3.8) reduces on the right to one type of term, namely, those for which $l_i = i$ $(i=1, \cdots, p)$

$$0 = \sum_{i=1}^{p} \left. \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} \right|_{t_0=t}.$$

And when $t_0 = t$, $dX_i/dt_0|_{t=t_0} = dX_i/dt = P_i(x_1, \cdots, x_n) = P_i$ so that condition (b) yields

$$0 = \sum_{i=1}^{p} \frac{\partial P_i}{\partial x_i}. \tag{3.9}$$

If we now choose a set of g_i as

$$\begin{aligned} x_1 &= 0, & & \\ x_i &= u_i, & i &= 2, \cdots, p+1, \\ x_i &= 0, & i &> p+1, \end{aligned}$$

we shall, by the same route as above, arrive at the relationship

$$\sum_{i=2}^{p+1} \frac{\partial P_i}{\partial x_i} = 0.$$

This equation combined with (3.9) yields $\partial P_1/\partial x_1 = \partial P_{p+1}/\partial x_{p+1}$ and in general we shall have (after suitable repetitions of this argument) that $\partial P_i/\partial x_i = \partial P_k/\partial x_k$. These combine with (3.9) to furnish the condition

$$\partial P_i/\partial x_i = 0, \qquad i = 1, \cdots, n. \tag{3.10}$$

(B) *Second choice of* g_i. Put

$$\begin{aligned} x_i &= u_i, & i &= 1, \cdots, p-1, \\ x_p &= x_{p+1} = u_p, & & \\ x_i &= 0, & i &> p+1. \end{aligned} \tag{3.11}$$

We note first of all that

$$\frac{\partial(x_{l_1}, \cdots, x_{l_p})}{\partial(u_1, \cdots, u_p)} = \begin{cases} 1, & l_i = i, \ i = 1, \cdots, p-1, \ l_p = p \text{ or } p+1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if we use (3.11) and these last results, (3.6) reduces to

$$(3.12)\quad 2 = \sum_{(l_i)} \left\{ \frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\}^2$$

and this is easily seen to verify condition (a). Namely when $t_0 = t$ the only nonzero terms are those for which

$$(l_1, \cdots, l_p) = \begin{cases} 1, 2, \cdots, p, \\ 1, 2, \cdots, p-1, p+1. \end{cases}$$

Proceeding as before, we verify (b) by differentiating (3.12)—and simplifying beforehand by dropping out all terms for which

$$(l_1, \cdots, l_p) \neq \begin{cases} 1, 2, \cdots, p, \\ 1, 2, \cdots, p-1, p+1, \end{cases}$$

since these will drop out when t_0 is put equal to t. This gives us

$$\begin{aligned} 0 = &\left\{ \frac{\partial(X_1, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_1, \cdots, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ &\cdot \sum_{i=1}^{p} \left\{ \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} \right. \\ &\qquad\qquad \left. + \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ &+ \left\{ \frac{\partial(X_1, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_1, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ &\cdot \sum_{i=1}^{p+1}{}' \left\{ \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_p)} \right. \\ &\qquad\qquad \left. + \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_{p-1}, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_p)} \right\} \end{aligned}$$

where $\sum'$ means the value $i = p$ is omitted. Putting $t = t_0$ reduces this to

$$\begin{aligned} 0 = &\sum_{i=1}^{p} \left\{ \frac{\partial(X_1, \cdots, P_i, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_1, \cdots, P_i, \cdots, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ (3.13)\qquad &+ \sum_{i=1}^{p+1}{}' \left\{ \frac{\partial(X_1, \cdots, P_i, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_p)} \right. \\ &\qquad\qquad \left. + \frac{\partial(X_1, \cdots, P_i, \cdots, X_{p-1}, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\}. \end{aligned}$$

The first term of the first brace is $\partial P_i/\partial x_i$ and the second term is zero except when $i=p$ when it is $\partial P_p/\partial x_{p+1}$; the first term of the second brace is zero except for $i=p+1$ when it is $\partial P_{p+1}/\partial x_p$ and the second term is $\partial P_i/\partial x_i$. Thus if we remove the $\partial P_i/\partial x_i$ which are zero, (3.13) gives us

$$\frac{\partial P_p}{\partial x_{p+1}} + \frac{\partial P_{p+1}}{\partial x_p} = 0.$$

A similar procedure based on corresponding choices of the g_i produces

$$\frac{\partial P_k}{\partial x_l} = -\frac{\partial P_l}{\partial x_k} \qquad \text{for all } k, l \tag{3.14}$$

(actually for $k\neq l$ but if we allow $k=l$ then (3.14) will include (3.10)).

From condition (3.14) we can derive very easily that the motion must be rigid, for differentiating (3.14) with respect to x_k shows that

$$\frac{\partial^2 P_l}{\partial x_k^2} = -\frac{\partial}{\partial x_k}\cdot\frac{\partial P_k}{\partial x_l} = -\frac{\partial}{\partial x_l}\cdot\frac{\partial P_k}{\partial x_k} \equiv 0.$$

Thus the P_i will be linear functions of the x_i and by (3.14) skew symmetric. Equation (1.1) written in vector form is then

$$\frac{dx}{dt} = Ax + c$$

where x, dx/dt, c are column vectors and A a skew-symmetric matrix ($A=(a_{ij})$ where $a_{ij}=-a_{ji}$). If one defines a new vector z by the relationship $Bz=x$, B being nonsingular, then z satisfies the vector differential equation

$$\frac{dz}{dt} = B^{-1}AB\cdot z + B^{-1}c.$$

And since A is skew-symmetric we can find an orthogonal B such that

$$B^{-1}AB = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & A_k \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & A_k & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

depending on whether n is even or odd and where

$$A_i = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}$$

where α_i is a constant (and real). The variables thus separate and the equations of the components of z can be integrated in pairs directly (trivially if $A_i = (0)$. For example, if $\alpha \neq 0$

$$\frac{dx_1}{dt} = \alpha x_2 + c_1, \qquad \frac{dx_2}{dt} = -\alpha x_1 + c_2$$

when integrated will give a rotation of angular velocity α about the point $(-c_2/\alpha, c_1/\alpha)$.

4. **Proof of Theorem II.** Let the correspondence $y_i = f_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$, $f_i \in C^3$, be an area-preserving map of E^3 onto itself. Since the surface S: $x_i = g_i(u_1, u_2)$ $(i = 1, 2, 3)$ $(g_i \in C^3$, $(u_1, u_2) \in R$, R a convex, open two-cell) and its image under this map have equal areas we must have

$$\iint_R \left\{ \sum_i \frac{\partial(g_i g_{i+1})^2}{\partial(u_1, u_2)} \right\}^{1/2} du_1 du_2$$
$$= \iint_R \left\{ \sum \frac{\partial(f_i(g_r) f_{i+1}(g_s))^2}{\partial(u_1, u_2)} \right\}^{1/2} du_1 du_2.$$

Moreover the integrands must be equal, hence, by the lemma of §2,

$$\sum_{i=1}^{3} \frac{\partial(g_i, g_{i+1})^2}{\partial(u_1, u_2)} = \sum_{j=1}^{3} \left(\sum_{k=1}^{3} \frac{\partial(f_j, f_{j+1})}{\partial(x_k, x_{k+1})} \frac{\partial(g_k, g_{k+1})}{\partial(u_1, u_2)} \right)^2.$$

This expression must be an identity in the g_i. Choosing $g_1 = u_1$, $g_2 = u_2$ and $g_3 = \text{const.}$ (and making two similar choices) yields

$$1 = \sum_{i=1}^{3} \frac{\partial(f_i, f_{i+1})^2}{\partial(x_j, x_{j+1})}, \qquad j = 1, 2, 3;$$

choosing $g_1 = u_1$, $g_2 = g_3 = u_2$ (and making two similar choices) yields

$$2 = \sum_{i=1}^{3} \left(\frac{\partial(f_i, f_{i+1})}{\partial(x_j, x_{j+1})} + \frac{\partial(f_i, f_{i+1})}{\partial(x_{j+2}, x_{j+3})} \right)^2, \qquad j = 1, 2, 3.$$

Combination of these last two restrictions yields

$$0 = \sum_{i=1}^{3} \frac{\partial(f_i, f_{i+1})}{\partial(x_j, x_{j+1})} \frac{\partial(f_i, f_{i+1})}{\partial(x_{j+2}, x_{j+3})}, \qquad j = 1, 2, 3.$$

The first and last of these relations tell us immediately that the 3 vectors $A_i = (\partial(f_1, f_2)/\partial(x_i, x_{i+1}), \partial(f_2, f_3)/\partial(x_i, x_{i+1}), \partial(f_3, f_1)/\partial(x_i, x_{i+1}))$ are normal and orthogonal. A_i represents the normal to a surface element which is the image of a plane element perpendicular to the $(i+2)$th coordinate axis. Thus we know that the families of planes parallel to the coordinate planes must be carried into a triply orthogonal family of surfaces. Thus excluding the rigid motions it follows from the proof of Liouville's theorem on three-dimensional conformal maps[2] that the map must be an inversion. But such maps do not preserve areas of spheres concentric to center of inversion and so are excluded—leaving the rigid motions (and reflections, of course).

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[2] Blaschke, *Differentialgeometrie*, 3d ed., Berlin, Springer, p. 100.

GENERALIZED CONVEX FUNCTIONS AND SECOND ORDER DIFFERENTIAL INEQUALITIES

MAURICIO MATOS PEIXOTO

1. **Introduction.** A well known theorem states that a necessary and sufficient condition in order that the twice differentiable function $y(x)$, $a<x<b$, be convex is that $y''\geqq 0$. The condition $y''>0$ is sufficient for the strict convexity of y.

In the present paper we show that if convexity is taken in the generalized sense of E. F. Beckenbach [1, 2],[1] a differential characterization of the above type can be obtained. As a particular case of a general theorem concerning second order differential inequalities we obtain a recent result of S. Tchaplygin, V. N. Petrov and J. E. Wilkins [3] concerning linear differential inequalities.

2. **Generalized convexity.** Let $\{F(x)\}$ be a family of real functions of the real variable x defined for $a<x<b$ and such that:

(1) Each member of the family is a continuous function of x.

(2) Given in the xy-plane two arbitrary points (x_1, y_1), (x_2, y_2) such that $a<x_1<x_2<b$, there is a unique member of the family passing through these two points, that is, such that its graph passes through these two points.

A function $\phi(x)$, $a<x<b$, is said to be *convex relative to the family* $\{F(x)\}$ *—a sub-*$\{F(x)\}$ *function* in Beckenbach's notation—if, for arbitrary x_1, x_2 such that $a<x_1<x_2<b$, the member of the family, $F_{12}(x)$, which pasess through $[x_1, \phi(x_1)]$, $[x_2, \phi(x_2)]$ is such that

$$\phi(x) \leqq F_{12}(x), \qquad x_1 \leqq x \leqq x_2. \tag{3}$$

If we have

$$\phi(x) < F_{12}(x), \qquad x_1 < x < x_2, \tag{4}$$

we say that $\phi(x)$ is *strictly convex relative to the family* $\{F(x)\}$ *or else that it is a strictly sub-*$\{F(x)\}$ *function.*

The ordinary convexity is obtained if we take as the family $\{F(x)\}$ the linear functions $mx+n$.

3. **An auxiliary theorem.** $\phi(x)$ being a sub-$\{F(x)\}$ function and $a<x_0<b$ we have proved elsewhere [4] the following theorem.

THEOREM 1. *There exist* $D(x)\in\{F(x)\}$, $E(x)\in\{F(x)\}$ *such that*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$D(x_0) = E(x_0) = \phi(x_0), \tag{5}$$

$$E(x) \leqq D(x) \leqq \phi(x), \qquad x_0 < x < b, \tag{6}$$

$$D(x) \leqq E(x) \leqq \phi(x), \qquad a < x < x_0. \tag{7}$$

(8) *If* $\phi'(x_0)$, $E'(x_0)$, $D'(x_0)$ *exist, then* $E'(x_0) = D'(x_0) = \phi'(x_0)$.

4. **The family $\{F(x)\}$ as solutions of a differential equation.** Consider on the strip S, $a<x<b$, $-\infty<y<+\infty$, the differential equation

$$y'' = G(x, y, y') \tag{9}$$

and let us suppose:

(10) the function $G(x, y, y')$, $(x, y) \in S$, $-\infty < y' < +\infty$, is continuous;

(11) to each point $(x_0, y_0) \in S$ and $-\infty < y_0' < +\infty$, there exists a unique solution $y(x)$ of (9) defined for $a<x<b$ such that

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'$$

and we shall assume the continuity of this solution with respect to the initial values y_0, y_0';

(12) given two distinct points belonging to S there is a unique solution of (9) passing through these points.

For some special types of $G(x, y, y')$ it is possible to assure that this condition (12) holds [5, Chap. V, VI].

From now on $\{F(x)\}$ will designate the family formed by the solutions of (9). Now, let $\phi(x)$, $a<x<b$, be a function with a continuous second derivative. Our main result is expressed by the following theorem.

THEOREM 2. *A necessary and sufficient condition that* $\phi(x)$ *be a sub-*$\{F(x)\}$ *function is that*

$$\phi'' \geqq G(x, \phi, \phi'), \qquad a < x < b. \tag{13}$$

PROOF. (a) *The condition is necessary.* Let $\varphi(x)$ be a sub-$\{F(x)\}$ function and consider a generic point $a<x_0<b$. Consider $\varphi(x) \in \{F(x)\}$ such that

$$\varphi(x_0) = \phi(x_0), \qquad \varphi'(x_0) = \phi'(x_0). \tag{14}$$

By Theorem 1 and (11) we have

$$\varphi(x) = D(x) = E(x) \leqq \phi(x), \qquad a < x < b. \tag{15}$$

We have by (14)

$$\begin{aligned}\phi(x) - \varphi(x) &= \phi[x_0 + (x - x_0)] - \varphi[x_0 + (x - x_0)] \\ &= \frac{(x - x_0)^2}{2}[\phi''(x_0) - \varphi''(x_0) + \epsilon]\end{aligned} \tag{16}$$

where ϵ is infinitesimal with $|x - x_0|$.

Suppose

$$\phi''(x_0) < \varphi''(x_0). \tag{17}$$

In this case we have, in a suitable neighborhood of x_0,

$$\phi(x) < \varphi(x) \tag{18}$$

which contradicts (15). As (17) is false, we must have, in virtue of (9) and (14),

$$\phi''(x_0) \geqq \varphi''(x_0) = G[x_0, \phi(x_0), \phi'(x_0)] \tag{19}$$

for every $a < x_0 < b$. Then (13) holds.

(b) *The condition is sufficient.* Suppose that (13) holds. The demonstration follows easily from the following lemmas:

LEMMA 1. *If* (13) *holds, to every* $a < x_0 < b$ *there exists a* $\nu > 0$ *such that*

$$\phi(x') - \bar{\varphi}(x') \geqq 0, \qquad |x' - x_0| \leqq \nu, \ |\bar{x} - x_0| \leqq \nu, \tag{20}$$

where $\bar{\varphi}(x) \in \{F(x)\}$ *is defined by*

$$\bar{\varphi}(\bar{x}) = \phi(\bar{x}), \qquad \bar{\varphi}'(\bar{x}) = \phi'(\bar{x}). \tag{21}$$

In fact, we have

$$\begin{aligned}\phi(x') - \bar{\varphi}(x') &= \phi[\bar{x} + (x' - \bar{x})] - \bar{\varphi}[\bar{x} + (x' - \bar{x})] \\ &= \frac{(x' - \bar{x})^2}{2}[\phi''(x^*) - \bar{\varphi}''(x^*)], \\ x^* &= \bar{x} + \theta(x' - \bar{x}), \qquad 0 < \theta < 1.\end{aligned} \tag{22}$$

As $\phi''(x)$ is a continuous function, given arbitrarily $\epsilon > 0$, there exists $h > 0$ such that

$$\phi''(x^*) > \phi''(x_0) - \epsilon/2, \qquad |x' - x_0| \leqq h, \ |\bar{x} - x_0| \leqq h. \tag{23}$$

By the supposed continuity of the solutions of (9) relative to the initial values, there is an $h_1 > 0$ such that

$$|\bar{\varphi}''(x^*) - \varphi''(x^*)| < \epsilon/4, \qquad |x' - x_0| \leqq h_1, \ |\bar{x} - x_0| \leqq h_1, \tag{24}$$

$\varphi(x) \in \{F(x)\}$ satisfying (14).

By the continuity of $\varphi''(x)$, there is an $h_2>0$ such that

$$(25)\qquad |\varphi''(x^*)-\varphi''(x_0)|<\epsilon/4,\qquad |x'-x_0|\leqq h_2,\ |\bar{x}-x_0|\leqq h_2.$$

If we put $\nu=\min(h, h_1, h_2, 2^{1/2}/2)$ we have from (24) and (25)

$$(26)\qquad \bar{\varphi}''(x^*)<\varphi''(x_0)+\epsilon/2,\qquad |x'-x_0|\leqq \nu,\ |\bar{x}-x_0|\leqq \nu.$$

From (22), (23), (26) and (19) we have

$$(27)\qquad \begin{aligned} \phi(x')-\bar{\varphi}(x') &> \frac{(x'-\bar{x})^2}{2}\left[\phi''(x_0)-\varphi''(x_0)-\epsilon\right] \\ &\geqq -\frac{(x'-\bar{x})^2}{2}\epsilon\geqq-\epsilon, \\ &\qquad |x'-x_0|\leqq\nu,\ |\bar{x}-x_0|\leqq\nu. \end{aligned}$$

As ϵ is an arbitrarily chosen positive number, (20) holds.

DEFINITION 1. Let us say that a closed interval of (a, b) has the property (P) when the graph of $\phi(x)$ in this interval lies above or on any integral curve of (9), which at a point of the arc of $\phi(x)$ corresponding to the interval has the same slope as $\phi(x)$.

In terms of this definition the above lemma states that any point $a<x_0<b$ is the center of a closed interval where the property (P) holds.

LEMMA 2. *The property* (P) *holds on every closed interval* $[\alpha, \beta]$, $a<\alpha\leqq x\leqq\beta<b$.

In fact, by the application of Lemma 1 and the familiar Heine-Borel theorem, we have that there are a finite number of closed inintervals covering $[\alpha, \beta]$ for which the property (P) holds. Observe that the closed interval intersection of two closed intervals for which the property (P) holds is another closed interval for which this property holds.

Then Lemma 2 is proved if we show that the closed interval union of two contiguous closed intervals having the property (P) is another closed interval for which this property holds. But this is easy to see. Indeed, let $[x_1, x_2]$, $[x_2, x_3]$, $x_1<x_2<x_3$, be the two closed intervals.

We shall show that the property (P) holds on $[x_1, x_3]$. Let, for instance, $x_1\leqq x_0\leqq x_2$, and $\varphi(x)\in\{F(x)\}$ be defined by (14). We have, as $[x_1, x_2]$ has the property (P),

$$(28)\qquad \varphi(x)\leqq\phi(x),\qquad x_1\leqq x\leqq x_2.$$

Suppose

$$\varphi(x_2) = \phi(x_2).$$

Let us show that we must have

$$\varphi'(x_2) = \phi'(x_2). \tag{29}$$

Indeed, according to (28), we must have

$$\varphi'(x_2) \geqq \phi'(x_2). \tag{30}$$

We shall show that the hypothesis

$$\varphi'(x_2) > \phi'(x_2) \tag{31}$$

is false. In fact, suppose that $\psi(x) \in \{F(x)\}$ is such that

$$\psi(x_2) = \phi(x_2), \qquad \psi'(x_2) = \phi'(x_2). \tag{32}$$

Then, by property (P), we have

$$\psi(x) \leqq \phi(x), \qquad x_1 \leqq x \leqq x_3. \tag{33}$$

We have also

$$\varphi(x) < \psi(x), \qquad x_1 \leqq x < x_2, \tag{34}$$

in virtue of (12) and of the fact that by (31) and (32) this inequality holds in some left neighborhood of x_2.

But from (14), (33) and (34) we have

$$\psi(x_0) \leqq \phi(x_0), \qquad \psi(x_0) > \phi(x_0) = \phi(x_0)$$

and this is absurd. Therefore (31) is false and (29) must hold. By (11) we must have

$$\varphi(x) = \psi(x) \leqq \phi(x), \qquad x_1 \leqq x \leqq x_3. \tag{35}$$

Suppose now

$$\varphi(x_2) < \phi(x_2).$$

Then it is easy to see, in virtue of (11), that there must be a point $x_0 < \xi < x_2$ such that

$$\varphi(\xi) = \psi(\xi). \tag{36}$$

Therefore we have, by (12),

$$\varphi(x) < \psi(x) \leqq \phi(x), \qquad \xi < x < x_2. \tag{37}$$

Combining (28) and (37) we have

$$\varphi(x) \leqq \phi(x), \qquad x_1 \leqq x \leqq x_3.$$

Then the property (P) holds on the closed interval $[x_1, x_3]$ and our lemma is demonstrated.

From this lemma follows immediately the proof of part (b) of our theorem. Indeed let α, β be two arbitrary numbers satisfying the condition $a<\alpha<\beta<b$. We must demonstrate that the $\theta(x)\in\{F(x)\}$ defined by

$$\theta(\alpha) = \phi(\alpha), \quad \theta(\beta) = \phi(\beta) \tag{38}$$

is such that

$$\phi(x) \leqq \theta(x), \qquad \alpha \leqq x \leqq \beta. \tag{39}$$

In fact suppose there is an $\alpha<\lambda<\beta$ such that

$$\phi(\lambda) > \theta(\lambda) \tag{40}$$

and consider the function $\omega(x)\in\{F(x)\}$ such that

$$\omega(\lambda) = \phi(\lambda), \qquad \omega'(\lambda) = \phi'(\lambda).$$

By Lemma 2 we know that

$$\omega(\alpha) \leqq \phi(\alpha) = \theta(\alpha), \qquad \omega(\beta) \leqq \phi(\beta) = \theta(\beta). \tag{41}$$

From (40) and (41) we conclude the existence of two numbers p, q, $\alpha<p<\lambda<q<\beta$, such that

$$\theta(p) = \omega(p), \qquad \theta(q) = \omega(q)$$

and this contradicts (12). Therefore (40) is false and (39) must hold. So condition (13) is sufficient to secure that $\phi(x)$ is a sub-$\{F(x)\}$ function. Hence Theorem 2 is proved.

Remark. This theorem generalizes obviously the well known property of ordinary convex functions considered in the beginning. It also generalizes the result that the condition

$$\phi''(x) + \phi(x) \geqq 0 \tag{42}$$

is necessary and sufficient in order that the twice differentiable function $\phi(x)$ be convex relative to the family of solutions of the equation

$$y'' + y = 0$$

(x varying in an interval shorter than π) which is equivalent [6, p. 98; 7, p. 281] to a result due to G. Pólya.

5. **A sharper result.** A sharper result is expressed by the following theorem.

Theorem 3. *A sufficient condition in order that the twice differentiable* *function* $\phi(x)$ *be a strictly sub-*$\{F(x)\}$ *function is that*

$$\phi'' > G[x, \phi, \phi'], \qquad a < x < b. \tag{13'}$$

PROOF. The proof of this theorem is but slightly different from that of Theorem 2, part (b), and we shall only insist on the modifications. It follows easily from the following lemmas.

LEMMA 1′. *If* (13′) *holds, to every* $a<x_0<b$ *there exists a number* $\nu>0$ *such that*

$$\phi(x') - \bar{\varphi}(x') > 0 \qquad |x' - x_0| \leqq \nu, \ |\bar{x} - x_0| \leqq \nu, \ x' \neq \bar{x}. \tag{20'}$$

To demonstrate this lemma it is sufficient to observe that in virtue of (13′), $\phi''(x_0)-\varphi''(x_0)>0$ so that by (27) and from the arbitrariness of ϵ we have (20′).

DEFINITION 2. Let us say that a closed interval has the property (P′) when the graph of $\phi(x)$ lies above any integral curve of (9) which at a point of the arc of $\phi(x)$ corresponding to the interval has the same slope as $\phi(x)$—with the unique exception of the point of contact.

The above lemma states that each point $a<x_0<b$ is the center of a closed interval where the property (P′) holds.

LEMMA 2′. *The property* (P′) *holds on every closed interval* (α, β), $a<\alpha\leqq x\leqq\beta<b$.

The demonstration is analogous to that of Lemma 2 and will not be given here.

The proof of Theorem 3 follows easily. Indeed let α, β be two arbitrary numbers satisfying the condition $a<\alpha<\beta<b$ and $\theta(x)\in\{F(x)\}$ defined by (38). We have shown that (39) holds. Now we shall show that we have

$$\phi(x) < \theta(x), \qquad \alpha < x < \beta. \tag{43}$$

In fact, suppose there is $\alpha<\xi<\beta$ such that

$$\phi(\xi) = \theta(\xi). \tag{44}$$

Then we have, by (39),

$$\phi'(\xi) = \theta'(\xi). \tag{45}$$

By Lemma 2′ we have that

$$\theta(x) < \phi(x), \qquad \alpha < x < \beta, \ x \neq \xi, \tag{46}$$

and (46) contradicts (39). So (44) is false and (43) holds. The theorem is demonstrated.

REMARK. The condition (13′) is not necessary in order that $\phi(x)$

be a strictly sub-$\{F(x)\}$ function. This is not true even when the convexity is taken in the ordinary sense. For instance, $\phi(x)=x^4$, $-\infty<x<+\infty$, is obviously a strictly convex function but $\phi''(0)=0$.

6. The theorem of Tchaplygin. Consider the differential equation

$$y'' = p_1y' + p_2y + q, \qquad x \geqq x_0, \tag{47}$$

where $p_1(x)$, $p_2(x)$ and $q(x)$ are continuous when $x \geqq x_0$ and let $y(x)$ be a solution of this equation such that

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'. \tag{48}$$

Suppose there exists a solution $u(x)$ of

$$u'' = p_1u' + p_2u \tag{49}$$

such that

$$u(x) \neq 0, \qquad x_0 < x < x_1. \tag{50}$$

Let $u_0(x)$ be a solution of (49) such that $u_0(x_0)=0$, $u_0'(x_0)=1$ and let $X(x_0)$ be the first zero of $u_0(x)$ to the right of x_0, if any such zero exists; otherwise let $X(x_0)=+\infty$. Then a recent theorem of S. Tchaplygin, successively generalized by N. V. Petrov and J. E. Wilkins [3] states that:

(I) *If* $\phi(x)$ *is such that*

$$\phi'' > p_1\phi' + p_2\phi + q, \qquad x \geqq x_0,\ \phi(x_0) = y(x_0),\ \phi'(x_0) = y'(x_0) \tag{51}$$

then

$$\phi(x) > y(x), \qquad x_0 < x \leqq x_1. \tag{52}$$

(II) *The interval* $x_0<x\leqq X(x_0)$ *is the largest one in which the inequality* (52) *can be asserted to hold.*

We shall prove that part (I) of this theorem can be deduced easily from Lemma 2′ and Lemma 1′.

In fact, from (50) it follows that in the strip $x_0<x<x_1$, $-\infty<y<+\infty$, the equation (47) has the property (12) that there is a unique solution passing through two arbitrary points with distinct x-coordinates. To see this known fact it is sufficient to observe that if $u(x)$ is a solution of (49) another independent solution of this equation is

$$u(x)\int_{x_0}^{x} u^{-2}(t) \exp\left[\int_{t_0}^{t} p_1(t)dt\right] dt.$$

Therefore, we have by Lemma 2′ that property (P′) holds on every closed interval contained in $x_0<x<x_1$.

According to (51), there is a number $\xi<x_1$ such that

$$\phi(x) > y(x), \qquad x_0 < x \leqq \xi. \tag{53}$$

Let $g(x)$ be the solution of (47) such that $g(\xi)=\phi(\xi)$, $g'(\xi)=\phi'(\xi)$. We have, by Lemma 2′, that

$$g(x) < \phi(x), \qquad x_0 < x < x_1,\ x \neq \xi. \tag{54}$$

If it is true that

$$g(x) \geqq y(x), \qquad x_0 \leqq x < \xi, \tag{55}$$

we conclude by (53) and (54) that $g'(x_0)=y'(x_0)$, which is absurd. Then (55) is false and from the property (12) it follows that there exists a ξ_0, $x_0<\xi_0<\xi$, such that

$$\begin{aligned} g(\xi_0) = y(\xi_0), \quad g(x) &< y(x) \qquad \text{for } x_0 < x < \xi_0, \\ g(x) &> y(x) \qquad \text{for } \xi_0 < x < x_1. \end{aligned} \tag{56}$$

From (53), (54) and (56) it follows that

$$y(x) < \phi(x), \qquad x_0 < x < x_1. \tag{57}$$

So it is sufficient to demonstrate that

$$y(x_1) < \phi(x_1) \tag{58}$$

in order to state that (52) holds. But from (54) it follows immediately that

$$g(x_1) \leqq \phi(x_1). \tag{59}$$

If $g(x_1)<\phi(x_1)$, then (58) follows from (56). Suppose now

$$g(x_1) = \phi(x_1). \tag{60}$$

Using for the point x_1 the reasoning of Lemma 1′ we see that there is a point $\xi_0<\xi_1<x_1$ such that the solution $h(x)$ of (47) determined by $h(\xi_1)=\phi(\xi_1)$, $h'(\xi_1)=\phi'(\xi_1)$ is such that

$$h(x) < \phi(x), \qquad \xi_1 < x \leqq x_1. \tag{61}$$

Then, by (54), (60) and (61), we have that there exists a point ξ_2, $\xi_1<\xi_2<x_1$, such that

$$g(\xi_2) = h(\xi_2). \tag{62}$$

Repeating the reasoning which showed the falseness of (55) we conclude the existence of a number ξ_3, $\xi<\xi_3<\xi_2$, such that

$$g(\xi_3) = h(\xi_3). \tag{63}$$

But (62) and (63) are in contradiction with property (12). So (60) is false and (52) holds. Our statement is proved.

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A RECURRENCE FORMULA FOR $\zeta(2n)$

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In the present paper I shall give a new recurrence formula for $\zeta(2n)$. It differs from other similar formulas in one very important respect. In order to calculate $\zeta(4n)$, for example, other formulas use the values of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, $\cdots$ up to $\zeta(4n-2)$, while mine only requires the values of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, $\cdots$, $\zeta(2n)$. So, for large values of n its advantage is manifest.

This new recurrence formula is

$$\begin{aligned} C_{2n} = {} & \frac{\pi(2\pi)^{2n-1}}{4\{(n-1)!\}^2(2n-1)} \\ (1) \qquad & + \frac{1}{(n-1)!}\sum_{k=0}^{[n/2]}(-1)^k\frac{C_{2k}(2\pi)^{2n-2k}}{(n-2k)!(2n-2k)} \\ & + \frac{1}{\pi}\sum_{k=0}^{[n/2]}\sum_{j=0}^{[n/2]}(-1)^{k+j}\frac{C_{2k}C_{2j}(2\pi)^{2n-2k-2j+1}}{(n-2k)!(n-2j)!(2n-2k-2j+1)}, \end{aligned}$$

where n is a positive integer, and where for simplicity we have put $C_0=-1/2$ and $C_{2k}=\zeta(2k)=\sum_{n=1}^{\infty}1/n^{2k}$, $k=1, 2, 3, \cdots$.

To prove it, let us note that

$$\frac{\pi-x}{2}=\sum_{n=1}^{\infty}\frac{\sin nx}{n}, \qquad 0<x<2\pi.$$

Repeated integration between limits 0 and x gives

$$(2) \qquad \frac{\pi x^{2m-2}}{2(2m-2)!}+\sum_{k=0}^{m-1}(-1)^k\frac{C_{2k}x^{2m-2k-1}}{(2m-2k-1)!}=(-1)^{m-1}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}},$$

$$(3) \qquad \frac{\pi x^{2m-1}}{2(2m-1)!}+\sum_{k=0}^{m}(-1)^k\frac{C_{2k}x^{2m-2k}}{(2m-2k)!}=(-1)^{m}\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2m}},$$

where the C's have the same meaning as above. An application of Parseval's theorem to (2) would yield (1) with $n=2m-1$ and a similar application to (3) would yield (1) with $n=2m$. Thus (1) is true for all positive integral values of n.

Now, let us apply the formula to calculate, for example, $\zeta(2)$ and $\zeta(4)$.

We have

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$$\zeta(2) = C_2 = \frac{\pi(2\pi)}{4} + \left(-\frac{1}{2}\right)\cdot\frac{(2\pi)^2}{2} + \frac{(-1/2)(-1/2)(2\pi)^3}{3\pi} = \frac{\pi^2}{6},$$

and

$$\begin{aligned}\zeta(4) = C_4 = {} & \frac{\pi(2\pi)^3}{4\cdot 3} + \left(-\frac{1}{2}\right)\frac{(2\pi)^4}{2\cdot 4} - \frac{\pi^2(2\pi)^2}{6\cdot 2} \\ & + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(2\pi)^5}{\pi\cdot 2\cdot 2\cdot 5} \\ & + (-1)\left(-\frac{1}{2}\right)\left(\frac{\pi^2}{6}\right)\frac{2\cdot(2\pi)^3}{2\cdot 3\pi} + \frac{\pi^4(2\pi)}{36\pi} = \frac{\pi^4}{90},\end{aligned}$$

which are in accordance with the usual results.

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ON A PROBLEM OF MAX A. ZORN

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1. **Introduction.** Max A. Zorn has proved[1] the following theorem.

THEOREM. *If every substitution $x=at$, $y=bt$ in which a and b are complex numbers transforms $\sum a_{ij}x^iy^j$ into a power series with a non-vanishing radius of convergence, the series $\sum|a_{ij}x^iy^j|$ converges for sufficiently small $|x|$ and $|y|$.*

He has also suggested the following problem. If $\sum a_{ij}x^iy^j$ is a power series which is transformed by every substitution of convergent power series $\sum_1^\infty a_it^i$ and $\sum_1^\infty b_it^i$ with real coefficients for x and y into a convergent power series in t, is the double series $\sum a_{ij}x^iy^j$ convergent?

The answer is yes. In fact, Zorn's theorem itself holds even when the coefficients a and b are restricted to take only real values. We can obtain a proof quite directly by Zorn's method, if we use an estimate for the coefficients of homogeneous polynomials in real variables.

2. **Homogeneous polynomials in real variables.** We shall prove a lemma which may easily be extended to the case of many variables.

LEMMA. *Let $P(x, y)=\sum_{i+j=n} a_{ij}x^iy^j$ be a homogeneous polynomial in real variables. If $|P(x, y)|\leqq M$ for $|x-x_0|\leqq 2\delta$, $|y-y_0|\leqq 2\epsilon$, then $|a_{ij}\delta^i\epsilon^j|\leqq M$.*

PROOF. Set $x=x_0+\delta(\xi+\xi^{-1})$. Then $\xi^nP(x, y)=\sum a_{ij}\xi^j(\xi x)^iy^j$ is a polynomial in ξ whose absolute value does not exceed M when ξ moves on the unit circle of the Gaussian plane. By Cauchy's inequality of function theory, and considering the coefficients of ξ^k in $\xi^nP(x, y)$, we have

$$\left|\sum_{j=0}^{k} a_{ij}c_iy^j\right| \leqq M,$$

where $0\leqq k\leqq n$, $i+j=n$, and c_i is the coefficient of ξ^{k-j} in $(\xi x)^i$.

Again set $y=y_0+\epsilon(\eta+\eta^{-1})$ and apply the Cauchy inequality to the constant term of $\eta^k\sum_{j=0}^k a_{ij}c_iy^j$. We have

$$|a_{lk}c_l\epsilon^k| \leqq M,$$

where $l+k=n$ and c_l equals δ^l by definition. This completes our proof.

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[1] Bull. Amer. Math. Soc. vol. 53 (1947) pp. 791–792.

3. Proof of Zorn's theorem in the real case. Now we can follow Zorn's method directly.

PROOF. Let $P_n(x, y)=\sum_{i+j=n} a_{ij}x^iy^j$. The set D of vectors (x, y) for which $\sum P_n(x, y)$ converges is of the second category. For every vector is in[2] mD for some positive integer m. If D were of the first category, the set mD and therefore the two-dimensional Euclidean space would be the same character.

By virtue of the continuity of the functions P_n there will exist a square $|x-x_0|\leqq 2p$, $|y-y_0|\leqq 2p$, $p>0$ and an M such that $|P_n(x, y)|\leqq M$ holds in the square for all n. Then by our lemma $|a_{ij}p^{i+j}|\leqq M$, Hence for $|x|$, $|y|\leqq p/2$, we have

$$|a_{ij}x^iy^j|\leqq M/2^{i+j}$$

which establishes the absolute convergence of the double series.

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[2] mD is the set of (mx, my) where $(x, y)\in D$.

A NOTE ON S-SPACES

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An S-space is a normal topological space in which each covering by open sets has a refinement which is star-finite, that is, each set of the refinement meets only a finite number of sets of the refinement. Thus a compact (=bicompact) space is an S-space, and an S-space is paracompact [1].[1]

In this note we discuss cartesian products in which one of the factors is an S-space. We show that if the other factor is compact, then the product is an S-space, and the dimension of the product does not exceed the sum of the dimensions of the factors. However, if both factors are S-spaces, the product need not be an S-space.

THEOREM. *Let A be an n-dimensional S-space and B an m-dimensional compact space. Then $A \times B$ is an S-space and $\dim(A \times B) \leqq n+m$.*

By the dimension of a space we mean, of course, the Lebesgue dimension (cf. [2, p. 206]).

Let $\mathfrak{W}_0$ be an arbitrary covering of $A \times B$. We are going to construct a locally-finite cell complex, D, with $\dim D \leqq n+m$, a mapping f of $A \times B$ onto D, and a covering $\mathfrak{Y}$ of D such that $f^{-1}(\mathfrak{Y})$ is a refinement of $\mathfrak{W}_0$.

Let a be any point of A. Each point of $a \times B$ is contained in an open set of the form $U \times V$, U open in A, V open in B, such that $U \times V$ is contained in an open set of $\mathfrak{W}_0$. For a fixed point $a \in A$, the set of all such V's is a covering of B, and hence a finite number of them, say $V_{a,1}, V_{a,2}, \cdots, V_{a,k(a)}$, form a covering $\mathfrak{V}_a$ of B. Let U_a be the intersection of the corresponding U's.

The collection of all such sets U_a is a covering of A. Hence there is a star-finite refinement $\mathfrak{U}$ of this covering whose order is no more than $n+1$. We may also assume [2, p. 210] that $\mathfrak{U}$ is *normal*, that is, that there is a mapping ϕ of A onto $N(\mathfrak{U})$ such that each open set of $\mathfrak{U}$ is the inverse image, under ϕ, of the star of a vertex of $N(\mathfrak{U})$.

We form a covering $\mathfrak{W}$ of $A \times B$ as follows: each set U of $\mathfrak{U}$ is contained in some U_a, and with each U_a is associated a covering $\mathfrak{V}_a$ of B. Form the product of U with each set of $\mathfrak{V}_a$. The totality of these products forms $\mathfrak{W}$, and by construction, $\mathfrak{W}$ is a refinement of $\mathfrak{W}_0$.

Let θ be the mapping of $A \times B$ onto $N(\mathfrak{U}) \times B$ defined by $\theta(a \times b) = \phi(a) \times b$, where ϕ is the above mapping of A onto $N(\mathfrak{U})$. Each ele-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

ment of $\mathfrak{W}$ is thus mapped by θ onto an open set of $N(\mathfrak{U})\times B$, so $\mathfrak{X}=\theta(\mathfrak{W})$ is a covering of $N(\mathfrak{U})\times B$.

Now let u_1 be a fixed vertex of $N(\mathfrak{U})$ and let S_1 be the closed star of u_1, and, inductively, let S_i be the closed star of S_{i-1}. Let $T_1=S_1$ and let T_i, $i>1$, be the closure of S_i-S_{i-1}, and let $R_i=T_{i-1}\cap T_i$.

Now $\bigcup_{i=1}^{\infty} S_i$ is a connected set which is both open and closed in $N(\mathfrak{U})$ and hence is a component of $N(\mathfrak{U})$. Since the constructions we make below can be made independently in each component, we may assume without loss of generality that $\bigcup_{i=1}^{\infty} S_i=N(\mathfrak{U})$.

Each vertex u of $N(\mathfrak{U})$ corresponds to an open set U of $\mathfrak{U}$ and, as above, each U is contained in a set U_a to which there corresponds a covering $\mathfrak{V}_a$ of B. Let $\mathfrak{V}_1'$ be a finite covering of B which is a common refinement of each $\mathfrak{V}_a$ which corresponds to a vertex of T_1. Let $\mathfrak{V}_1$ be a normal finite covering of B which is of order not greater than $m+1$ and which is a star-refinement of $\mathfrak{V}_1'$, that is, each set consisting of an element $\mathfrak{V}_1$ together with all the elements of $\mathfrak{V}_1$ which meet it is in an element of $\mathfrak{V}_1'$.

In general, having obtained $\mathfrak{V}_{i-1}$, we obtain $\mathfrak{V}_i$ as follows: let $\mathfrak{V}_i'$ be a common finite refinement of $\mathfrak{V}_{i-1}$ and of each $\mathfrak{V}_a$ which corresponds to a vertex of T_i. Let $\mathfrak{V}_i$ be a normal finite covering of B, of order not greater than $m+1$, which is a star-refinement of $\mathfrak{V}_i'$.

Let C_i be the finite cell-complex $T_i\times N(\mathfrak{V}_i)$. Since $\mathfrak{V}_i$ is a refinement of $\mathfrak{V}_{i-1}$, there is a projection π_i, a simplicial mapping, of $N(\mathfrak{V}_i)$ into $N(\mathfrak{V}_{i-1})$. For each i, identify the subcomplex $R_i\times N(\mathfrak{V}_i)$ of $T_i\times N(\mathfrak{V}_i)$ with the subcomplex $R_i\times\pi_i N(\mathfrak{V}_i)$ of $T_{i-1}\times N(\mathfrak{V}_{i-1})$. The result of these identifications is the cell-complex D. Since $N(\mathfrak{U})$ is at most n-dimensional, and $N(\mathfrak{V}_i)$, for each i, is at most m-dimensional, the highest possible dimension for a cell of D is $n+m$.

Since each $\mathfrak{V}_i$ is normal, there is a corresponding mapping ζ_i of B onto $N(\mathfrak{V}_i)$. Let ζ be the transformation of $N(\mathfrak{U})\times B$ onto D defined by setting $\zeta(p\times b)=p\times\zeta_i(b)$ for $p\in T_i-R_i$. Since each π_i is continuous, so is ζ. Now $f=\zeta\theta$ is a mapping of $A\times B$ onto D.

To construct the covering $\mathfrak{Y}$ of D, let u be any vertex of $N(\mathfrak{U})$. Then u is in some R_i. Let v be a vertex of $N(\mathfrak{U}_{i-1})$, and consider $u\times v$ as a vertex of C_{i-1}. Let K be the star, in C_{i-1}, of $u\times v$. Then consider u as a vertex of T_i, and let $v_1, v_2, \cdots, v_s$ be all the vertices of $N(\mathfrak{U}_i)$ which are mapped onto v by π_i. Let L be the union of the stars of $u\times v_1, \cdots, u\times v_s$ in C_i. Then the set $K\cup L$ of $C_{i-1}\cup C_i$ becomes, after the identifications made in defining D, an open set of D containing $u\times v$. The collection of all such sets constitutes the covering $\mathfrak{Y}$. Since each $\mathfrak{V}_i$ is a star-refinement of $\mathfrak{V}_{i-1}$, it is easy to see that $\zeta^{-1}(\mathfrak{Y})$ is a refinement of $\mathfrak{X}$ and hence that $f^{-1}(\mathfrak{Y})$ is a refinement of $\mathfrak{W}$.

It is now easy to finish the proof of the theorem. First we make a barycentric subdivision of D, thus obtaining a simplicial complex E. Let e_1 be a vertex of E, and let $\bar{S}_i$ have the same meaning for E as S_i has for $N(\mathfrak{U})$ above. Next we subdivide $\bar{S}_2$ simplicially until the star of each vertex in the induced subdivision of $\bar{S}_1$ is contained in some element of $\mathfrak{Y}$. Then we subdivide $\bar{S}_3$ simplicially, without introducing any new vertices in $\bar{S}_1$, until each vertex of the induced subdivision of $\bar{S}_2$ has its star contained in some element of $\mathfrak{Y}$.

Continuing in this fashion, all of E is subdivided in such a way that each cell of D is divided into a finite number of simplexes.

Now let $\mathfrak{Z}$ be the covering of D by the stars of the vertices of the subdivision of E. By construction, $\mathfrak{Z}$ is a refinement of $\mathfrak{Y}$. Since each cell of D is of dimension at most $n+m$, the same is true of E and of its subdivision. Hence, order $\mathfrak{Z} \leqq n+m+1$. Clearly $\mathfrak{Z}$ is star-finite. Hence $f^{-1}(\mathfrak{Z})$ is a star-finite covering of $A \times B$, of order not greater than $n+m+1$, and a refinement of $\mathfrak{W}_0$, which proves the theorem.

To show that the product of two S-spaces need not be an S-space, we appeal to an example, constructed by Sorgenfrey [4], of a paracompact space whose product with itself is not paracompact. It is only necessary to observe that this space is actually an S-space, as is easily seen by an inspection of his proof.

Finally, we remark that Hemmingsen [3] has shown that the dimension theorem holds for the product of two compact spaces, and Dieudonné [1] has shown that the product of a compact space and a paracompact space is paracompact. Thus, the only unsettled question in this direction is that concerning the dimension of the product of a compact and paracompact space. It is clear that the method used above cannot be used in this case.

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SOME REMARKS ABOUT LIE GROUPS TRANSITIVE ON SPHERES AND TORI

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The present note pertains principally to two papers of D. Montgomery and H. Samelson [1, 2],[1] in which the authors study compact Lie groups transitive on tori [1] and spheres [2]. I will here prove in another way, generalize, and sharpen a part of their results. §1 contains the remarks to [1], §2 to [2]; they are independent of one another and the methods used in both are quite different.

I recall first the definition and some simple properties of homogeneous spaces. A manifold W is a homogeneous space under the Lie group[2] G if to each element a of G there corresponds a differentiable transformation $T_a: x \rightarrow T_a(x)$ of W into itself such that:

(1) $T_a(x)$ depends continuously on the pair $a \in G$, $x \in W$.

(2) To the product (ab) corresponds the mapping $x \rightarrow T_{(ab)}(x) = T_a[T_b(x)]$.

(3) Given any two points x, y in W, there exists $a \in G$ such that $T_a(x) = y$ (that is, G is *transitive* on W).

G is said to be *effective* on W if only the identity element e of G induces the identity transformation of W.

Let us choose an arbitrary point x of W. The set of elements h in G for which $T_h(x) = x$ is a closed subgroup H of G, called the *associated group*. As is well known [3, no. 29], W may be identified with the space of left cosets G/H, the mappings T_a being then: $xH \rightarrow (ax)H$. Actually, H depends on the choice of $x \in W$ and should be denoted H_x, but I shall in general drop the index x as there will be no danger of confusion and also because all the groups H_x $(x \in W)$ are conjugate to each other in G.

When considering a homogeneous space as the space of left cosets, it is quite easy to prove that *every subgroup of H which is invariant in G induces the identity mapping of W*, and, conversely, *a subgroup of H, each element of which induces the identity of W, is invariant in G.*

1. The n-dimensional torus as a homogeneous space.

In [1], D. Montgomery and H. Samelson proved that a Lie group which acts transitively and effectively on the n-dimensional torus is itself the n-dimensional toral group T^n. Actually, as they remark at the end of

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The manifolds and Lie groups considered here are always *compact*.

their note, their proof gives at the same time the stronger theorem:

Let W be an n-dimensional homogeneous space under a compact connected Lie group G, the first Betti number of W being n.

Then W is homeomorphic to the n-dimensional torus, and if G is effective on W, it is isomorphic to T^n.[3]

I shall prove here the more general theorem:

THEOREM I. *Let W be an n-dimensional homogeneous space under the compact connected Lie group G. Let us suppose that for one index j $(1 \leqq j \leqq n-1)$ the jth Betti number of W equals the binomial coefficient $C_{n,j}$. Then:*

(a) *W is homeomorphic to the n-dimensional torus;*

(b) *if G is effective on W, G is isomorphic to the n-dimensional toral group T^n.*[3]

The demonstration is quite different from that given in [1] in the case $j=1$, and employs the theory of integral invariants on a homogeneous space [4], the main theorems of which I review now.

Let us denote by p_j the jth Betti number of W and by n_j the number of linearly independent differential exterior forms of degree j on W which are invariant under all transformations of G. Then we always have:

$$p_j \leqq n_j \leqq C_{n,j}.$$

The first inequality follows from the theorems of G. de Rham [5] and from the fact that every closed form is equivalent to an invariant one [4, Theorem I]. To obtain the second inequality one needs only to remark that an invariant form is completely determined by its value at one point of W.

Let now x_0 be a definitely chosen point of W, $H=H_{x_0}$ the associated group; we can take in a neighborhood $U(e)$ of e in G canonical coordinates $x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{n+s}$ such that $H \cap U(e)$ is the s-plane of the last s coordinates; $x_1, \cdots, x_{n+s}$ may also be taken as coordinates in the tangent space to G at e. The transformations: $x \rightarrow (a^{-1}xa)$, where $a \in G$, $x \in U$, are linear, and form the adjoint linear group of G, which I shall denote by Ad G. G being compact, Ad G may be assumed to be orthogonal. The representation of H contained in Ad G splits then into two parts, one of which is a linear group γ leaving invariant the set of variables $x_1, \cdots, x_n$. But $x_1, \cdots, x_n$ can be taken as coordinates in a neighborhood $V(x_0)$ of x_0 in W (or as coordinates in the vector space $L(x_0)$ tangent to W at x_0), so that γ indicates how H acts on $V(x_0)$ (or on $L(x_0)$).

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[3] G being then abelian, H reduces to the identity element if G is effective; W may be identified with the manifold of G.

To γ there corresponds a linear group γ_j of degree $C_{n,j}$: the group of transformations of j-dimensional elements of $L(x_0)$ induced by the operations of γ. The following theorem allows us to compute, at least theoretically, n_j with the help of γ_j (see [4, nos. 25, 28]).

The number of linearly independent invariant differential forms of degree j equals the number of times the trivial representation[4] *of H occurs in γ_j.*

If W is the manifold of a group G', one takes as transformation group of W the left and the right translations of G'; then $G=G'\times G'$, and the differential forms invariant under G are the doubly (left *and* right) invariant forms. The associated group H_e is isomorphic to Ad G' and the number n_j of doubly invariant independent forms is also given by the previous theorem, where γ is replaced by Ad G' and γ_j by the corresponding group $(\text{Ad } G')_j$ of transformations of j-dimensional elements (see [4, no. 53]).

Theorem I will be an immediate consequence of the results mentioned above and of the following rather trivial lemma:

LEMMA. *Let A be a regular $n\times n$ matrix, A_j the matrix of degree $C_{n,j}$ giving the transformation of j-dimensional planes induced by A.*

If for one index j $(1\leqq j\leqq n-1)$ $A_j=E$ (identity matrix), then $A=\pm E$.

PROOF. The coefficients of A_j are the determinants of degree j of A, and especially the diagonal terms of A_j are the principal j-minors of A.

If $A\neq cE$, then there is at least one vector $\vec{x}$ which is not eigenvector of the linear transformation: $\vec{x}\rightarrow A\vec{x}$ given by A, that is, $\vec{x}$ and $A\vec{x}$ are linearly independent. Let π_j be a j-dimensional plane containing $\vec{x}$ but not $A\vec{x}$ (such a plane exists, since $j\leqq n-1$). π_j is certainly not invariant under A_j and $A_j\neq E$, which contradicts the assumption. Therefore we must have $A=cE$; but then each diagonal term of A_j equals c^j; if $A_j=E$, one has $c=\pm 1$ and $A=\pm E$.

PROOF OF THEOREM I. Let W be a n-dimensional homogeneous space, one Betti number p_j of which equal $C_{n,j}$. Then we know that $n_j=C_{n,j}$ and that γ_j reduces to the identity matrix. The previous lemma shows that γ consists either of E or of $+E$ and $-E$. In the former case, every element of the associated group H induces the identity mapping of a neighborhood $V(x)$ of x in W, and therefore on the whole of W.[5] H is then invariant in G and W is homeomorphic to the manifold of a group $G'=G/H$. We also see that, if G is effective on W, $H=\{e\}$ and $G'=G$.

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[4] That is, the representation of degree 1 which assigns the number one to each element of H.

In the second case $(-E\in\gamma)$, H possesses a subgroup H_1 of index two represented by $+E$ in γ. H_1 is invariant in G, $\overline{W}=G/H_1$ is the manifold of a group G' and $p_j(\overline{W})\leqq C_{n,j}$. But on the other hand $\overline{W}$ is a two-fold covering space of W and therefore, as is known, $p_j(\overline{W})\geqq p_j(W)$. Thus $p_j(\overline{W})=C_{n,j}$.

We know now that, if $p_j(W)=C_{n,j}$, then W is either homeomorphic to or twice covered by the manifold of a group G', and that $G=G'$ if G is effective on W. The latter case cannot occur when G' is abelian (see footnote 3).

We have seen that $p_j(G')=C_{n,j}$. Theorem I will therefore be completely proved if we establish the proposition:

Let W be the manifold of a compact connected n-parameter Lie group G. For one index j $(1\leqq j\leqq n-1)$ let $p_j(W)=C_{n,j}$.

Then G is isomorphic to the n-dimensional toral group T^n.

Proof. This could be deduced from theorems of E. Cartan and H. Hopf on the Poincaré polynomials of compact Lie groups, but we can also follow the same method as above: if $p_j=C_{n,j}$ then $n_j=C_{n,j}$, $(\mathrm{Ad}\, G)_j=E$ and $\mathrm{Ad}\, G=E$ (Ad G is connected and contains only one element if it is discrete). That means that $(a^{-1}xa)=x$ for $a\in G$, $x\in U(e)$, and therefore also for every $x\in G$, since an element of a connected topological group may be written as the product of a finite number of elements taken in an arbitrary neighborhood $U(e)$ of the identity. G is then abelian; being compact and connected, it is isomorphic to T^n according to a well known theorem [**3**, no. 43]).

2. Even-dimensional spheres as homogeneous spaces.

In [**2**], Montgomery and Samelson study spheres of arbitrary dimensions; their results and demonstrations show that the cases of even and odd dimensionality have to be treated separately. Here I shall consider only the simpler one: even-dimensional spheres.

It is first shown in [**2**] that a compact connected Lie group acting transitively and effectively on an even-dimensional sphere S^n is simple. S^n being simply connected and having, for n even, an Euler-characteristic $\chi(S^n)$ equal to two, that theorem is contained in the following statement:

Theorem II. *Let G be a compact connected Lie group acting transi-*

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[5] This last point may be for instance deduced from the fact that the transformations T_a are isometric mappings of W onto itself in a complete Riemannian metric [**3**, no. 36].

tively and effectively on a simply connected space W which has an Euler-characteristic equal to a prime number.

Then G is simple.

The proof is based on a theorem of H. Hopf and H. Samelson [6] which I shall formulate a little later, but first I must recall some points of the theory of compact Lie groups.

All maximal abelian subgroups of a compact connected Lie group are toral groups and conjugate to each other (see for example [6, no. 4]). Their common dimension r defines the *rank* $r(G)$ of G. Let T^r be a maximal toral group; the normalizer $N(T^r)$ of T^r (that is, the totality of elements $x \in G$ for which $x^{-1}T^rx \subset T^r$) has also the dimension r and consists of a finite number of cosets of T^r [6, Hilfssatz 2]; each coset defines one automorphism of T^r and the group $N(T^r)/T^r$ is isomorphic to the group of automorphisms of T^r obtained by means of the inner automorphisms of G leaving T^r invariant; this group plays a fundamental role in the theory of semi-simple Lie groups. I shall call it $\Phi(G)$; it is independent of the choice of T^r since all maximal toral groups of G are conjugate to each other. If H is a proper subgroup of G having the same rank as G, the group $\Phi(H)$ is of course a subgroup of $\Phi(G)$. *If H is a proper connected subgroup of same rank as G, then $\Phi(H)$ is a proper subgroup of $\Phi(G)$.* This is not explicitly stated, but follows easily from the theory of singular elements in a compact group (see, for example [7], especially §2, nos. 5, 7).

The theorem of Hopf and Samelson I need is:

Let W be a homogeneous space under a compact connected Lie group.

Then $\chi(W) \geqq 0$; it is positive if and only if the rank of the associated group H equals the rank of G; in that case, $\chi(W)$ is equal to the index of $\Phi(H)$ in $\Phi(G)$.[6]

Proof of Theorem II. Let W be a homogeneous space possessing the properties listed in Theorem II, and let H be the associated group; then $r(H) = r(G)$; moreover, W being simply connected, H is connected [3, no. 31],[7] and we see, by the way, that $\chi(W) > 1$. Let us call a connected subgroup of G *maximal* if it is not contained in another *connected* proper subgroup of G. Then, if $\chi(W)$ is a prime number, H is maximal, for if there were a connected group H' such that $H \subset H' \subset G$, $H \neq H' \neq G$, we should have $\Phi(H) \subset \Phi(H') \subset \Phi(G)$ with $\Phi(H) \neq \Phi(H')$

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[6] That is, the quotient of the order of $\Phi(G)$ by the order of $\Phi(H)$.

[7] In our special case, the converse is also true: *If H is connected and if $r(H) = r(G)$, then G/H is simply connected.* This follows from the fact that H contains a toral group T^r maximal in G and that every closed curve in G is homotopic to a closed curve in T^r.

$\neq \Phi(G)$ (see the previous paragraph), and the index of $\Phi(H)$ in $\Phi(G)$ could not be a prime number.

Let us suppose now that G is not simple. Then $G=\overline{G}/N$, where N is a finite group and $\overline{G}$ a direct product $G_1\times G_2\times \cdots \times G_k$ of compact simple groups [3, no. 52]; W may be considered in an evident way as a homogeneous space under $\overline{G}$, the associated group $\overline{H}$ being the reciprocal image of H in $\overline{G}$. If G is effective then $\overline{G}$ is "almost effective," that is, only a finite number of elements in $\overline{G}$ induce the identity mapping of W. It is clear that $\overline{H}$ is maximal in $\overline{G}$ and that $r(\overline{H})=r(\overline{G})$; from the last equality it may be deduced readily that $\overline{H}$ is itself a direct product $H_1\times H_2\times \cdots \times H_k$ $(H_i\subset G_i$, $i=1, \cdots, k)$. One H_i at least must be different from the G_i in which it lies; let us suppose that $H_1\neq G_1$, then, H being maximal in G, we have $H_i=G_i$ $(i=2, 3, \cdots, k)$.

$G_2 \times G_3 \times \cdots \times G_k$ is now a *connected* subgroup of $\overline{H}$ which is invariant in $\overline{G}$; it must contain only the identity element if $\overline{G}$ is almost effective; therefore, G is isomorphic to G_1/N and is simple, q.e.d.

In [2] D. Montgomery and H. Samelson also determined the simple groups which act transitively on S^n. Their method is of topological nature and requires the knowledge of the homology rings of simple groups; it could not be applied to the exceptional groups.

Another method is suggested by the previous considerations; it consists in finding directly the associated group H. We have seen that if G/H is homeomorphic to S^n (n even) then H is connected, maximal (in the sense of the proof of Theorem II), has the same rank as G and a group $\Phi(H)$ of index two in $\Phi(G)$.

In a paper I wrote with J. de Siebenthal (Lausanne), which will appear in the Comment. Math. Helv.,[8] we study the subgroups of maximal rank of compact Lie groups and we give, for each simple group of the Killing-Cartan classification, a list of all types of connected maximal subgroups having the same rank as the group itself. On the other hand, the orders of the groups Φ may be easily computed: for the simple groups, they are to be found for example in [8], for the others, they are given by the relation $\Phi(G_1\times G_2)=\Phi(G_1)\times\Phi(G_2)$. By studying that list of maximal subgroups, I found that the index of $\Phi(H)$ in $\Phi(G)$ equals 2 only in the following cases:

(a) D_r in B_r, $r=1, 2, \cdots$;[9]

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[8] A summary is given in a note published in C. R. Acad. Sci. Paris vol. 226 (1948) pp. 1662–1664.

[9] I follow the usual notations: B_r and D_r are the unimodular orthogonal groups of respectively $2r+1$ and $2r$ variables, A_r the unimodular unitary group, C_r the unitary symplectic group of $2r$ variables, G_2, F_4 the exceptional groups of 14 and 52 parameters.

(b) A_2 in G_2.

According to the theorem of Hopf and Samelson, the characteristic of the spaces B_r/D_r and G_2/A_2 is two. But it is well known that these spaces are really homeomorphic to spheres (G_2 is the automorphism-group of the Cayley numbers and acts transitively on the purely imaginary Cayley numbers of norm one, which are in a one-to-one correspondence with the points of S^6). Thus we have proved the following two theorems, the first of which is slightly stronger than the result obtained in [**2**, Theorem II, p. 462].

THEOREM III. *The only compact connected simple Lie group acting transitively on the even-dimensional sphere* S^{2r} *is locally isomorphic to* B_r $(r=1, 2, \cdots)$, *and also, for* $r=3$, *to* G_2.

THEOREM IV. *The even-dimensional spheres are, up to a homeomorphism, the only simply-connected spaces of characteristic two on which compact connected Lie groups act transitively.*

Theorem III gives thus an infinity of simply-connected homogeneous spaces of characteristic 2. This fact occurs only for the prime number 2. More precisely, we can assert the following theorem:

THEOREM V. *For each prime number* $p>2$, *there are only a finite number of simply-connected spaces of characteristic* p *on which compact connected Lie groups act transitively.*[10] *These spaces are homeomorphic to*:

(1) $A_{p-1}/A_{p-2}\times T^1$ (*dimension* $2(p-1)$),

(2) $C_p/C_{p-1}\times C_1$ (*dimension* $4(p-1)$),

and, for $p=3$:

F_4/B_4 (*dimension* 16) *and* $G_2/A_1\times A_1$ (*dimension* 8).

This can be checked with the help of the list of maximal subgroups already cited.

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4. ———, *Sur les invariants intégraux de certains espaces homogènes clos . . .* , Polskie Towarzystwo matematyczne, Krakow, Rocznik vol. 8 (1929) pp. 182–225;

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[10] The space G/H is automatically of even dimension if $r(H)=r(G)$, for we have for every compact n-parameter Lie group the relation $n\equiv r(G)$ (modulo 2) (see [**7**, p. 359]).

also *Selecta*, Paris, 1939, pp. 203–233.

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A NOTE ON LEAST COMMON LEFT MULTIPLES

B. M. STEWART

1. **Introduction.** Consider n-by-n matrices $A, B, \cdots$ with elements in a principal ideal ring and recall the following definitions. If $A = BC$, then A is a *left multiple* of C and C is a *right divisor* of A. If $A = RD$ and $B = PD$, then D is a *common right divisor* of A and B; if, furthermore, D is a left multiple of every common right divisor of A and B, then D is a *greatest common right divisor* of A and B. If $M = PA = QB$, then M is a *common left multiple* of A and B; if, furthermore, M is a right divisor of every common left multiple of A and B, then M is a *least common left multiple* of A and B. If $FE = I$, where I is the identity matrix, then E is a *unimodular* matrix. If E is unimodular, then EA is a *left associate* of A.

The basic tool in the following constructions is the theorem[1] that any given matrix A is the left associate of a uniquely determined matrix A_1, known as the Hermite canonical triangular form, having zeros above the main diagonal, having elements below the main diagonal in a prescribed residue class modulo the diagonal element above, having each diagonal element in a prescribed system of non-associates, and if a diagonal element is zero, having the corresponding row all zero.

C. C. MacDuffee has presented the following method,[2] due in essence to E. Cahen and A. Chatelet, for finding a greatest common

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[1] C. C. MacDuffee, *Matrices with elements in a principal ideal ring*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 570–573.

[2] C. C. MacDuffee, loc. cit. p. 573.

right divisor D and a least common left multiple M of two given matrices A and B. Consider $2n$-by-$2n$ matrices written in the form of n-by-n blocks. Use the theory of the Hermite form, as applied to the $2n$-by-$2n$ matrices, to write the following equations.

$$\begin{gathered}\begin{pmatrix} R & S \\ P & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}\begin{pmatrix} R & S \\ P & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.\end{gathered} \tag{1}$$

Since first: $A = R'D$ and $B = P'D$, and second: $RA + SB = D$, it follows that D is a greatest common right divisor of A and B, with no restrictions on A and B.

With the rather severe restriction that both A and B be nonsingular, it has been shown[3] that the matrix $M = PA = -QB$, obtained from (1), is a least common left multiple of A and B.

It is the purpose of this note to show by a different proof that the stated method for finding M is correct for a much larger set of matrices A and B—namely, whenever D is nonsingular, that is, whenever the row space of the union of A and B has the rank n, thus including, in addition to the case that both A and B are nonsingular, the case when either A or B is nonsingular and many other possibilities. Furthermore, in the exceptional case when D is singular, there is presented a method for finding an M that may be considered a natural modification of the method based on the equations (1). In conclusion it is noted that a theorem of E. Steinitz guarantees that both D and M are unique up to a unimodular left factor.

For completeness it should be noted that a method for finding the least common left multiple of two matrices when the matrices have elements in a field has been given by C. C. MacDuffee,[4] but the method is not applicable to the case where the elements are limited to a principal ideal ring.

2. **When D is nonsingular.** In terms of the notation explained in the introduction the following theorem may be stated:

THEOREM 1. *In the matric equation*

$$\begin{pmatrix} R & S \\ P & Q \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

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[3] C. C. MacDuffee, loc. cit. p. 574.

[4] C. C. MacDuffee, *Vectors and matrices*, Carus Monograph, 1943, pp. 100–102.

where the matrix involving R, S, P, Q is unimodular, the matrix D is an all cases a greatest common right divisor of A and B; and if D is nonsingular, then the matrix $M=PA=-QB$ is a least common left multiple of A and B.

That D is in all cases a greatest common right divisor has been shown above. To show that M, which is obviously a common left multiple of A and B, is a least common left multiple, let $M_1=UA=-VB$ be any common left multiple of A and B. Use the matrices determined in (1) to write the following equations:

$$\begin{pmatrix} R & S \\ U & V \end{pmatrix}\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

$$(2) \qquad \begin{pmatrix} R & S \\ U & V \end{pmatrix}\begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the most general solution of the equation

$$\begin{pmatrix} X & W \\ Y & T \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Here W and T are arbitrary, but X and Y must be chosen so that

$$(3) \qquad XD = D, \qquad YD = 0.$$

Subject to the conditions (3) the following equations must hold:

$$\begin{pmatrix} R & S \\ U & V \end{pmatrix}\begin{pmatrix} R' & S' \\ P' & Q' \end{pmatrix}=\begin{pmatrix} X & W \\ Y & T \end{pmatrix}, \quad \begin{pmatrix} R & S \\ U & V \end{pmatrix}=\begin{pmatrix} X & W \\ Y & T \end{pmatrix}\begin{pmatrix} R & S \\ P & Q \end{pmatrix}.$$

In particular, it appears that U must have the form

$$(4) \qquad U = YR + TP.$$

But under the hypothesis that D is nonsingular the only solution of $YD=0$ is $Y=0$. Hence it follows in this case that $U=TP$; then from $M_1=UA=TPA=TM$ it follows that $M=PA=-QB$ is indeed a least common left multiple of A and B and the proof of the theorem is complete.

3. **When D is singular.** If the matrix D is singular it is easy to construct examples in which the matrix PA of Theorem 1 is not a least common left multiple of A and B. Hence the construction proposed in the following theorem has some interest.

THEOREM 2. *Let A_1 and B_1 be the Hermite forms of A and B, respectively. In the matric equation*

$$(1') \qquad \begin{pmatrix} R^* & S^* \\ P^* & Q^* \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ B_1 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

where the matrix involving R^*, S^*, P^*, Q^* is unimodular and the matrix involving D is in Hermite form, let it be agreed, since this is a possible procedure, that no operations are performed upon the rows corresponding to the zero rows of D. Then the matrix $M^* = P^*A_1 = -Q^*B_1$ is a least common left multiple of A and B.

It follows readily that if E_1 and E_2 are unimodular, then a least common left multiple M^* of $A_1 = E_1A$ and $B_1 = E_2B$ is also a least common left multiple of A and B. First: from $M^* = KA_1 = LB_1$ it follows that $M^* = KE_1A = LE_2B$; second: from $M_1 = UA = VB$ it follows that $M_1 = UF_1A_1 = VF_2B_1$, where $F_1E_1 = I = F_2E_2$, but since M^* is a least common left multiple of A_1 and B_1, it follows that there exists a matrix G such that $M_1 = GM^*$. In particular A_1 and B_1 may be taken to be the Hermite forms of A and B, respectively.

In the equation (1′) the * notation has been adopted in describing the component blocks of the transforming matrix for two reasons: first: to indicate that the operations are to be made on the matrix involving A_1 and B_1 rather than A and B; and second: to indicate that the operations are to be made in what may be described as one of the "simplest" ways. Certainly no zero row can appear in D unless it is also a zero row of A_1 (the converse, of course, does not hold). Consequently, in obtaining the Hermite form involving D, it is quite unnecessary to operate on the rows (z) which are zero rows of D, and this agreement results in a "simplest" transforming matrix. In fact the matrix R^* must have each of its rows (z) all zero except for a 1 in the diagonal position. Furthermore, the most general matrix Y such that $YD = 0$ must have the form $Y = ZY^*$, where Z is arbitrary but Y^* has a structure just like R^* in its rows (z) and zero rows elsewhere. Thus $Y^*R^* = Y^*$ and

$$(5) \qquad YR^* = ZY^*R^* = ZY^* = Y.$$

As observed above A_1 has zero rows in at least the rows (z), hence from $YD = 0$ it follows that $YA_1 = 0$, and from (5) it follows that

$$(6) \qquad YR^*A_1 = YA_1 = 0.$$

Let $M_1 = UA_1 = -VB_1$ be any common left multiple of A_1 and B_1. Starting from (1′) instead of (1), obtain the relations analogous to (2), (3), and (4). In particular, consider

$$(4') \qquad U = YR^* + TP^*.$$

From (6) it follows that $M_1 = UA_1 = (YR^* + TP^*)A_1 = TP^*A_1$. Hence with $M^* = P^*A_1 = -Q^*B_1$ it follows that $M_1 = TM^*$ and that M^* is a least common left multiple of A_1 and B_1. Automatically M^* is a least common left multiple of A and B and the proof of the theorem is complete.

It should be noted that Theorem 2 works in all cases and does not *require* D to be singular. However the construction of Theorem 1, when applicable, is much shorter, since the separate computation of A_1 and B_1 is avoided. A working procedure would be to start from A and B and compute D in Hermite form; if D has no zero rows, apply Theorem 1; if D has zero rows (z), return to compute A_1 and B_1, find D anew starting from A_1 and B_1 but not operating on the rows (z), and then apply Theorem 2.

4. Uniqueness.

LEMMA. *If $A = UB$ and $B = VA$, then there exists a unimodular matrix E such that $B = EA$.*

This lemma has been proved by E. Steinitz[5] in a much more general setting—namely, for matrices whose elements are in a domain of integrity over the rational field, not necessarily a principal ideal ring. The proof is considerably simpler for the case of principal ideal rings. In fact by taking the proof for the case of matrices with elements in a field[6] and merely replacing the word nonsingular by the word unimodular, a suitable proof for the case of principal ideal rings is easily made.

THEOREM 3. *The greatest common right divisor D and the least common left multiple M of two given matrices A and B are uniquely determined up to unimodular left factors.*

If D and D_1 are two greatest common right divisors of A and B, then each is a common left multiple of the other, say $D = UD_1$ and $D_1 = VD$. Then by the lemma, D and D_1 are left associates.

If M and M_1 are two least common left multiples of A and B, then each is a common right divisor of the other, say $M_1 = UM$ and $M = VM_1$. Then by the lemma, M and M_1 are left associates.

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[5] E. Steinitz, Math. Ann. vol. 71 (1911) pp. 328–354.

[6] C. C. MacDuffee, *Vectors and matrices*, pp. 45–46.

INTEGRAL EXTENSIONS OF A RING

HAROLD CHATLAND AND H. B. MANN

Introduction. Let R be a commutative ring with a unit element and let $a, b \subset R$.

DEFINITIONS. (1) a and b are said to be coprime in R if $\tau \subset R$, τ/a, τ/b implies $\tau/1$.

(2) A ring R' is called an integral extension of R if $R' \supset R$ and $a = br$, $a, b \subset R$, $r \subset R'$ implies there exists an element $\bar{r} \subset R$ such that $a = b\bar{r}$.

(3) a and b are said to be absolutely coprime if they are coprime in every extension R' of R. In this paper it is shown that to every set of ideals of a commutative ring there exists an extension of the ring such that every ideal of the set is the intersection of the ring and a principal ideal of its extension. This is the main result and is given in Theorem 2. In a particular case of Theorem 2 it is shown in Theorem 1 that $a, b \subset R$ are absolutely coprime if and only if there exist elements $x, y \subset R$ such that $ax + by = 1$.

Similar results for algebraic integers are well known [1].[1] In the special case where the domains considered are completely integrally closed and the ideals have finite bases, a different extension fulfilling the conditions of Theorem 2 was obtained by Kronecker [2].

The extension of R. An extension of R, in the sense of this paper, may be obtained in the following manner. We first form the ring $R(u)$ by adjoining to R the elements u^n, $n = \pm 1, \pm 2, \cdots$, transcendental over R and such that $u^n a = au^n$, $a \subset R$. Let $\mathfrak{a}$ be the ideal generated by the set A of elements $a, b, \cdots$ of R. Then the subring R' of $R(u)$ consisting of all "polynomials"

$$a_{-m}u^{-m} + a_{-m+1}u^{-m+1} + \cdots + a_{-1}u^{-1} + a_0 + a_1 u + \cdots + a_n u^n$$

with $a_i \subset R$ and $a_{-r} \subset \mathfrak{a}^r$, $r > 0$, is an integral extension of R for $R' \supset R$. Moreover if $c = dh$, $c, d \subset R$, $h \subset R'$, then

$$h = e_{-m}u^{-m} + \cdots + e_{-1}u^{-1} + e_0 + e_1 u + \cdots + e_n u^n$$

$e_i \subset R$, $e_{-r} \subset \mathfrak{a}^r$. Multiplying by d we have

$$c = de_{-m}u^{-m} + \cdots + de_{-1}u^{-1} + de_0 + de_1 u + \cdots + de_n u^n.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Since $c \subset R$, all terms involving u must vanish. Hence $c = de_0$ and R' is an integral extension of R.

Results.

THEOREM 1. *a, $b \subset R$ are absolutely coprime if and only if there exist elements x, $y \subset R$ such that $ax + by = 1$.*

Clearly the condition is sufficient. Use the above extension R' of R where A consists of a and b. In $R' = R(au^{-1}, bu^{-1}, u)$, u is a common divisor of a and b. If a and b are absolutely coprime, u must be a unit divisor. Hence

$$u^{-1} = \alpha_{-m}u^{-m} + \cdots + \alpha_{-1}u^{-1} + \alpha_0 + \alpha_1 u + \cdots + \alpha_n u^n$$

$\alpha_i \subset R$, $\alpha_{-r} \subset \mathfrak{a}^r$. Since u is trancendental over R all terms in the sum must vanish except $\alpha_{-1}u^{-1}$. But $\alpha_{-1} \subset \mathfrak{a}$ and hence is of the form $ax + by$. Multiplying by u we have

$$1 = ax + by.$$

REMARK. If, $\bar{x}$, $\bar{y}$ are elements of R such that

$$a\bar{x} + b\bar{y} = 1$$

then the pair x, $y \subset R$ is also a solution if and only if $x = \bar{x} + b\mu$, $y = \bar{y} - a\nu$ where $ab\mu = ab\nu$.

Substitution shows the condition to be sufficient. Moreover if $ax + by = 1$, $a\bar{x} + b\bar{y} = 1$ then $a(x - \bar{x}) = b(\bar{y} - y)$ and so $a\bar{x}(x - \bar{x}) = b(\bar{y} - y)\bar{x}$. Adding $b\bar{y}(x - \bar{x})$ to each side of the last equation we have

$$x - \bar{x} = b(\bar{y} - y)\bar{x} + b\bar{y}(x - \bar{x}) = b\mu.$$

Similarly $y - \bar{y} = -a\nu$.

But $a(x - \bar{x}) = b(\bar{y} - y)$. Hence

$$ab\mu = ab\nu.$$

LEMMA. *If $\mathfrak{a}$ is an ideal in R, then there exists an extension R' of R such that $\mathfrak{a}$ is the intersection of R and a principal ideal of R'.*

Consider the extension R' of R with $\mathfrak{a} = A$. In R' the ideal (u) is principal and contains $\mathfrak{a}$ and no other elements of R, for every element of $\mathfrak{a}$ is obtained from the set of products $au^{-1}xu$. Also for $\lambda \subset R'$ suppose $\lambda u = c$, $c \subset R$. Then $\lambda = cu^{-1}$. Hence $c \subset \mathfrak{a}$ and $\mathfrak{a} = R \wedge (u)$.

THEOREM 2. *To every set of ideals of R there exists an extension R' of R such that every ideal of the set is the intersection of R and a principal ideal of R'.*

To each ideal $\mathfrak{a}$ in the set let there correspond an element $u_\mathfrak{a}$ transcendental over R. Form the ring $R(u)$ by adjoining $u_\mathfrak{a}^n$ to R where $u_\mathfrak{a}^n a = a u_\mathfrak{a}^n$, $a \subset R$, $n = \pm 1, \pm 2, \cdots$. The subring R' of $R(u)$ consisting of the "polynomials"

$$\sum a_{r_1 r_2 \ldots} u_1^{r_1} u_2^{r_2} \cdots, \qquad a_{r_1 r_2 \ldots} \subset R,$$

with the condition that $a_{r_1 r_2 \cdots r_i}$ belong to $\mathfrak{a}_i^{-r_i}$ if r_i is negative, is an integral extension of R. This follows by the method demonstrated above. For if $l = mn$, l, $m \subset R$, $n \subset R'$ then

$$n = \sum \beta_{r_1 r_2 \ldots} u_1^{r_1} u_2^{r_2} \cdots$$

where $\beta_{r_1 r_2 \ldots}$ are certain $a_{r_1 r_2 \ldots}$. Hence

$$l = mn = \sum m \beta_{r_1 r_2 \ldots} u_1^{r_1} u_2^{r_2} \cdots .$$

But since the indeterminates u_i are transcendental over R and $l \subset R$, all terms in the sum, except the constant term $m\beta_0$, must vanish. Hence $b = m\beta_0$.

We proceed as in the lemma. Consider the principal ideal $(\mathfrak{a})$. All of the elements $a^{(i)}$ of $\mathfrak{a}$ may be obtained as products $a^{(i)} u_\mathfrak{a}^{-1} \cdot u_\mathfrak{a}$. Moreover only the elements $a^{(i)} \subset R$ may be so obtained for suppose $\nu \subset R'$ and $\nu u_i = d \subset R$. Then $\nu = d u_i^{-1}$. Hence $\mathfrak{a} = R \wedge (u_i)$.

Remark. In the case of non-commutative rings a result analogous to Theorem 2 holds for two-sided ideals.

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A THEOREM ON DIFFERENCE POLYNOMIALS

RICHARD M. COHN

We shall prove the following analogue of a theorem of Kronecker's: *Let $\mathcal{J}$ be a difference field*[1] *containing an element t which is distinct from its transforms of any order. Every perfect ideal in the ring $\mathcal{J}[y_1, \cdots, y_n]$ has a basis consisting of $n+1$ difference polynomials.*

The corresponding theorem for differential polynomials was proved by J. F. Ritt.[2] We shall follow in all but details a brief and elegant proof of the theorem for differential equations which he has given recently.[3]

Let Λ be a perfect ideal in $\mathcal{J}[y_1, \cdots, y_n]$. We know[4] that Λ has a finite basis $A_1, \cdots, A_r$. We adjoin to $\mathcal{J}[y_1, \cdots, y_n]$ the $(n+1)r$ additional variables u_{ij}, $i=1, \cdots, n+1; j=1, \cdots, r$, and consider the polynomials[5]

$$L_i = \sum_{j=1}^{r} u_{ij}A_j, \qquad i = 1, \cdots, n+1. \tag{1}$$

We shall show that there exists a nonzero polynomial H in the u_{ij} only, which vanishes for all solutions of the system (1) which do not annul every A_i. Our theorem will follow immediately from the existence of H. For I showed in my Dissertation[6] that the presence of t in $\mathcal{J}$ implies that we may assign to the u_{ij} values α_{ij} in $\mathcal{J}$ which do not annul H. Let L_i become M_i, $i=1, \cdots, n+1$, when the u_{ij} are

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[1] For a definition of the terms used in the statement of this theorem the reader is referred to J. F. Ritt and H. W. Raudenbush, *Ideal theory and algebraic difference equations*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 445–452. The reader's attention is also called to the paper *Complete difference ideals* by J. F. Ritt, Amer. J. Math. vol. 63 (1941) p. 681, in which the definition of the term "difference ideal" is modified. It is the latter definition which is now in use. The symbol $\mathcal{J}[y_1, \cdots, y_n]$ denotes the ring of all difference polynomials in the unknowns $y_1, \cdots, y_n$, whose coefficients lie in the difference field $\mathcal{J}$.

[2] J. F. Ritt, *Differential equations from the algebraic standpoint*, Amer. Math. Soc. Colloquium Publications, vol. 14, pp. 50–58.

[3] To appear in the forthcoming revised edition of *Differential equations from the algebraic standpoint*. This shorter proof does not furnish all details of the theorem established in the paper mentioned in footnote 2.

[4] This is one of the principal results of the paper of Ritt and Raudenbush referred to in footnote 1.

[5] We use the term "polynomial" as an abbreviation for "difference polynomial."

[6] *Manifolds of difference polynomials*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 133–172. This paper is referred to below as M.D.P.

thus specialized. Then every solution of the system $M_i=0$ can be extended to a solution of (1) by letting $u_{ij}=\alpha_{ij}$, $i=1, \cdots, n+1$, $j=1, \cdots, r$. Since H does not vanish it follows that the A_i are all zero. Thus the M_i are a basis for Λ containing $n+1$ polynomials.

It remains only to prove the existence of H. We make successive transformally transcendental adjunctions[7] to $\mathcal{J}$ of elements v_i, $i=1, \cdots, n$; w_{ij}, $i=1, \cdots, n+1, j=2, \cdots, r$. We let each $y_i=v_i$, and each $u_{ij}=w_{ij}$, $j>1$. The u_{i1} may then be computed from the equations $L_i=0$. Let Σ be the reflexive[8] prime ideal consisting of all polynomials in the u_{ij} and y_i which are annulled by this solution.

Σ contains no difference polynomial in the y_i, $i=1, \cdots, n$, and u_{ij}, $j>1$, alone; for no such polynomial is annulled by the solution just described. Each L_i is in Σ, so that, for each i from 1 to $n+1$, Σ contains a polynomial in u_{i1} which is free of the other u_{j1}. Thus the u_{ij}, $j>1$, and y_i may be chosen as a set of arbitrary unknowns for Σ. The u_{ij} are more numerous than the arbitrary unknowns so that, by Theorem IV of M.D.P., Σ contains a nonzero polynomial H_1 in the u_{ij} only.

It is evident that the L_i constitute a basic set[9] of Σ when we choose for the arbitrary unknowns the set indicated above. The remainder of H_1 with respect to this set is zero. It follows, since A_1 is the initial of each L_i, that there exists a product P of powers of transforms of A_1 such that PH_1 is a linear combination of the L_i and their transforms. Then H_1 vanishes for all solutions of (1) which do not annul A_1.

We may show in the same way that there exist nonzero polynomials H_i, $i=2, \cdots, r$, in the u_{ij} only, such that H_k vanishes for every solution of (1) which does not annul A_k. Then $H=\prod_{i=1}^{r} H_i$ is the polynomial whose existence we wished to prove. The theorem is established.

It is perhaps surprising that $n+1$ polynomials should suffice for a basis of a perfect difference ideal in n unknowns, when it is known that many more polynomials may be required in a basic set. As an illustration of this phenomenon we give the following example of an ideal with a basis which is a subset of its basic set.[10]

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[7] This is to say that the v_i and w_{ij} annul no difference polynomial with coefficients in $\mathcal{J}$. See M.D.P., §6.

[8] A difference ideal is called reflexive if the inclusion in the ideal of the transform C_1 of a difference polynomial C implies that C itself is in the ideal. See the paper *Complete difference ideals* referred to in footnote 1.

[9] Basic sets of systems of difference polynomials are defined by J. F. Ritt and J. L. Doob in their paper, *Systems of algebraic difference equations*, Amer. J. Math. vol. 55 (1933) pp. 506–508. It should be noted that the definition of ascending sets, which is preliminary to that of basic sets, is substantially different from that of ascending sets of differential polynomials.

Let $\mathcal{J}$ be the field of all rational functions of x with transforming defined as the operation of replacing x by $x+1$. We consider the polynomials

$$y_1 - y, \qquad z^2 - y, \qquad z_1 - z. \tag{2}$$

Using Theorem IX of M.D.P. one shows easily that (2) is a basic set of a prime reflexive ideal Λ with coefficients in $\mathcal{J}$. Since the initials of the polynomials of (2) are unity, it follows that it is a basis for Λ. But the equations $z_1-z=0$, $z^2-y=0$, imply that $y_1=z_1^2=z^2=y$; so that any solution of these equations is also a solution of $y_1-y=0$. Thus z^2-y, z_1-z is itself a basis for Λ.

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[10] This example emerged during a conversation of the author with E. R. Kolchin.

INVERSIVE DIFFERENCE FIELDS

RICHARD M. COHN

It may happen that the functions of an analytic difference field admit not only the operation of replacing z by $z+1$, but also its inverse. Since these two operations have essentially the same properties it is to be expected that to each theorem concerning difference equations there will be a corresponding theorem, valid in fields of the type we have just described, in which the rôles of the highest and lowest transforms of the unknowns are interchanged. For example, J. F. Ritt has shown[1] that the number of ordinary manifolds of a first-order difference polynomial in the unknown y does not exceed its degree in y_1; and he observes that in fields where the inverse substitution is always possible, the number of ordinary manifolds is also limited by the degree in y.

The study of abstract difference fields enables us to apply this principle in every case; for, as we shall show, every abstract difference field can be extended to a difference field in which there exists, for every element h of the field, an element g such that h is the trans-

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[1] J. F. Ritt, *Algebraic difference equations*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 303–308.

form of g. We shall call a field with this property an ***inversive difference field.***

The first part of this note, then, will be devoted to the brief proof that every difference field has an inversive extension. In the second part we apply this theorem to complete certain earlier results on the structure of the manifold of a difference polynomial.

PART I

THEOREM. *Every difference field may be imbedded in an inversive difference field.*

We first prove that any difference field F may be imbedded in a field F' isomorphic to F and such that every element of F is a transform of an element of F'. Let F_1 be the set consisting of first transforms of elements of F. It is easily seen that F_1 is a difference field isomorphic to F by the correspondence which maps any element α of F on its transform α_1. Now we may extend[2] this isomorphism to an isomorphism between the extension F of F_1, and an extension F' of F. Then F' has the desired properties.

Let us now construct, by this process, a sequence of fields F, F', F'', $\cdots$, such that $F^{(k)}$ is isomorphic to $F^{(k+1)}$, and every element of $F^{(k)}$ is a transform of an element of $F^{(k+1)}$. Let $\bar{F}$ be the difference field which is the union of the $F^{(k)}$, $k=0, 1, 2, \cdots$. Then $\bar{F}$ is evidently the inversive extension of F required by the theorem.

PART II

We have shown elsewhere[3] that the essential irreducible manifolds of a difference polynomial A in unknowns $y_1, \cdots, y_n$ may be divided into ordinary and essential singular manifolds. No ordinary manifold annuls a polynomial of lower effective order than A in any y_k, while each singular manifold of A annuls such a polynomial for every y_k, $1 \leqq k \leqq n$. Concerning these two types of manifolds we may make the following statements:

(1) *There is always at least one ordinary manifold; but their number does not exceed the degree of A in the lowest, nor in the highest transform of any y_i which appears effectively.*

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[2] Let H be isomorphic to, but contain no element in common with, F. Let H_1 be the subfield of H corresponding to F_1. By the obvious extension to difference fields of a theorem in van der Waerden, *Moderne Algebra*, 2d ed., New York, F. Ungar, p. 42, the extension F' of F exists isomorphic to H, with H_1 corresponding to F under the isomorphism. Evidently F' has the properties we require.

[3] *A note on the singular manifolds of a difference polynomial*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 917–922. This note is referred to henceforth as S.M.

(2) *The essential singular manifolds of A annul its formal first partial derivatives with respect to the lowest and with respect to the highest transforms of each y_i which appear effectively.*

Except for the reference to transforms of lowest order these statements have been proved previously.[4] We now complete the proof in this respect.

We consider an inversive difference field F, and denote by F^* the difference field consisting of the elements of F with the operation of transforming defined as the inverse of that in F. Let A be any polynomial in unknowns $y, y_1, \cdots, y_n$, with coefficients in F. Then we denote by A^* a polynomial obtained from A by replacing each y_{ik} appearing in A by $y'_{i,m-k}$, where m is an integer sufficiently large so that $m-k$ is always non-negative. In other respects A is unaltered except that its coefficients are now to be regarded as elements of F^* rather than F. The operation $*$ will be applied in what follows to only a finite number of polynomials at any one time. We may therefore always choose an m which can be used for all the polynomials under consideration, and we shall assume in each case that this has been done.

We prove a lemma concerning the operation $*$.

LEMMA 1. *Let F be an inversive difference field, and let B and A be polynomials in F. Then B holds A if, and only if, B^* holds A^*.*

We assume first that B^* holds A^*, but that A has a solution $y_i = \alpha_i$, in an extension G of F, which does not annul B.

Let $\bar{G}$ be an inversive extension of G, and let $\bar{G}^*$ be the field consisting of the elements of $\bar{G}$ with the transforming operation defined as the inverse of that of $\bar{G}$. Then $\bar{G}^*$ is an extension of F^*. The α_i lie in $\bar{G}^*$, but in this field α_{ik} is the transform of $\alpha_{i,k+1}$ if $k \geq 0$, and transforms of the α_{i0} exist. When we substitute the α_{ik} for the $y'_{i,m-k}$, where m is the integer used in defining $*$, it is evident that A^* vanishes while B^* does not. This contradicts the statement that B^* holds A^*. Thus the implication is proved in one direction.

We next assume that B holds A, but that A^* has a solution $y'_i = \beta_i$ in an extension G^* of F^*, which does not annul B^*. Let $\bar{G}^*$ be an inversive extension of G^*, and let $\bar{G}$ be the field consisting of the elements of $\bar{G}^*$, with the transforming operation defined as the inverse of that in $\bar{G}^*$. Then $\bar{G}$ is an extension of F. When we replace each y_{ik} by $\beta_{i,m-k}$, where m is the integer used in defining $*$, it is evident that A vanishes, while B does not. This is a contradiction which establishes the remaining implication. Thus the lemma is proved.

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[4] *Manifolds of difference polynomials*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 133-172 and S.M.

It is convenient to prove the next two lemmas together. We retain the notation of Lemma 1. In a previous paper[5] we defined the y_i-separant of a difference polynomial as its formal first partial derivative with respect to the highest transform of y_i present. We now introduce the term ***inverse y_i-separant*** for the partial derivative with respect to the lowest transform of y_i present. We may now state the lemmas.

LEMMA 2. *The polynomials A and A^* have the same number of ordinary manifolds.*

LEMMA 3. *The essential singular manifolds of A annul its inverse separants. (Note that the coefficient field F is still assumed to be inversive.)*

Let M be any essential irreducible manifold of A, and let Λ be the prime ideal consisting of all polynomials with coefficients in F which are annulled by all solutions in M. Let G be the field which results from F by adjoining the general point $y_i = \alpha_i$ of Λ. Let $\overline{G}$ denote the inversive extension of G formed as in Part I, and finally let $\overline{G}^*$ be the field consisting of the elements of $\overline{G}$, but with the inverse transforming operation. Then $\overline{G}^*$ is an extension of F^*.

The substitution $y'_{ik} = \alpha_{i,m-k}$ annuls A^*. We denote by Λ^* the reflexive prime ideal consisting of all polynomials with coefficients in F^* which vanish when these values, or their appropriate transforms, are assigned to the y'_{ik}.

Let T be any polynomial with coefficients in F^*. We shall investigate under what circumstances T is in Λ^*.

By an operation similar to * we convert T into a polynomial T'. Specifically, we replace each y'_{ik} by $y''_{i,p-k}$, where p is a sufficiently great integer so that the subscripts are all non-negative, and we consider the coefficients of the resulting polynomial as elements of F. The substitutions $y''_{i,k+p-m} = \alpha_{i,k}$ annul T' if and only if T holds Λ^*. In this substitution the α_{ik} with k negative are to be understood as the corresponding positive transforms of α_{i0} in F^*.

Let $\overline{T}$ be formed by replacing $y''_{i,k+p-m}$ by y_{ik} in a transform of T' which is such that only non-negative values of k appear. Then T holds Λ^* if and only if $\overline{T}$ is annulled by the substitution $y_{ik} = \alpha_{ik}$, in other words, if and only if $\overline{T}$ is in Λ.

To prove Lemma 2 we let M be an ordinary manifold of A. We claim that the manifold of Λ^* is an ordinary manifold of A^*. For, if it is not, there is a T of effective order less than that of A^* in y_1 which holds Λ^*. Then the corresponding $\overline{T}$ holds Λ. Since $\overline{T}$ is obviously of effective order less than A in y_1 this is impossible, so that our statement is justified. Thus there corresponds to each ordinary manifold of A an ordinary manifold of A^*.

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[5] S.M., §1.

Let M_1, M_2 be two distinct ordinary manifolds of A (if such exist), and Λ_1, Λ_2 the corresponding prime ideals. We form Λ_1^*, Λ_2^* from Λ_1, Λ_2 as Λ^* was formed from Λ. Then Λ_1^*, Λ_2^* are distinct for large m. For let C be in Λ_1 but not in Λ_2. If the m used in defining * is sufficiently large we may form C^*. Then C^* holds Λ_1^* but not Λ_2^*, so that these are distinct. It follows that, for large m, A^* has at least as many ordinary manifolds as A. It remains to prove this statement for all m.

We consider integers m_1 and $m_2 > m_1$. Let us use * to denote specifically the operator previously so designated, but with m fixed as m_1, and let us use ** to denote the corresponding operator with $m = m_2$. Then A^{**} results from A^* by replacing each y'_{ij} by its transform of order $m_2 - m_1$. Let A' denote the $(m_2 - m_1)$th transform of A^*. Then A' and A^* have the same number of ordinary manifolds. Consider the isomorphism which carries F^* into the field F' consisting of the transforms of order $m_2 - m_1$ of elements of F^*. This isomorphism carries A^{**} into A'. Thus A^{**} and A' have the same number of ordinary manifolds. We conclude that A^{**} and A^* have the same number of ordinary manifolds. Now by choosing m_2 sufficiently large we may show, by the method of the preceding paragraph, that A^{**} possesses at least as many ordinary manifolds as A. Then this is also true of A^*.

Now if we let $T = A^*$, and form $\overline{T}$ as above, we see, by an argument similar to that of the preceding paragraphs, that $\overline{T}$ possesses at least as many ordinary manifolds as A^*. But, if we choose $p = m$, we find that $\overline{T} = A$. Combining this result with that of the preceding paragraph we verify Lemma 2.

We now turn to the proof of Lemma 3. We let M denote an essential singular manifold of A, and form its prime ideal Λ and the corresponding prime ideal Λ^* of A^* as at the beginning of the discussion of Lemma 2. Let $M_1, M_2, \cdots, M_r$ denote the ordinary manifolds of A, $\Lambda_1, \Lambda_2, \cdots, \Lambda_r$ their corresponding prime ideals, and $\Lambda_1^*, \Lambda_2^*, \cdots, \Lambda_r^*$ the prime ideals of A^* formed from the Λ_i. The proof of Lemma 2 shows that, at least for large m, the Λ_i^* are distinct and their manifolds constitute the totality of ordinary manifolds of A^*. Let C_i, $i = 1, \cdots, r$, be a polynomial of Λ_i which does not hold Λ. We may assume that m is large enough so that we can form the C_i^*, $i = 1, \cdots, r$. Each C_i^* is in Λ_i^* but not in Λ^*. It follows that the manifold of Λ^* is contained within no ordinary manifold of A^*. Then any essential irreducible manifold of A^* which contains the manifold of Λ^* is an essential singular manifold, and therefore annuls each y_i-separant of A^*. We denote the y_1-separant by T, and form $\overline{T}$ as at the beginning of the proof of Lemma 2, using $p=m$. By what precedes $\overline{T}$ is annulled by a general point of Λ, so that it holds Λ. But $\overline{T}$ is the inverse y_1-separant of A, and we may prove similarly that each inverse y_i-separant holds Λ. Thus the lemma is proved.

We now complete the proofs of statements (1) and (2) at the beginning of Part II. For inversive fields the previously unproved part of statement (1) is an immediate consequence of Lemma 2. Now let F be any difference field, and A an algebraically irreducible difference polynomial with coefficients in F. Let $M_1, M_2, \cdots, M_r$ be the ordinary manifolds of A, and $\Lambda_1, \Lambda_2, \cdots, \Lambda_r$ the corresponding prime ideals. We let $y_i=\alpha_i$ be a general point of Λ_1 and denote by G_1 the field obtained by adjoining the α_i to F. In a similar way we form fields G_i containing general points of the remaining Λ_i.

There exists, by the theorem of Part I, an inversive extension $\overline{G}_i$ of G_i, $i=1, \cdots, r$. The elements of any $\overline{G}_i$ whose transforms of any order lie in F form a subfield $\overline{F}_i$. Now it is easy to see that the $\overline{F}_i$ are all isomorphic. We denote any one of them by $\overline{F}$. It is possible to construct,[6] for each i, an extension of $\overline{F}$ isomorphic to $\overline{G}_i$ and therefore containing a general point of Λ_i. We shall assume that the $\overline{G}_i$ above are actually extensions of the inversive field $\overline{F}$.

Let Σ_1 be the reflexive prime ideal consisting of those polynomials with coefficients in $\overline{F}$ which are annulled by the substitution $y_i=\alpha_i$. Σ_1 must be the prime ideal corresponding to an ordinary manifold of A considered as a polynomial with coefficients in $\overline{F}$, or of an irreducible factor of A in $\overline{F}$ which is of the same effective order as A. For otherwise some polynomial D of Σ_1 is of lower effective order than A in some y_i, say y_1. A transform, D_k, of D has coefficients in F. Then D_k is in Λ_1, since it is annulled by the substitution of the α_i for the y_i, and is of lower effective order than A in y_1. This is impossible, so that our statement is verified.

We now construct Σ_i, $i=2, \cdots, r$, corresponding to each Λ_i. The construction can be carried out similarly to that of Σ_1. The Σ_i are all prime ideals corresponding to ordinary manifolds of irreducible factors of A considered as a polynomial with coefficients in $\overline{F}$. Furthermore they are all distinct. For Λ_i contains a polynomial C which is not annulled by the general point of Λ_j, $j\neq i$. Then C is in Σ_i, but not in Σ_j. Thus we see that there are at least as many ordinary manifolds of irreducible factors of A, considered as a polynomial in $\overline{F}$, as of A considered as a polynomial in F. Then the truth of statement (1) for the inversive field $\overline{F}$ implies its truth for F.

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[6] The method of footnote 2 is to be used.

We turn now to statement (2). Let M be an essential singular manifold of the algebraically irreducible difference polynomial A with coefficients in the field F, and let Λ be the reflexive prime ideal consisting of all polynomials annulled by M. Let $y_i = \alpha_i$ be a general point of Λ. We form the inversive extension $\overline{F}$ of F as in Part I. Let Σ be the set of polynomials with coefficients in $\overline{F}$ which are annulled when the y_i are replaced by the α_i. Obviously Σ is a reflexive prime ideal held by A. Let N be an essential irreducible manifold of A (as a polynomial in $\overline{F}$) which contains the manifold of Σ. Let ψ be the prime ideal consisting of those polynomials with coefficients in $\overline{F}$ which are annulled by the solutions of N. We form the system ψ' consisting of those polynomials of ψ with coefficients in F. ψ' is held by a polynomial of effective order in y_1 lower than that of A. For otherwise its manifold is an ordinary manifold of A as a polynomial in F. But this is impossible because the general point $y_i = \alpha_i$ of the ideal Λ, whose manifold is an essential singular manifold, annuls ψ'.

It follows that N is an essential singular manifold of an irreducible factor of A considered as a polynomial in $\overline{F}$. Lemma 3 shows that each inverse separant of this factor is in ψ. It follows that the inverse separants of A vanish for the solution $y_i = \alpha_i$, and so are in Λ. This completes the proof of statement 2.

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THE WEDDERBURN PRINCIPAL THEOREM FOR ALTERNATIVE ALGEBRAS

R. D. SCHAFER

Except for a generalization of the so-called Wedderburn principal theorem, the structure theory of alternative algebras over an arbitrary field is as complete as that for associative algebras. It is our purpose here to fill this one gap in the alternative theory.

1. **The principal theorem.** A non-associative algebra $\mathfrak{A}$ of order n over an arbitrary field $\mathfrak{F}$ is called *alternative* in case

$$ax^2 = (ax)x, \qquad x^2a = x(xa)$$

for all a, x in $\mathfrak{A}$. It is clear that associative algebras are alternative.

The most famous examples of alternative algebras which are not associative are the so-called Cayley-Dickson algebras of order 8 over $\mathfrak{F}$. Let $\mathfrak{Z}$ be an algebra of order 2 over $\mathfrak{F}$ which is either a separable quadratic field over $\mathfrak{F}$ or the direct sum $\mathfrak{F}\oplus\mathfrak{F}$. There is one automorphism $z\rightarrow\bar{z}$ of $\mathfrak{Z}$ (over $\mathfrak{F}$) which is not the identity automorphism. The associative algebra $\mathfrak{Q}=\mathfrak{Z}+u\mathfrak{Z}$ with elements

$$q = z_1 + uz_2, \qquad z_i \text{ in } \mathfrak{Z}, \tag{1}$$

and multiplication defined by

$$(z_1 + uz_2)(z_3 + uz_4) = (z_1z_3 + \beta z_4\bar{z}_2) + u(\bar{z}_1z_4 + z_3z_2) \tag{2}$$

for $\beta\neq 0$ in $\mathfrak{F}$ is called a *quaternion algebra*. For q in the form (1), the correspondence

$$q \rightarrow \bar{q} = \bar{z}_1 - uz_2 = t(q) - q \tag{3}$$

is an involution of $\mathfrak{Q}$. The Cayley-Dickson algebras $\mathfrak{C}=\mathfrak{Q}+v\mathfrak{Q}$ are obtained by repetition of this process: the elements of $\mathfrak{C}$ are

$$x = q_1 + vq_2, \qquad q_i \text{ in } \mathfrak{Q}, \tag{4}$$

and multiplication in $\mathfrak{C}$ is defined by

$$(q_1 + vq_2)(q_3 + vq_4) = (q_1q_3 + \gamma q_4\bar{q}_2) + v(\bar{q}_1q_4 + q_3q_2) \tag{5}$$

for $\gamma\neq 0$ in $\mathfrak{F}$, where $q\rightarrow\bar{q}$ is the involution (3) of $\mathfrak{Q}$.

Most of our knowledge of alternative algebras is due to M. Zorn.[1]

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[1] See references [6], [7], [9]. Numbers in brackets refer to the references cited at the end of the paper.

The *radical* $\mathfrak{N}$ of an alternative algebra $\mathfrak{A}$ is the set of properly nilpotent elements of $\mathfrak{A}$, and is the maximal nilpotent ideal[2] of $\mathfrak{A}$. The difference algebra $\mathfrak{A}-\mathfrak{N}$ is semi-simple: it is the direct sum $\mathfrak{S}_1 \oplus \cdots \oplus \mathfrak{S}_m$ of simple components $\mathfrak{S}_i$. Any simple alternative algebra is central simple (that is, simple for all scalar extensions) over its center, and all central simple alternative algebras are either associative or Cayley-Dickson algebras.[3] In the sense of equivalence there is (over a given field) exactly one Cayley-Dickson algebra with divisors of zero.[4]

A non-associative algebra $\mathfrak{A}$ over $\mathfrak{F}$ is called *separable* in case $\mathfrak{A}_{\mathfrak{K}}$ is semi-simple (that is, a direct sum of simple algebras) for every scalar extension $\mathfrak{K}$ of $\mathfrak{F}$. It is easy to see that an alternative algebra $\mathfrak{A}$ is separable if and only if it is the direct sum of simple components whose centers are separable fields (over $\mathfrak{F}$). Furthermore, if $\mathfrak{A}$ is separable, there exists a scalar extension $\mathfrak{K}$ of finite degree over $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{K}}$ is a direct sum of components each of which is either a total matric algebra or a Cayley-Dickson algebra with divisors of zero.[5] Such a scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ we call a *splitting field* of $\mathfrak{A}$, and we use the term *split algebra* for a total matric algebra or a Cayley-Dickson algebra with divisors of zero. The number of total matric components of $\mathfrak{A}_{\mathfrak{K}}$ is the same for all splitting fields $\mathfrak{K}$ of $\mathfrak{A}$, and is the sum of the degrees over $\mathfrak{F}$ of the centers of the associative simple components of $\mathfrak{A}$. Similarly the number of Cayley-Dickson components of $\mathfrak{A}_{\mathfrak{K}}$ is independent of the particular splitting field $\mathfrak{K}$, and is the sum of the degrees over $\mathfrak{F}$ of the centers of the simple components of $\mathfrak{A}$ which are not associative.

We shall prove the following.

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[2] It is asserted in the conclusion of [9] that $\mathfrak{N}$ is nilpotent, and that this may be shown by methods similar to those employed in [8]. Since no proof has subsequently appeared in the literature, and since we require the result, we give a proof in Lemma 1 below.

[3] We have given in [5, §§1, 2] a complete account of the structure of simple alternative algebras over an arbitrary field. We assume some familiarity on the part of the reader with this account and with the proof of the Wedderburn principal theorem for associative algebras as given, say, in [1, §3.8].

[4] This corresponds to the fact that there is exactly one quaternion algebra over $\mathfrak{F}$ with divisors of zero: namely, the total matric algebra $\mathfrak{M}_2$ of degree two. For $\mathfrak{Q}$ is a division algebra (that is, has no divisors of zero) if and only if $\mathfrak{Z}$ is a field and β in (2) is not the norm $z\bar{z}$ of an element z in $\mathfrak{Z}$; otherwise $\mathfrak{Q}$ is the total matric algebra $\mathfrak{M}_2$. If $\mathfrak{Q}$ is a division algebra, and if γ in (5) is not the norm $q\bar{q}$ of an element q in $\mathfrak{Q}$, then $\mathfrak{C}$ is a division algebra; otherwise $\mathfrak{C}$ is the unique Cayley-Dickson algebra over $\mathfrak{F}$ with divisors of zero.

[5] Since any semi-simple alternative algebra has a unity element, these statements are immediate consequences of [3, §4] and [1, §3.7].

THEOREM. *Let* $\mathfrak{A}$ *be an alternative algebra over an arbitrary field* $\mathfrak{F}$, *and let* $\mathfrak{N}$ *be the radical of* $\mathfrak{A}$. *If* $\mathfrak{A}-\mathfrak{N}$ *is separable, then* $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$ *where* $\mathfrak{S}$ *is equivalent to* $\mathfrak{A}-\mathfrak{N}$.

This theorem generalizes to the case of alternative algebras the well-known *Wedderburn principal theorem* for associative algebras [1, p. 47]. Its proof resembles the associative one.

2. **The case** $\mathfrak{N}^2\neq 0$. We use the fact[6] that the "associator"

$$[a_1, a_2, a_3] = (a_1a_2)a_3 - a_1(a_2a_3) \tag{6}$$

"alternates" in an alternative algebra $\mathfrak{A}$; that is,

$$[a_1, a_2, a_3] = \epsilon[a_{i_1}, a_{i_2}, a_{i_3}], \qquad a_i \text{ in } \mathfrak{A}, \tag{7}$$

for any permutation i_1, i_2, i_3 of 1, 2, 3, where ϵ is 1 in case the permutation is even, -1 in case it is odd. Furthermore, any subalgebra of $\mathfrak{A}$ generated by only two elements is associative.[7] This implies that powers x^j of an element x in $\mathfrak{A}$ are unambiguously defined, and that we have

$$R_{x^j} = R_x^j, \qquad L_{x^j} = L_x^j, \qquad L_xR_x = R_xL_x \tag{8}$$

of all x in $\mathfrak{A}$, where R_x and L_x are the right and left multiplications[8] of $\mathfrak{A}$ determined by x. Equation (7) is equivalent to the equalities

$$\begin{aligned} R_xR_y - R_{xy} &= L_{xy} - L_yL_x = L_yR_x - R_xL_y = L_xL_y - L_{yx} \\ &= R_yL_x - L_xR_y = R_{yx} - R_yR_x \end{aligned} \tag{9}$$

for all x, y in $\mathfrak{A}$.

A non-associative algebra $\mathfrak{N}$ is called *nilpotent*[9] in case there exists an integer t such that every product of t elements in $\mathfrak{N}$, no matter how associated, is zero. It follows from an observation of Etherington [4, p. 2] that $\mathfrak{N}$ is nilpotent if and only if the associative algebra $\mathfrak{N}^*$ generated by the right and left multiplications of $\mathfrak{N}$ is nilpotent (for a simple induction suffices to prove that, if $\mathfrak{N}^{*k}=0$, then every product of $t=2^k$ elements of $\mathfrak{N}$ is zero). It is an immediate consequence of the known structure theory for alternative algebras and of the following lemma that the radical $\mathfrak{N}$ of an alternative algebra $\mathfrak{A}$ is the maximal nilpotent ideal in $\mathfrak{A}$, and is the minimal ideal such that $\mathfrak{A}-\mathfrak{N}$ is semi-simple.

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[6] [6, equation (5), p. 126].

[7] The so-called Theorem of Artin [6, p. 127].

[8] See [2, §2].

[9] The term "strongly nilpotent" has been used for this concept recently, and "nilpotent" used in case the associative algebra $\mathfrak{N}^*$ were nilpotent. However, Etherington's theorem shows the equivalence of the two notions.

LEMMA 1 (ZORN). *Let $\mathfrak{N}$ be an alternative algebra, each of whose elements is nilpotent. Then $\mathfrak{N}$ is nilpotent.*

If $\mathfrak{C}$ is any linear subset of $\mathfrak{N}$, we denote by $\mathfrak{C}^*$ the subalgebra of $\mathfrak{N}^*$ generated by the right and left multiplications of $\mathfrak{N}$ corresponding to elements of $\mathfrak{C}$. We shall prove by an induction on the number of generating elements of $\mathfrak{B}$ that $\mathfrak{B}^*$ is nilpotent for all subalgebras $\mathfrak{B}$; hence, in particular, for $\mathfrak{B}=\mathfrak{N}$. If $\mathfrak{B}$ is generated by one element x, then by (8) any T in $\mathfrak{B}^*$ is a linear combination of transformations of the form

$$R_x^{j_1}, \qquad L_x^{j_2}, \qquad R_x^{j_3}L_x^{j_4} \tag{10}$$

for $j_i \geq 1$. Then, if $x^j=0$, we have $T^{2j-1}=0$, $\mathfrak{B}^*$ is nilpotent. Hence, by the assumption of the induction, we may take a maximal proper subalgebra $\mathfrak{B}$ of $\mathfrak{N}$ and know that $\mathfrak{B}^*$ is nilpotent. But then there exists an element x not in $\mathfrak{B}$ such that

$$x\mathfrak{B}^* \leqq \mathfrak{B}. \tag{11}$$

For $\mathfrak{B}^{*k}=0$ implies that $\mathfrak{N}\mathfrak{B}^{*k}=0\leqq\mathfrak{B}$, and there exists a smallest integer $m\geqq 1$ such that $\mathfrak{N}\mathfrak{B}^{*m}\leqq\mathfrak{B}$. If $m=1$, take x in $\mathfrak{N}$ but not in $\mathfrak{B}$; if $m>1$, take x in $\mathfrak{N}\mathfrak{B}^{*m-1}$ but not in $\mathfrak{B}$. Then (11) is satisfied. Since $\mathfrak{B}$ is maximal, the subalgebra generated by $\mathfrak{B}$ and x is $\mathfrak{N}$ itself. It follows from (11) that $\mathfrak{N}=\mathfrak{B}+\mathfrak{F}[x]$. Put $y=b$ in (9) for any b in $\mathfrak{B}$. Then (11) implies that

$$\begin{aligned} R_xR_b &= R_{b_1} - R_bR_x, & R_xL_b &= L_bR_x + R_bR_x - R_{b_2}, \\ L_xR_b &= R_bL_x + L_bL_x - L_{b_3}, & L_xL_b &= L_{b_1} - L_bL_x \end{aligned} \tag{12}$$

for b_i in $\mathfrak{B}$. Equations (12) show that, in each product of right and left multiplications in $\mathfrak{B}^*$ and $(x\mathfrak{F})^*$, the transformation R_x or L_x may be systematically passed from the left to the right of R_b or L_b in a fashion which, although it may change signs and introduce new terms, preserves the number of factors from $\mathfrak{B}^*$ and does not increase the number of factors from $(x\mathfrak{F})^*$. Hence any T in $\mathfrak{N}^*=(\mathfrak{B}+x\mathfrak{F})^*$ may be written as a linear combination of terms of the form (10) and others of the form

$$B_1, \qquad B_2R_x^{m_1}, \qquad B_3L_x^{m_2}, \qquad B_4R_x^{m_3}L_x^{m_4}$$

for B_i in $\mathfrak{B}^*$, $m_i\geqq 1$. Then, if $\mathfrak{B}^{*k}=0$ and $x^j=0$, we have $T^{k(2j-1)}=0$; for every term in the expansion of $T^{k(2j-1)}$ contains either at least

k factors B_i or an uninterrupted sequence of at least $2j-1$ factors from $(x\mathfrak{F})^*$. Hence $\mathfrak{N}^*$ is nilpotent, and so is $\mathfrak{N}$.

Let $\mathfrak{B}$ be an ideal of a non-associative algebra $\mathfrak{A}$. We define a family of subalgebras $\mathfrak{B}^i$ $(i=1, 2, \cdots)$ of $\mathfrak{A}$ inductively as follows:

$$(13) \qquad \mathfrak{B}^1=\mathfrak{B}, \qquad \mathfrak{B}^i=(\mathfrak{B}\mathfrak{B}^{i-1}, \mathfrak{B}^{i-1}\mathfrak{B}), \qquad i=2, 3, \cdots,$$

the linear subset of $\mathfrak{A}$ spanned by elements of $\mathfrak{B}\mathfrak{B}^{i-1}$ and $\mathfrak{B}^{i-1}\mathfrak{B}$. For any $\mathfrak{A}$, the $\mathfrak{B}^i$ form a descending chain

$$\mathfrak{B}=\mathfrak{B}^1 \geqq \mathfrak{B}^2 \geqq \cdots \geqq \mathfrak{B}^r \geqq \cdots$$

of ideals of $\mathfrak{B}$. For alternative algebras $\mathfrak{A}$, we have the following stronger statement.

LEMMA 2. *Let* $\mathfrak{B}$ *be an ideal of the alternative algebra* $\mathfrak{A}$. *Then the subalgebras* $\mathfrak{B}^i$ *in* (13) *are ideals of* $\mathfrak{A}$.

The proof is by induction: $\mathfrak{B}^1$ is an ideal by hypothesis. We assume that $\mathfrak{B}^{i-1}$ is an ideal of $\mathfrak{A}$. An arbitrary element of $\mathfrak{A}\mathfrak{B}^i$ is the sum of elements of the form of $a(xy)$ and $a(yx)$ for a in $\mathfrak{A}$, x in $\mathfrak{B}$, y in $\mathfrak{B}^{i-1}$. Then

$$(14) \qquad \begin{aligned} a(xy) &= (ax)y-[a, x, y]=(ax)y-[y, a, x] \\ &= (ax)y-(ya)x+y(ax) \end{aligned}$$

by (7). The right-hand side of (14) is in $\mathfrak{B}^i$ by the assumption of the induction. Similarly $a(yx)$ is in $\mathfrak{B}^i$, and we have $\mathfrak{A}\mathfrak{B}^i \leqq \mathfrak{B}^i$. The reciprocal relationship in alternative algebras gives $\mathfrak{B}^i\mathfrak{A} \leqq \mathfrak{B}^i$, or $\mathfrak{B}^i$ is an ideal of $\mathfrak{A}$.

Let $\mathfrak{N}$ be the radical of an alternative algebra $\mathfrak{A}$. Since $\mathfrak{N}$ is nilpotent by Lemma 1, there is a smallest integer r such that $\mathfrak{N}^r=0$. Moreover, in the chain

$$\mathfrak{N}=\mathfrak{N}^1 > \mathfrak{N}^2 > \cdots > \mathfrak{N}^r = 0$$

the inclusions are proper inclusions, since $\mathfrak{N}^{i-1}=\mathfrak{N}^i$ for $2 \leqq i \leqq r$ would imply $\mathfrak{N}^{i-1}=\mathfrak{N}^i= \cdots =\mathfrak{N}^r=0$, a contradiction. By Lemma 2 the $\mathfrak{N}^i$ are ideals of $\mathfrak{A}$.

An inductive argument based on the dimension of $\mathfrak{N}$ suffices to reduce the proof of the principal theorem to the case $\mathfrak{N}^2=0$. For the preliminaries disposed of in this section allow us to use the argument of the associative proof [1, p. 47].

3. The case $\mathfrak{N}^2=0$. In §4 we shall prove the principal theorem for $\mathfrak{A}$ having a unity element, with $\mathfrak{N}^2=0$ and $\mathfrak{A}-\mathfrak{N}$ a split algebra (that is, either a total matrix algebra or a Cayley-Dickson algebra with divisors of zero). At this point let us assume this result and show how the case $\mathfrak{N}^2 = 0$ of the principal theorem may be proved.

First the theorem may be reduced to the case where $\mathfrak{A}$ has a unity element. If $\mathfrak{A}$ has no unity element, adjoin a unity element e to obtain $\mathfrak{A}_1 = e\mathfrak{F} + \mathfrak{A}$. Then $\mathfrak{A}_1$ is alternative. Also the radical $\mathfrak{N}$ of $\mathfrak{A}$ is the radical of $\mathfrak{A}_1$. For $\mathfrak{N}$ is a nilpotent ideal of $\mathfrak{A}_1$ and is therefore contained in the radical $\mathfrak{N}_1$ of $\mathfrak{A}_1$. Conversely if a_1 is in $\mathfrak{N}_1$, then $a_1 = \alpha e + a$ for α in $\mathfrak{F}$, a in $\mathfrak{A}$, and there exists an integer t such that $a_1^t = \alpha^t e + a' = 0$ for a' in $\mathfrak{A}$. Hence $\alpha^t = 0$, $\alpha = 0$, $a_1 = a$ is in $\mathfrak{A}$, $\mathfrak{N}_1 \leqq \mathfrak{A}$. Then $\mathfrak{N}_1$ is a nilpotent ideal of $\mathfrak{A}$, $\mathfrak{N}_1 \leqq \mathfrak{N}$. If we can obtain a Wedderburn decomposition $\mathfrak{A}_1 = \mathfrak{S}_1 + \mathfrak{N}$ of $\mathfrak{A}_1$, we have $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$ where $\mathfrak{S} = \mathfrak{S}_1 \cap \mathfrak{A} \cong \mathfrak{A} - \mathfrak{N}$.

Throughout the remainder of this paper we assume that $\mathfrak{A}$ has a unity element e, and that $\mathfrak{N}^2 = 0$. We denote residue classes modulo $\mathfrak{N}$ by the customary bracket symbol [ ].

LEMMA 3. *If* $[u_1], \cdots, [u_s]$ *are pairwise orthogonal idempotents in* $\mathfrak{A} - \mathfrak{N}$, *there exist pairwise orthogonal idempotents* $e_1, \cdots, e_s$ *in* $\mathfrak{A}$ *such that*

$$[e_i] = [u_i], \qquad i = 1, \cdots, s. \tag{15}$$

Furthermore, if $[e] = [u_1] + \cdots + [u_s]$, *then* $e = e_1 + \cdots + e_s$.

We prove the first assertion by induction. The element

$$e_1 = 3u_1^2 - 2u_1^3 \tag{16}$$

is an idempotent in $\mathfrak{A}$. For, since $u_1^2 - u_1$ is in $\mathfrak{N}$, $\mathfrak{N}^2 = 0$, we have $(u_1^2 - u_1)^2 = 0$, and $e_1^2 = e_1$ by (16). Also

$$[e_1] = [u_1] \tag{17}$$

by (16), so that $e_1 \neq 0$, and we have proved the case $s = 1$. If f_1 is any idempotent in $\mathfrak{A}$, write $f_0 = e - f_1$, so the Peirce decomposition of $\mathfrak{A}$ relative to f_1 takes the form

$$\mathfrak{A} = f_1\mathfrak{A}f_1 + f_1\mathfrak{A}f_0 + f_0\mathfrak{A}f_1 + f_0\mathfrak{A}f_0,$$

where products $f_i a f_j$ $(i, j = 0, 1)$ for a in $\mathfrak{A}$ are trivially seen to be associative. The $f_i\mathfrak{A}f_i$ $(i = 0, 1)$ are orthogonal subalgebras of $\mathfrak{A}$. The corresponding Peirce decomposition of $\mathfrak{A} - \mathfrak{N}$ is

$$\begin{aligned}\mathfrak{A} - \mathfrak{N} = {} & [f_1](\mathfrak{A} - \mathfrak{N})[f_1] + [f_1](\mathfrak{A} - \mathfrak{N})[f_0] \\ & + [f_0](\mathfrak{A} - \mathfrak{N})[f_1] + [f_0](\mathfrak{A} - \mathfrak{N})[f_0].\end{aligned}$$

We assume the existence of pairwise orthogonal idempotents $e_2, \cdots, e_s$ in $\mathfrak{A}$ such that $[e_i] = [u_i]$, $i = 2, \cdots, s$, and write $f_1 = e_2 + \cdots + e_s$. Then $[u_1]$ is orthogonal to $[f_1] = [u_2] + \cdots + [u_s]$, and we have $[u_1]$ in $[f_0](\mathfrak{A}-\mathfrak{N})[f_0]$. Hence we may choose u_1 in the subalgebra $f_0\mathfrak{A}f_0$ of $\mathfrak{A}$, so that e_1 in (16) is in $f_0\mathfrak{A}f_0$ and is therefore orthogonal to $f_1=e_2+\cdots+e_s$. That is, $e_1, e_2, \cdots, e_s$ are pairwise orthogonal idempotents in $\mathfrak{A}$, (15) holds, and the inductive proof is complete. Since $e_1+\cdots+e_s$ is idempotent, $e-(e_1+\cdots+e_s)$ is either zero or idempotent. But $[e]=[u_1]+\cdots+[u_s]$ implies that $e-(e_1+\cdots+e_s)$ is in $\mathfrak{N}$ and cannot be idempotent; hence $e=e_1+\cdots+e_s$.

The reduction of the principal theorem to the case $\mathfrak{A}-\mathfrak{N}$ simple is now easy. Let $\mathfrak{A}-\mathfrak{N}=\mathfrak{B}_1\oplus\cdots\oplus\mathfrak{B}_s$ for simple algebras $\mathfrak{B}_i$. By Lemma 3, there exist pairwise orthogonal idempotents e_i in $\mathfrak{A}$ such that the unity element of $\mathfrak{B}_i$ is $[e_i]$. In the refined Peirce decomposition

$$\mathfrak{A}=\sum\mathfrak{A}_{ij}\qquad (i, j=1, \cdots, s) \tag{18}$$

we have $\mathfrak{A}_{ij}=e_i\mathfrak{A}e_j$ (associative products!), and the radical $\mathfrak{N}_i$ of $\mathfrak{A}_{ii}$ is the intersection of $\mathfrak{N}$ and $\mathfrak{A}_{ii}$ [9, §§4, 5]. It is easy to see that $\mathfrak{A}_{ii}-\mathfrak{N}_i\cong\mathfrak{B}_i$. If we have Wedderburn decompositions $\mathfrak{A}_{ii}=\mathfrak{S}_i+\mathfrak{N}_i$, $\mathfrak{S}_i\cong\mathfrak{B}_i$, then (since the subalgebras $\mathfrak{A}_{ii}$ of $\mathfrak{A}$ are pairwise orthogonal) the sum of the subalgebras $\mathfrak{S}_i$ of $\mathfrak{A}$ is the direct sum $\mathfrak{S}=\mathfrak{S}_1\oplus\cdots\oplus\mathfrak{S}_s\cong\mathfrak{A}-\mathfrak{N}$, and we have $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$.

We can now prove the principal theorem. Since $\mathfrak{A}-\mathfrak{N}$ is separable, there exists a splitting field $\mathfrak{K}$ such that $(\mathfrak{A}-\mathfrak{N})_\mathfrak{K}=\mathfrak{A}_\mathfrak{K}-\mathfrak{N}_\mathfrak{K}$ is a direct sum of split algebras. Then $\mathfrak{N}_\mathfrak{K}$ is the radical of $\mathfrak{A}_\mathfrak{K}$ (since it is a nilpotent ideal containing the radical). By the result in the preceding paragraph, we may assume that $(\mathfrak{A}-\mathfrak{N})_\mathfrak{K}$ has only one simple component (a split algebra). Then by §4 below, $\mathfrak{A}_\mathfrak{K}$ contains a subalgebra $\mathfrak{B}\cong(\mathfrak{A}-\mathfrak{N})_\mathfrak{K}$. The remaining steps are those of the associative proof [1, p. 48], since no form of associativity is used there.

4. **The case $\mathfrak{A}-\mathfrak{N}$ a split algebra.** We assume throughout that $\mathfrak{A}$ has a unity element e, and that $\mathfrak{N}^2=0$. Suppose that $\mathfrak{A}-\mathfrak{N}$ is a total matric algebra $\mathfrak{M}_s$ of degree s. Then the principal theorem is an immediate consequence of Lemma 4, which we state in the following slightly more general form for use in the proof of Lemma 5.

LEMMA 4. *Let $\mathfrak{A}-\mathfrak{N}$ contain a total matric algebra $\mathfrak{M}_s$ with unity element $[e]$. Then $\mathfrak{A}$ contains a total matric algebra $\mathfrak{M}$ of degree s with unity element e, and $\mathfrak{M}_s$ is the image of $\mathfrak{M}$ under the homomorphism $x\rightarrow[x]$ of $\mathfrak{A}$ onto $\mathfrak{A}-\mathfrak{N}$.*

For let $\mathfrak{M}_s$ have basal elements $[u_{ij}]$ $(i, j=1, \cdots, s)$ with the customary multiplication table

$$[u_{ij}][u_{kl}]=\delta_j^k[u_{il}]\qquad \text{(Kronecker delta).} \tag{19}$$

Then $[e]=[u_{11}]+\cdots+[u_{ss}]$, and by Lemma 3 there exist pairwise orthogonal idempotents e_{ii} in $\mathfrak{A}$ such that $[u_{ii}]=[e_{ii}]$, $e=e_{11}+\cdots+e_{ss}$. In the Peirce decomposition (18) of $\mathfrak{A}$ relative to this set of idempotents we have $\mathfrak{A}_{ij}=e_{ii}\mathfrak{A}e_{jj}$.

We may take $u_{11}=e_{11}$. Since $[u_{i1}]=[e_{ii}][u_{i1}][e_{11}]$, we may also take u_{i1} in $\mathfrak{A}_{i1}$ $(i=2,\cdots,s)$, and similarly take u_{1j} in $\mathfrak{A}_{1j}$ $(j=2,\cdots,s)$. It follows from (19) that

$$u_{1j}u_{j1}=e_{11}+a_j \tag{20}$$

where a_j is in $\mathfrak{N}\cap\mathfrak{A}_{1j}\mathfrak{A}_{j1}\leqq\mathfrak{N}\cap\mathfrak{A}_{11}$ since

$$\mathfrak{A}_{ij}\mathfrak{A}_{jk}\leqq\mathfrak{A}_{ik} \tag{21}$$

[9, equation (4.5.1)]. Put

$$e_{1j}=(e_{11}-a_j)u_{1j}, \qquad e_{i1}=u_{i1} \qquad (i,j=2,\cdots,s).$$

Then e_{1j} is in $\mathfrak{A}_{11}\mathfrak{A}_{1j}\leqq\mathfrak{A}_{1j}$ by (21), and e_{i1} is in $\mathfrak{A}_{i1}$. Since

$$(x_{ij}y_{jk})z_{ki}=x_{ij}(y_{jk}z_{ki}) \qquad \text{unless } (i,j,k)=(i,i,i) \tag{22}$$

for $x_{\alpha\beta}$, $y_{\alpha\beta}$, $z_{\alpha\beta}$ in $\mathfrak{A}_{\alpha\beta}$ [9, Theorem (4.7.1)], we obtain the formula

$$e_{1j}e_{j1}=e_{11} \qquad (j=1,\cdots,s). \tag{23}$$

The case $j=1$ of (23) is trivial, and for $j\neq 1$ we have $e_{1j}e_{j1}=\{(e_{11}-a_j)u_{1j}\}u_{j1}=(e_{11}-a_j)(u_{1j}u_{j1})=(e_{11}-a_j)(e_{11}+a_j)=e_{11}$ by (22), (20), and the fact that e_{11} is the unity element of $\mathfrak{A}_{11}$, while $a_j^2=0$ since $\mathfrak{N}^2=0$.

Define

$$e_{ij}=e_{i1}e_{1j} \qquad (i\neq j;\ i,j=2,\cdots,s) \tag{24}$$

in $\mathfrak{A}_{i1}\mathfrak{A}_{1j}\leqq\mathfrak{A}_{ij}$. Since $[e_{i1}]=[u_{i1}]$ and $[e_{1j}]=[e_{11}-a_j][u_{1j}]=[u_{11}][u_{1j}]=[u_{1j}]$, we have $[e_{ij}]=[u_{ij}]$. Note that (24) holds if $i=1$ or $j=1$. Also (24) holds if $i=j$. For $(e_{i1}e_{1i})^2=e_{i1}\{(e_{1i}e_{i1})e_{1i}\}=e_{i1}e_{1i}$ by (23) and the Theorem of Artin; since $[e_{i1}e_{1i}]=[u_{ii}]=[e_{ii}]$, we may apply the second part of Lemma 3 to $\mathfrak{A}_{ii}$ to obtain $e_{ii}=e_{i1}e_{1i}$. Formula (24) is established for all $i,j=1,\cdots,s$. Now

$$e_{ij}e_{hk}=0 \qquad \text{for } j\neq h. \tag{25}$$

For $\mathfrak{A}_{ij}\mathfrak{A}_{hk}=0$ $(j\neq h)$ unless $h=i$, $k=j$ [9, Theorem (4.5.3)]. But

$$x_{ij}^2=0 \qquad \text{for } x_{ij} \text{ in } \mathfrak{A}_{ij},\ i\neq j \tag{26}$$

[6, p. 132, formula (15'')], which disposes of the remaining case in (25). We have

$$e_{1j}e_{jk}=e_{1k}, \qquad e_{kj}e_{j1}=e_{k1} \qquad (j,k=1,\cdots,s). \tag{27}$$

For, since we have proved (23), we may take $k \neq 1$. Then by (24), (6), (7), (23), and (25), we have $e_{1j}e_{jk} = e_{1j}(e_{j1}e_{1k}) = (e_{1j}e_{j1})e_{1k} - [e_{1j}, e_{j1}, e_{1k}] = e_{1k} - [e_{1k}, e_{1j}, e_{j1}] = e_{1k} - (e_{1k}e_{1j})e_{j1} + e_{1k}(e_{1j}e_{j1}) = e_{1k}$. The second equation in (27) is the reciprocal relationship. With (27) we can prove

$$e_{ij}e_{jk} = e_{ik} \qquad (i, j, k = 1, \cdots, s). \tag{28}$$

For $e_{ij}e_{jk} = (e_{i1}e_{1j})e_{jk} = e_{i1}(e_{1j}e_{jk}) - [e_{i1}, e_{1j}, e_{jk}] = e_{ik} - [e_{1j}, e_{jk}, e_{i1}] = e_{ik} - e_{1k}e_{i1} + e_{1j}(e_{jk}e_{i1})$. If $i = k$, this gives $e_{ij}e_{ji} = e_{ii} - e_{11} + e_{1j}e_{j1} = e_{ii}$; if $i \neq k$, it gives $e_{ij}e_{jk} = e_{ik}$ by (25). Equations (25) and (28) imply that the subalgebra $\mathfrak{M}$ of $\mathfrak{A}$ with basis e_{ij} $(i, j = 1, \cdots, s)$ is a total matric algebra, and the lemma is proved.

Consider the total matric algebra $\mathfrak{M}_2$ of degree two over $\mathfrak{F}$; it is a quaternion algebra with divisors of zero.[10] Taking the usual matric basis e_{ij} $(i, j = 1, 2)$ for $\mathfrak{M}_2$, the unity element is $e_{11} + e_{22}$ and the involution $q \to \bar{q} = t(q)(e_{11} + e_{22}) - q$ of $\mathfrak{M}_2$ is defined by

$$\bar{e}_{ii} = e_{jj}, \qquad \bar{e}_{ij} = -e_{ij} \qquad (i \neq j;\ i, j = 1, 2). \tag{29}$$

The algebra $\mathfrak{C} = \mathfrak{M}_2 + w\mathfrak{M}_2$ with elements $q_1 + wq_2$ (for q_i in $\mathfrak{M}_2$) and multiplication

$$(q_1 + wq_2)(q_3 + wq_4) = (q_1q_3 + q_4\bar{q}_2) + w(\bar{q}_1q_4 + q_3q_2) \tag{30}$$

is a Cayley-Dickson algebra (we have written w for v in (5) and set $\gamma = 1$). Since the Cayley-Dickson algebra with divisors of zero is unique, it may be taken in this form.

LEMMA 5. *Let* $\mathfrak{A} - \mathfrak{N} = \mathfrak{C}$, *a Cayley-Dickson algebra with divisors of zero. Then* $\mathfrak{A} = \mathfrak{C}_1 + \mathfrak{N}$ *where* $C_1 \cong C$.

For we may take $\mathfrak{C}$ in the form $\mathfrak{C} = \mathfrak{M}_2 + [w]\mathfrak{M}_2$, where, for elements $[q]$ in $\mathfrak{M}_2$, multiplication is given by (30) with brackets. By Lemma 4, $\mathfrak{A}$ contains a total matric algebra $\mathfrak{M}$ of degree two such that e is the unity element of $\mathfrak{M}$ and the matric basis e_{ij} of $\mathfrak{M}$ gives the matric basis $[e_{ij}]$ of $\mathfrak{M}_2$. Note that $\overline{[q]} = [\bar{q}]$ for q in $\mathfrak{M}$. In the proof of [5, Theorem 2], it is shown that the multiplication formula (5) follows from the alternative law and formulas $v^2 = \gamma e$, $qv = v\bar{q}$ for q in $\mathfrak{Q}$. Therefore, in order to prove the lemma, it is sufficient to show the existence of an element v not in $\mathfrak{M}$ such that

$$v^2 = e, \qquad qv = v\bar{q} \qquad \text{for } q \text{ in } \mathfrak{M}. \tag{31}$$

Write

$$[f_{ij}] = [w][e_{jj}] \qquad \text{for } i \neq j \ (i, j = 1, 2). \tag{32}$$

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[10] See footnote 4.

We may take f_{ij} in $\mathfrak{A}_{ij}$ $(i\neq j)$, for $[e_{ii}]([f_{ij}][e_{jj}])=[e_{ii}]([w][e_{jj}]^2)$ $=[w][\bar{e}_{ii}e_{jj}]=[w][e_{jj}]=[f_{ij}]$ by (32), (30), (29). Now $[e_{ji}][f_{ij}]$ $=[e_{ji}]([w][e_{jj}])=-[w]([e_{ji}][e_{jj}])=[0]$ implies that

$$e_{ji}f_{ij}=c_j \qquad (i\neq j;\ i, j=1, 2)$$

for c_j in $\mathfrak{N}\cap\mathfrak{A}_{jj}$. Write

$$h_{ij}=f_{ij}-e_{ij}c_j.$$

Then h_{ij} is in $\mathfrak{A}_{ij}$, $[h_{ij}]=[f_{ij}]$, and

$$e_{ji}h_{ij}=h_{ij}e_{ji}=0 \qquad (i\neq j;\ i, j=1, 2). \tag{33}$$

For $e_{ji}h_{ij}=c_j-e_{ji}(e_{ij}c_j)=c_j-(e_{ji}e_{ij})c_j=0$ by (22). Moreover, $e_{ij}c_j$ $=e_{ij}(e_{ji}f_{ij})=(e_{ij}e_{ji})f_{ij}-[e_{ij}, e_{ji}, f_{ij}]=f_{ij}+[f_{ij}, e_{ji}, e_{ij}]=f_{ij}+(f_{ij}e_{ji})e_{ij}$ $-f_{ij}(e_{ji}e_{ij})=(f_{ij}e_{ji})e_{ij}$ by (7), and we have $h_{ij}e_{ji}=f_{ij}e_{ji}-\{(f_{ij}e_{ji})e_{ij}\}e_{ji}$ $=0$ by (22). Now $[h_{ij}][h_{ji}]=[f_{ij}][f_{ji}]=[e_{ii}][\bar{e}_{jj}]=[e_{ii}]$ implies that

$$h_{ij}h_{ji}=e_{ii}+a_i$$

for a_i in $\mathfrak{N}\cap\mathfrak{A}_{ii}$ $(i=1, 2)$. Since $\mathfrak{N}^2=0$, we have $a_i^2=0$, and

$$(e_{ii}-a_i)(e_{ii}+a_i)=e_{ii}=(e_{ii}+a_i)(e_{ii}-a_i). \tag{34}$$

Write

$$p_{12}=(e_{11}-a_1)h_{12}, \qquad p_{21}=h_{21}.$$

Then p_{ij} is an $\mathfrak{A}_{ij}$, $[p_{ij}]=[f_{ij}]$, and we shall prove

$$p_{ij}p_{ji}=e_{ii} \qquad (i\neq j;\ i, j=1, 2). \tag{35}$$

For $p_{12}p_{21}=\{(e_{11}-a_1)h_{12}\}h_{21}=(e_{11}-a_1)(h_{12}h_{21})=(e_{11}-a_1)(e_{11}+a_1)=e_{11}$ by (22) and (34). But $a_ih_{ij}=(h_{ij}h_{ji}-e_{ii})h_{ij}=h_{ij}(h_{ji}h_{ij})-h_{ij}$ $=h_{ij}(e_{jj}+a_j)-h_{ij}=h_{ij}a_j$, so that

$$p_{12}=h_{12}-a_1h_{12}=h_{12}-h_{12}a_2=h_{12}(e_{22}-a_2) \tag{36}$$

and $p_{21}p_{12}=h_{21}\{h_{12}(e_{22}-a_2)\}=(h_{21}h_{12})(e_{22}-a_2)=(e_{22}+a_2)(e_{22}-a_2)=e_{22}$ by (22) and (34). It follows from (33), (36), and (22) that

$$e_{ij}p_{ji}=p_{ji}e_{ij}=0 \qquad (i\neq j;\ i, j=1, 2). \tag{37}$$

Finally write

$$v=p_{12}+p_{21}.$$

Then $[v]=[f_{12}]+[f_{21}]=[w]$, and $v^2=(p_{12}+p_{21})^2=p_{12}p_{21}+p_{12}p_{21}=e_{11}$ $+e_{22}=e$ by (26) and (35). For $i\neq j$ we have $e_{ii}v=e_{ii}(p_{ij}+p_{ji})=p_{ij}$ $=(p_{ij}+p_{ji})e_{jj}=ve_{jj}$; also (26) implies that $(e_{ij}+p_{ij})^2=e_{ij}p_{ij}+p_{ij}e_{ij}=0$ so that $e_{ij}v=e_{ij}(p_{ij}+p_{ji})=e_{ij}p_{ij}=-p_{ij}e_{ij}=-(p_{ij}+p_{ji})e_{ij}=-ve_{ij}$ by

(37). Hence $qv=v\bar{q}$ for all q in $\mathfrak{M}$ by (29), and we have shown the existence of an element v in $\mathfrak{A}$, not in $\mathfrak{M}$, satisfying (31). This completes the proof of the lemma, and of the theorem.

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THE INSTITUTE FOR ADVANCED STUDY

A NOTE ON THE MINIMUM OF A BINARY FORM

N. C. ANKENY

Let $f(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ be a binary form of degree n with real coefficients. The simplest algebraic invariant of such a form is its discriminant, defined by

$$D(f) = a_0^{2(n-1)} \prod_{i<j} (\theta_i - \theta_j)^2, \tag{1}$$

where[1]

$$f(x, y) = a_0(x - \theta_1 y) \cdots (x - \theta_n y). \tag{2}$$

$D(f)$ is a homogeneous polynomial of degree $2(n-1)$ in $a_0, \cdots, a_n$. The simplest arithmetical invariant of such a form is its minimum $L(f)$, defined as the lower bound of $|f(x, y)|$ for all integral values of x, y except 0, 0.

For $n = 2$ and 3, it is known that if $D(f) \neq 0$, then

$$L^{2(n-1)}(f) \leqq c \, | D(f) |, \tag{3}$$

where c is a constant. The best possible value of c is known for each type of form with $n = 2$ or 3. We say that a form is of *type* (r, s) if $\theta_1, \cdots, \theta_n$ consist of r real numbers, and s pairs of conjugate complex numbers (so that $r + 2s = n$). If we denote by $c(r, s)$ the least value of c for which (3) holds, then

$$c(0, 1) = 1/3, \quad c(2, 0) = 1/5, \quad c(3, 0) = 1/49, \quad c(1, 1) = 1/23.$$

The first two results are classical, and the last two are due to Mordell.

The object of this note is to prove that *no inequality of the type* (3) *is valid for all binary forms of degree n, and of a given type, when n is greater than* 3. This fact is probably well known to those who have followed recent work on the geometry of numbers, but no simple direct proof seems to have been given. One argument which suggests why (3) cannot be expected to hold when $n > 3$ is the following. When $n = 2$ or 3, a corollary to the existing results tells us that if $D(f) = 0$ then $L(f) = 0$, that is, $f(x, y)$ then takes values which are arbitrarily small numerically. This is not true for $n = 4$ as we see from the simple example $(x^2 - 2y^2)^2$. We shall show that if $n > 3$ there exist

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[1] The definition fails if $a_0 = 0$. In this case, we must first transform f by a linear substitution of determinant ± 1 into a form whose leading coefficient is not zero.

forms for which $|D(f)|$ is arbitrarily small, but not zero, and $L(f)$ is not small.

I am indebted to Prof. Harold Davenport for suggesting this topic to me and to both him and Prof. L. J. Mordell for comments on the manuscript.

In what follows, all small Latin letters denote rational integers. We recall that any irrational number α, with $1<\alpha<2$, has a continued fraction representation

$$\alpha = 1 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\cdots$$

where $a_1, a_2, \cdots$ are positive. If p_n/q_n is the nth convergent to α, so that

$$\frac{p_n}{q_n} = 1 + \frac{1}{a_1 +}\cdots\frac{1}{+ a_n},$$

we have the following well known results:

LEMMA 1.

$$\frac{1}{3a_{n+1}q_n^2} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}.$$

LEMMA 2. *If $q_n \leqq y < q_{n+1}$, then $|p_n - \alpha q_n| \leqq |x - \alpha y|$ for all x.*

Lemmas 1 and 2 imply:

LEMMA 3. *If $q_n \leqq y < q_{n+1}$, then $|x-\alpha y| > 1/3a_{n+1}q_n \geqq 1/3a_{n+1}y$ for all x.*

We now prove:

LEMMA 4. *If $\alpha > 1$ and $\beta > 1$ are both irrationals whose continued fraction expansions have all their partial quotients less than N, then*

$$|(x-\alpha y)(x+\beta y)| > 1/3N$$

for all x, y not both 0.

PROOF. We may certainly assume $xy \neq 0$. Suppose first $xy > 0$. Then $|x+\beta y| > |\beta y| > |y|$. Also, $|x-\alpha y| > 1/3N|y|$ from Lemma 3, so the result follows by multiplying these two inequalities. The case when $xy < 0$ follows from the case $xy > 0$ on interchanging α and β.

LEMMA 5. *Let θ be a real quadratic irrational integer, and let θ' be its conjugate. Suppose $\theta > 1$ and $\theta' > 1$. Let α satisfy the condition of Lemma*

4. *Then*

$$|(x-\alpha y)(x+\theta y)(x+\theta' y)| > 1/3N$$

for all x, y not both zero.

Proof. Again we may assume $xy \neq 0$. If $xy<0$, we have $|x-\alpha y|>1$, and $|(x+\theta y)(x+\theta' y)| \geqq 1$ since it is the absolute norm of an algebraic integer not zero. If $xy>0$, $|x+\theta y|>|y|$, $|x+\theta' y|>|y|$ and $|x-\alpha y| > 1/3N|y|$ by Lemma 3. This proves the result.

We can now prove the main theorem by constructing forms $F(x, y)$ of given type (r, s) where $r+2s=n\geqq 4$.

Case 1. $r \geqq 4$, $r\neq 5$. Let $G(x, y)$ be any binary form of type $(r-4, s)$ with integral coefficients, and $D(G)\neq 0$, such that the $r-4$ real roots of $G(\xi, 1)=0$ are all irrational. Since $r-4\neq 1$; such a form can be constructed by multiplying together various forms of the type

$$x^2-my^2, \qquad x^2+my^2, \qquad x^3+2x^2y-xy^2-y^3$$

using several different values of m if necessary, and not more than one cubic form. (If $r=4$, $s=0$, we take the empty product, that is, $G(x, y)=1$.) Then $L(G)\geqq 1$. Let

$$F(x, y) = (x-\alpha_1 y)(x+\beta_1 y)(x-\alpha_2 y)(x+\beta_2 y)G(x, y)$$

where α_1, β_1 and α_2, β_2 satisfy the hypotheses of Lemma 4 and $\alpha_1\neq\alpha_2$, $\beta_1\neq\beta_2$, and none of α_1, α_2, $-\beta_1$, $-\beta_2$, coincides with a root of $G(\xi, 1)=0$, Then

$$L(F) \geqq (3N)^{-2}$$

by Lemma 4. But, if the α's and β's are bounded,

$$0<|D(F)|<A(G)(\alpha_1-\alpha_2)^2(\beta_1-\beta_2)^2$$

where $A(G)$ is bounded for fixed G if the α's and β's are bounded.

We fix α_1, β_1, β_2 and choose α_2 to have the same first $k+1$ partial quotients as α_1. Then by Lemma 1,

$$|\alpha_1-\alpha_2| \leqq \left|\alpha_1-\frac{p_k}{q_k}\right| + \left|\alpha_2-\frac{p_k}{q_k}\right| < \frac{2}{a_{k+1}q_k^2} \leqq \frac{2}{k^2}\cdot$$

Hence $D(F)\to 0$ as $k\to\infty$, which proves the result.

Case 2. $r=5$. Let $G(x, y)$ be a form as above, of type $(0, s)$, which can be taken to be a product of forms x^2+my^2. Then $L(G)\geqq 1$. Let

$$F(x, y) = (x-\alpha_1 y)(x+\theta y)(x+\theta' y)(x-\alpha_2 y)(x+\beta_2 y)G(x, y);$$

where α_1, α_2, β_2 satisfy the same hypotheses as before and θ satisfies the hypotheses of Lemma 5, and $-\theta$, $-\theta'$, α_1, α_2, $-\beta_2$ are distinct. By Lemmas 4 and 5,

$$L(F) \geqq (3N)^{-2}.$$

Again $D(F)$ may be made arbitrarily small, but not zero, by choice of α_2.

Case 3. $r \leqq 3$. This presents less difficulty, and we take $F(x, y)$ to have the following form:

$$r = 0,\ s \geqq 2, \qquad F(x, y) = (x^2 + y^2)(x^2 + (1 + \epsilon)y^2)G(x, y),$$
$$L(F) \geqq L(G) \geqq 1,$$
$$r = 1,\ s \geqq 2, \qquad F(x, y) = (x - \alpha y)(x^2 + y^2)(x^2 + (1 + \epsilon)y^2)G(x, y),$$
$$|F(x, y)| \geqq \frac{1}{3N|y|}\, y^2 \geqq \frac{1}{3N} \qquad \text{for } y \neq 0,$$
$$r = 2,\ s \geqq 1, \qquad F(x, y) = (x - \alpha_1 y)(x - \alpha_2 y)(x^2 + y^2)G(x, y),$$
$$|F(x, y)| \geqq \frac{1}{(3Ny)^2}\, y^2 = \frac{1}{(3N)^2} \qquad \text{for } y \neq 0,$$
$$r = 3,\ s \geqq 1, \quad F(x, y) = (x - \alpha_1 y)(x + \beta_1 y)(x - \alpha_2 y)(x^2 + y^2)G(x, y),$$
$$|F(x, y)| \geqq \frac{1}{(3N)}\, \frac{1}{(3N|y|)}\, y^2 \geqq \frac{1}{(3N)^2} \qquad \text{for } y \neq 0.$$

In the first two cases $D(F) \to 0$ with ϵ; in the other two cases $D(F)$ is arbitrarily small, but unequal to zero, by choice of α_2.

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ON A THEOREM OF H. CARTAN

RICHARD BRAUER

As an application of the Galois theory of skew fields, H. Cartan[1] obtained recently the following theorem: If K is a skew field of finite rank over its center C, the only skew fields H, $C \subseteq H \subseteq K$, which are mapped into themselves by every inner automorphism of K are K and C.

I give a very short and direct proof removing at the same time the finiteness assumption, In fact, we have:

THEOREM. *If H is a skew field contained in the skew field K, and if every inner automorphism of K maps H into itself, then H is either K, or H belongs to the center of K.*

PROOF. If $a \in K$, $b \in H$, the assumption shows that an equation

$$ba = ab_1 \tag{1}$$

with $b_1 \in H$ holds (for $a=0$ this is true with $b_1 = b$). Also,

$$b(1 + a) = (1 + a)b_2 \tag{2}$$

with $b_2 \in H$. On subtracting (1) from (2), we find

$$b - b_2 = a(b_2 - b_1).$$

If a does not lie in H, this implies $b_2 = b_1$ and hence $b = b_2$. Then $b_1 = b$, that is, $ba = ab$. Every element a of K which does not belong to H commutes therefore with every element of H.

Suppose that H does not belong to the center of K. There exists an element b of H and an element c of K such that

$$bc \neq cb. \tag{3}$$

The remark above shows that $c \in H$. If $H \neq K$, there exist elements a outside of H in K. Then $a+c$ does not belong to H either. Hence $a+c$ and a both commute with $b \in H$,

$$b(a + c) = (a + c)b, \; ba = ab.$$

These two equations are not consistent with (3), and the theorem is proved.

The same argument applies under much weaker assumptions. For instance, it is sufficient to assume that K is a (not necessarily associa-

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[1] C. R. Acad. Sci. Paris vol. 224 (1947) pp. 249–251.

tive) ring, H a subring of K and that (α) H has a 1-element; (β) the equation $xh = h_1$ with h, $h_1 \in H$, $x \in K$, $h \neq 0$ implies that $x \in H$, (γ) for every $a \in K$, $b \in H$, there exists an element b_1 in H with $ba = ab_1$. If $H \neq K$, it follows that every element of H commutes with every element of K.

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A THEOREM ON INTEGRAL SYMMETRIC MATRICES[1]

B. W. JONES

Though the following theorem yields important results in the theory of quadratic forms, its statement and proof are independent of such theory and seem to possess significance in their own right.

THEOREM. *Let A and B be symmetric integral nonsingular matrices with respective dimensions n and m ($n > m$) and S an n by m matrix of rank m with rational elements such that s is the* l.c.m. *of the denominators and $S^TAS = B$. Then there is an n by n matrix T with rational elements the prime factors of whose denominators all divide s, whose determinant is* 1 *and which takes A into an integral matrix A_0 which represents B integrally, that is, $U^TA_0U = B$ for some integral matrix U.*

To prove this we first, for brevity's sake, define an *s-matrix* or *s-transformation* to be one with rational elements the prime factors of whose denominators all divide s. Then write $R = sS$, and, by elementary divisor theory, determine unimodular matrices P and Q such that

$$PRQ = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = s\begin{bmatrix} S_1 \\ 0 \end{bmatrix} = sS'$$

where R_1 is the diagonal matrix $r_1 \dotplus r_2 \dotplus \cdots \dotplus r_m$, $\dotplus$ denotes direct sum, r_i divides r_{i+1} for $i = 1, 2, \cdots, m-1$ and S' and S_1 are defined by the equations. Write $r_i/s = u_i/s_i$ where the latter fraction is in lowest terms and $s_i > 0$. Then s_i is divisible by s_{i+1} and hence s_i is prime to u_j for $j \leq i$.

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[1] This paper was written while on sabbatical leave from Cornell University with the aid of a grant from the Research Corporation.

If we write $A' = P^{IT}AP^{I}$ and $B' = Q^TBQ$, the equality $S^TAS = B$ becomes $S'^TA'S' = B'$ which, with $A' = (a_{ij})$ and $B' = (b_{ij})$, implies

$$(1) \qquad a_{ij}r_ir_j/s^2 = a_{ij}u_iu_j/s_is_j = b_{ij}, \qquad i, j = 1, 2, \cdots, m.$$

Suppose $s_i = 1 = |u_i|$ for $i = 1, 2, \cdots, h-1$ but not for $i = h$. Then (1) and B' integral implies that s_h divides a_{ih} for $1 \leqq i \leqq h$ and s_h^2 divides a_{hh}. Moreover u_h divides b_{ih} for $1 \leqq i \leqq m$ and u_h^2 divides b_{hh}. Write A' in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

where A_{11} is an h by h matrix. Let D_h be the matrix obtained from the h-rowed identity matrix by replacing its last diagonal element by s_h and U_h the matrix obtained from the m-rowed identity matrix by replacing its hth diagonal element by u_h. Write $F_h = D_h^I \dotplus K_h$ where K_h is an integral $(n-h)$-rowed square matrix of determinant s_h later to be determined. We then have

$$F_h^T A' F_h = \begin{bmatrix} D_h^I A_{11} D_h^I & D_h^I A_{12} K_h \\ K_h^T A_{12}^T D_h^I & K_h^T A_{22} K_h \end{bmatrix} = A_h.$$

Then S_h, defined by the equation $S_h = F_h^I S' U_h^I$, is the direct sum of the integral matrix $u_1 \dotplus u_2 \dotplus \cdots \dotplus u_{h-1} \dotplus 1$, whose elements have absolute value 1 and an $n-h$ by $m-h$ s-matrix. Furthermore $S'^TA'S' = B'$ becomes $S_h^TA_hS_h = B_h$ where the equality $U_h^IB'U_h^I = B_h$ defines B_h. Now B_h is integral since, as shown above, u_h divides b_{ih} for $1 \leqq i \leqq m$ and u_h^2 divides b_{hh}. Furthermore B_h represents B' integrally. Similarly $D_h^I A_{11} D_h^I$ is integral.

Next we determine K_h consistent with the above conditions so that A_h is integral, that is, so that $D_h^I A_{12} K_h$ is integral. In fact, in view of the definition of D_h, we need only make αK_h divisible by s_h where α is the hth row of A_{12}. This is easily done by finding a unimodular matrix W so that $\alpha W \equiv (w, 0, \cdots, 0) \pmod{s_h}$ and choosing W^IK_h to be the diagonal matrix $s_h \dotplus 1 \dotplus \cdots \dotplus 1$ of determinant s_h.

If now we diagonalize the last $n-h$ rows of S_h replacing A_h and B_h by equivalent matrices (that is, matrices obtained from them by unimodular transformations) we may continue along the above lines to derive inductively a sequence of integral matrices A_h obtained from A by s-transformations of determinant 1 and taken by transformations S_h into integral matrices B_h which represent B' integrally, each S_h being the direct sum of an h-rowed integral matrix and an s-matrix.

Then S_m is integral and taking $A_m = A_0$, $B_m = B_0$ we see that A_0 is obtained from A by an s-transformation of determinant 1, represents B_0 integrally and hence B integrally.

Since two quadratic forms with integral coefficients may be defined to be in the same genus if one may be taken into the other by a transformation of determinant ± 1 with rational elements whose denominators are prime to twice the determinant of the form, we have the following

COROLLARY 1. *If f and g are two quadratic forms with integral coefficients, having nonzero determinants and n and m variables respectively, $n>m$, and if there is an n by m matrix with rational elements whose denominators are prime to $2|f|$ and taking f into g, then there is a form f_0 in the same genus as f whose matrix has integral elements and which represents g integrally.*

This corollary has been proved using the theory of quadratic forms by C. L. Siegel,[2] whose proof contained some ideas in common with the above proof, by Hel Braun[3] and probably follows from the much earlier work of Minkowski. However in all these sources the condition that the denominators be prime to $2|f|$ is essential. It is then of interest to note that this restriction is not necessary for the theorem of this note.

In fact, a direct consequence of our theorem (as noted by the referee) is

COROLLARY 2. *If a classic quadratic form f (that is, a form whose matrix has integer elements) represents an integer or classic form g rationally, then some classic form f' of the same determinant as f represents g integrally.*

Thus for example the conditions for integral representation by the form $x_1^2 + x_2^2 + \cdots + x_n^2$, $1 \leq n \leq 7$, are the same conditions as those for rational representation since there is only one class of forms of determinant 1 in these cases.

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[2] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. (2) vol. 36 (1935) pp. 527–606, Lemma 24.

[3] Hel Braun, *Geschlechter quadratischer Formen*, J. Reine Angew. Math. vol. 182 (1940) pp. 32–49.

INVARIANTS OF INTERSECTION OF CERTAIN PAIRS OF SPACE CURVES

CHUAN-CHIH HSIUNG

Introduction. Projective invariants of certain pairs of curves and surfaces have been derived and characterized geometrically by the author and others. In this paper we shall continue these investigations by studying in ordinary space some pairs of curves intersecting at a nonsingular point with distinct tangents. For each pair of such curves we can determine an independent projective invariant by the use of neighborhoods up to and including the third order of the curves at the common point. Within these neighborhoods of the curves at the point there is no other affine invariant which is not a projective invariant.[1]

I. TWO CURVES INTERSECTING AT AN ORDINARY POINT WITH DISTINCT TANGENTS AND OSCULATING PLANES

1. Derivation of an invariant I. Let two curves C, $\bar{C}$ in ordinary space intersect at a nonsingular point 0 with distinct tangents t, $\bar{t}$ and osculating planes τ, $\bar{\tau}$ respectively. We shall first confine our attention to the case in which the line p of intersection of the osculating planes τ, $\bar{\tau}$ coincides with neither t nor $\bar{t}$. Let x, y, z represent the nonhomogeneous coordinates obtained from the projective homogeneous coordinates $x_1, \cdots, x_4$ by means of the relations

$$x = x_2/x_1, \qquad y = x_3/x_1, \qquad z = x_4/x_1.$$

If we choose the point 0 to be the origin, the tangents t, $\bar{t}$ and the line p to be respectively the x-, y-, and z-axes, then the power series expansions of the curves C, $\bar{C}$ in the neighborhood of the point 0 may be written in the form

$$C: \quad y = rx^3 + \cdots, \qquad z = ax^2 + \cdots; \tag{1}$$

$$\bar{C}: \quad x = \rho y^3 + \cdots, \qquad z = \alpha y^2 + \cdots. \tag{2}$$

The most general projective transformation which leaves the tangents t, $\bar{t}$ and the line p unchanged is expressed in terms of the nonhomogeneous coordinates by the equations

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[1] For some pairs of plane curves and surfaces in ordinary space L. A. Santaló has recently found some affine invariants which are not projective invariants. See his paper, *Affine invariants of certain pairs of curves and surfaces*, Duke Math. J. vol. 14 (1947) pp. 559–574.

(3) $$x = a_{22}x^*/D, \qquad y = a_{33}y^*/D, \qquad z = a_{44}z^*/D,$$

where

(4) $$D \equiv 1 + a_{12}x^* + a_{13}y^* + a_{14}z^*,$$

and the a's are arbitrary. The effect of this transformation on equations (1), (2) is to produce two other pairs of equations of the same form whose coefficients, indicated by stars, are given by the formulas

(5) $$\begin{aligned} a_{33}r^* &= a_{22}^{3}r, & a_{44}a^* &= a_{22}^{2}a, \\ a_{22}\rho^* &= a_{33}^{3}\rho, & a_{44}\alpha^* &= a_{33}^{2}\alpha. \end{aligned}$$

By eliminating the a's from equation (5) it is easily seen that *the expression*

(6) $$I = r\alpha^2/\rho a^2$$

is a projective invariant determined by the neighborhoods up to and including the third order of the two curves C, $\overline{C}$ at the point 0.

In view of the manner in which the a's appear in equations (5), it is evident that all other projective invariants depending on such neighborhoods of the two curves C, $\overline{C}$ at the point 0 are expressible in terms of the invariant I.

2. **Metric and projective characterizations of the invariant** I. For the purpose of giving a simple metric characterization of the invariant I, we make a projective transformation which leaves the point 0 unchanged and carries the x-, y-, and z-axes into three mutually perpendicular new axes. Let Γ, $\overline{\Gamma}$ be the transformed curves of C, $\overline{C}$, and x', y', z' be the nonhomogeneous Cartesian coordinates of a point in space referred to the new orthogonal coordinate system; then the power series expansions of the curves Γ, $\overline{\Gamma}$ in the neighborhood of the point 0 may be written in the form

(7) $$\Gamma: \quad y' = r'x'^3 + \cdots, \qquad z' = a'x'^2 + \cdots,$$

(8) $$\overline{\Gamma}: \quad x' = \rho'y'^3 + \cdots, \qquad z' = \alpha'y'^2 + \cdots,$$

and the invariant I takes the form

(9) $$I = r'\alpha'^2/\rho'a'^2.$$

Let R', $\overline{R}'$, and T', $\overline{T}'$ be respectively the radii of curvature and of torsion of the curves Γ, $\overline{\Gamma}$ at the point 0, then

(10) $$\begin{aligned} R' &= 1/2a', & T' &= a'/3r', \\ \overline{R}' &= 1/2\alpha', & \overline{T}' &= -\alpha'/3\rho', \end{aligned}$$

and therefore

$$I = -R'\bar{T}'/\bar{R}'T'. \tag{11}$$

Thus we obtain the following metric characterization of the invariant I associated with the point 0 of intersection of the two curves C, $\bar{C}$.

Let R, $\bar{R}$, *and* T, $\bar{T}$ *be respectively the radii of curvature and of torsion of the curves* C, $\bar{C}$ *at the point* 0, *then*

$$I = -R\bar{T}/\bar{R}T. \tag{12}$$

In order to interpret the invariant I projectively we observe that among the five-point quadric cones of the curve C at the point 0 we may determine a unique one which has $\bar{t}$ for a generator and whose tangent plane along $\bar{t}$ passes through the line p. From expansions (1) follows immediately the equation of this cone, namely,

$$a^2xy - rz^2 = 0. \tag{13}$$

Similarly, with the roles of the curves C, $\bar{C}$ interchanged we have another quadric cone:

$$\alpha^2xy - \rho z^2 = 0. \tag{14}$$

The two cones (13), (14) determine a pencil P of quadric cones, to which belong the pair of planes $x=0$, $y=0$ and the plane $z=0$, counted twice. Hence *the invariant* I *is equal to the cross ratio of the cones* (13), (14) *and these two pairs of planes in the pencil* P *of quadric cones.*

3. **Derivation of an invariant** J. We next consider in ordinary space two curves C, $\bar{C}$ intersecting at a nonsingular point 0 with distinct tangents t, $\bar{t}$ and osculating planes τ, $\bar{\tau}$, respectively, the line p of intersection of the osculating planes τ, $\bar{\tau}$ coinciding with the tangent $\bar{t}$. If we choose the point 0 to be the origin, the tangents t, $\bar{t}$ to be the x- and y-axes respectively, and any line through the point 0 and in the osculating plane $\bar{\tau}$ to be the z-axis, then the power series expansions of the curves C, $\bar{C}$ in the neighborhood of the point 0 may be written in the form

$$C\colon \quad z = rx^3 + \cdots, \qquad y = ax^2 + \cdots; \tag{15}$$

$$\bar{C}\colon \quad x = \rho y^3 + \cdots, \qquad z = \alpha y^2 + \cdots. \tag{16}$$

As in §1, we make the most general projective transformation of coordinates

$$x = a_{22}x^*/D, \qquad y = (a_{33}y^* + a_{34}z^*)/D, \qquad z = a_{44}z^*/D, \tag{17}$$

which leaves the tangents t, $\bar{t}$ and the osculating plane $\bar{\tau}$ unchanged, where D is defined by expression (4) and the a's are arbitrary. The effect of this transformation on equations (15), (16) is to produce two other pairs of equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$
\begin{aligned}
a_{44}r^* &= a_{22}^{3}r, & a_{33}a^* &= a_{22}^{2}a, \\
a_{22}\rho^* &= a_{33}^{3}\rho, & a_{44}\alpha^* &= a_{33}^{2}\alpha.
\end{aligned} \tag{18}
$$

Eliminating the a's from equations (18) we may obtain *an independent projective invariant*

$$J = r^5\rho/a^7\alpha^5 \tag{19}$$

determined by the neighborhoods up to and including the third order of the two curves C, $\bar{C}$ at the point 0.

4. Metric and projective characterizations of the invariant J. Let R, $\bar{R}$ and T, $\bar{T}$ be respectively the radii of curvature and of torsion at the point 0 of the curves C, $\bar{C}$ in the preceding section, then it is easily seen that

$$J = \left(\frac{2}{3}\right)^6 \frac{R^2\bar{R}^4}{T^5\bar{T}}\,. \tag{20}$$

Among the five-point quadric cones of the curve C at the point 0, we may determine a unique one which has the z-axis as a generator and whose tangent plane along the z-axis is the plane $x=0$. The equation of this cone is easily found, from expansions (15), to be

$$a^2xz - ry^2 = 0. \tag{21}$$

Similarly, the equation of the unique five-point quadric cone of the curve $\bar{C}$ at the point 0 having the x-axis as a generator and the xz-plane as tangent plane along that axis is

$$\alpha^2xy - \rho z^2 = 0. \tag{22}$$

The z-axis and the three lines of intersection, other than the x-axis, of the two quadric cones (21), (22) determine three planes through the z-axis, which intersect the plane $z=0$ in the three lines

$$z = a^4\alpha^2x^3 - r^2\rho y^3 = 0. \tag{23}$$

On the other hand, from expansions (15), (16) we may easily obtain the equation of any quadric which has the x-axis as a generator, and which has four-point contact with the curve C at the point 0 and three-point contact with the curve $\bar{C}$ at the point 0; the result is

$$rxy + b_{13}xz + a\alpha y^2 + b_{23}yz + b_{33}z^2 - az = 0, \tag{24}$$

where the b's are arbitrary. The plane $z=0$ cuts the quadric (24) in the x-axis and the line u:

$$z = rx + a\alpha y = 0. \tag{25}$$

Let v be any one of the three lines (23), then *the cross ratio of the four lines* t, $\bar{t}$, u, v *is equal to* $-J^{1/3}$.

II. TWO CURVES INTERSECTING AT AN ORDINARY POINT WITH DISTINCT TANGENTS BUT THE SAME OSCULATING PLANE

5. An invariant and its metric and projective characterizations. Finally, we consider in ordinary space two curves C, $\bar{C}$ intersecting at a nonsingular point 0 with distinct tangents t, $\bar{t}$, respectively, but the same osculating plane τ. If we choose the point 0 to be the origin, the tangents t, $\bar{t}$ to be respectively the x- and y-axes, and any line through the point 0 but not in the common osculating plane τ to be the z-axis, then the power series expansions of the curves C, $\bar{C}$ in the neighborhood of the point 0 may be written in the form

$$C\colon\quad z = rx^3 + \cdots, \qquad y = ax^2 + \cdots; \tag{26}$$

$$\bar{C}\colon\quad z = \rho y^3 + \cdots, \qquad x = \alpha y^2 + \cdots. \tag{27}$$

In a way similar to the foregoing we can easily show that *the expression*

$$K = r\alpha/a\rho \tag{28}$$

is an independent projective invariant determined by the neighborhoods up to and including the third order of the two curves C, $\bar{C}$ *at the point* 0.[2]

Let T, $\bar{T}$ be respectively the radii of torsion of the curves C, $\bar{C}$ at the point 0, then it follows immediately that

$$K = -\bar{T}/T. \tag{29}$$

It is known[3] that associated with the point 0 of the curves C, $\bar{C}$ there are three principal lines whose equations are

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[2] For two plane curves intersecting at an ordinary point no projective invariant can be determined by the neighborhoods up to and including the third order of the curves at the common point. See the author's paper, *Theory of intersection of two plane curves*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 786–792.

[3] E. Bompiani, *Invarianti d'intersezione di due curve sghembe*, Rendiconti R. Accademia Nazionale dei Lincei (6) vol. 14 (1931) p. 459.

$$z = rx^3 + \rho y^3 = 0. \tag{30}$$

On the other hand, let Q ($\bar{Q}$) be any three-point quadric of the curve C ($\bar{C}$) at the point 0 satisfying the condition that the tangent plane of Q ($\bar{Q}$) at the other point $\bar{S}$ (S) of intersection of Q ($\bar{Q}$) with the tangent $\bar{t}$ (t) passes through the point S ($\bar{S}$). Then the lines joining the point 0 to the other three points of intersection of the two conics determined by the quadrics Q, $\bar{Q}$ and the common osculating plane τ have the equations[4]

$$z = ax^3 - \alpha y^3 = 0. \tag{31}$$

If u be any one of the principal lines (30) and v be any one of the three lines (31), then *the projective invariant K associated with the point* 0 *of intersection of the curves C, $\bar{C}$ is, except for sign, equal to the cube of the cross ratio* (t, $\bar{t}$, u, v).

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[4] C. C. Hsiung, *Note on the intersection of two space* curves, Tôhoku Math. J. vol. 47 (1940) pp. 206–207.

ON ESSENTIALLY ABSOLUTELY CONTINUOUS PLANE TRANSFORMATIONS

TIBOR RADÓ

1. **Introduction.** Let Q be the unit square $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, in the uv-plane (u, v are Cartesian coordinates), and let $x(u, v)$, $y(u, v)$ be real-valued continuous functions in Q. These functions determine a continuous mapping T: $x = x(u, v)$, $y = y(u, v)$, $(u, v) \in Q$, from Q into the xy-plane, where x, y are again Cartesian coordinates. If we introduce, for convenience, the complex variables $w = u + iv$, $z = x + iy$, then T appears in the form

$$T: \quad z = f(w), \qquad w \in Q, \tag{1}$$

where $f(w) = x(u, v) + iy(u, v)$. In connection with various problems in geometry and in analysis, there arises the problem of defining the concepts of *bounded variation* and of *absolute continuity* for continuous mappings T given as in (1). A detailed review of the extensive relevant literature may be found in Chapter IV.5 of the writer's book *Length and area*. This book will be referred to as LA, and will be used as general reference both for technical details and for literature. The present note is concerned with a line of thought that led to the concepts eBV (*essential bounded variation*) and eAC (*essential absolute continuity*). The theory of these concepts, as presented in LA, is based on the work of the writer and P. V. Reichelderfer. During the war years, Cesari in Italy developed an analogous theory based on apparently different basic concepts. However, as the writer has shown (see Bibliography [2]), the basic concepts used by Cesari are equivalent to those used in LA (see Bibliography). Thus the theories developed in LA and in the work of Cesari respectively can be combined into a single theory whose aim it is to provide two-dimensional concepts of bounded variation and absolute continuity comparable in scope and in utility to the corresponding classical one-dimensional concepts for functions of a single real variable. The present status of the theory seems to justify the assumption that the two-dimensional concepts eBV and eAC (see above) represent adequate generalizations of the corresponding one-dimensional concepts. However, it seems desirable to put the definitions of the two-dimensional concepts into a form which reveals the fundamental analogy with the one-

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dimensional concepts as clearly as possible. The purpose of the present note is to make a contribution in this direction for the concept eAC.

2. The one-dimensional situation. Let I denote the unit interval $0 \leqq u \leqq 1$ on a u-axis, and let

$$T: \quad x = f(u), \qquad u \in I, \tag{2}$$

be a continuous mapping from I into a (real) x-axis. Naturally, T is then termed BV (of bounded variation) or AC (absolutely continuous) if and only if the function $f(u)$ is BV or AC respectively. However, the geometrical picture yields an important re-wording as follows. Let us define a multiplicity function $N(x)$ by the agreement: for $-\infty < x_0 < +\infty$, $N(x_0)$ is equal to the number (possibly infinite) of those points $u \in I$ that are mapped by T into x_0. The following facts are well known.

(a) T is BV if and only if $N(x)$ is summable.

(b) If T is BV, then it is AC if and only if for every set E of measure zero in I the image set $T(E)$ is also of measure zero.

3. The two-dimensional situation. In dealing with the two-dimensional situation, we shall use LA as a reference. However, in LA a more general situation is considered, namely the case where the mapping T operates from a general bounded domain. In the special case considered in this note, the domain is taken as the interior of the unit square Q, and T is assumed to be defined and continuous on the perimeter also. These restrictions are made only to simplify the presentation, and our main result is readily seen to remain valid in the general case considered in LA.

Given T as in (1), there is associated with T an *essential multiplicity function* $\kappa(z, T, Q)$ (see LA, IV.1.4). Furthermore, with each point z there is associated a class (possibly empty or infinite) of continua in the interior Q^0 of Q, namely the class of the *essential maximal model continua* (abbreviated to e.m.m.c.) of z under T in Q (see LA IV.1.27). As a matter of fact, $\kappa(z, T, Q)$ is equal to the number of the e.m.m.c.'s of z (see LA IV.1.39). The sum of all the e.m.m.c.'s, corresponding to all the points z, is the *complete essential set* $\mathcal{E}^*$, while the *restricted essential set* $\mathcal{E}$ is the sum of all those e.m.m.c.'s that reduce to single points (see LA IV.1.56). Both $\mathcal{E}^*$ and $\mathcal{E}$ are Borel sets (see LA IV.1.58, IV.1.59). In analogy with the statements (a), (b) in §2 above, we have the following statements in the two-dimensional case.

(a) *The mapping T, given as in* (1), *is eBV if and only if the essen-*

tial multiplicity function $\kappa(z, T, D)$ *is summable.*

(b) *If* T *is eBV, then it is eAC if and only if for every set* E *of measure zero in the complete essential set* $\mathcal{E}^*$ *the image set* $T(E)$ *is also of measure zero.*

The statement (a) is in fact the *definition* of the concept eBV (see LA, IV.4.1). The statement (b) is the new result that we propose to prove in the present note.

4. **Background.** In an interesting paper on absolutely continuous mappings, L. Giuliano (see Bibliography [1]) established the following facts: If the mapping T, given as in (1), is eBV and maps subsets of measure zero of Q^0 into sets of measure zero, then T is eAC. However, examples show that this condition is *not necessary* for T to be eAC.

Thus it would seem that the characterization (b) of AC mappings in the one-dimensional case (see §2) admits of no analogue for the two-dimensional eAC concept. And yet, our theorem (b) in §3 reveals a fundamental analogy. This theorem also reveals the importance of the complete essential set $\mathcal{E}^*$. The basic importance of the *complete* essential set $\mathcal{E}^*$, as against the *restricted* essential set $\mathcal{E}$, was first recognized by P. V. Reichelderfer (see Bibliography [3]).

5. **Proof of the theorem.** *Necessity.* Suppose that T, given as in (1), is eAC. By definition (see LA IV.4.1) T is then AC$\mathcal{E}^*$ (see LA IV.2.39). Hence, by LA IV.2.42, for every set E of measure zero in $\mathcal{E}^*$ the image set $T(E)$ is of measure zero.

Sufficiency. Suppose that T, given as in (1), satisfies the following conditions.

(i) T is eBV (see LA IV.4.1).

(ii) For every set E of measure zero in $\mathcal{E}^*$, the image set $T(E)$ is of measure zero.

We have to show that T is then eAC. To this end, we verify the following fact.

(iii) T is BV$\mathcal{E}^*$ (see LA IV.2.11).

Indeed, by LA IV.4.1, IV.2.42, the statements (ii) and (iii) imply that T is eAC, and hence it is sufficient to verify (iii). Now, by LA IV.2.13, the statement (iii) is equivalent to the following fact.

(iv) the function $N(z, \mathcal{E}^*)$ is summable (where $N(z, \mathcal{E}^*)$ is the number, possibly infinite, of those points $w \in \mathcal{E}^*$ whose image under T is z).

To establish (iv), we note that by assumption (i) the essential multiplicity function $\kappa(z, T, Q)$ is summable (see LA IV.4.1). Hence (iv) will be proved if we can prove the following fact.

(v) $N(z, \mathcal{E}^*)=\kappa(z, T, Q)$ a.e. (almost everywhere) in the z-plane.

To establish (v), let us denote for $0\leqq r\leqq 1$, r rational, by E_r' the intersection of $\mathcal{E}^*$ with the line $u=r$, and by E_r'' the intersection of $\mathcal{E}^*$ with the line $v=r$. Then E_r', for fixed r, is a set of measure zero in $\mathcal{E}^*$, and hence by (ii) the image set $T(E_r')$ is of measure zero. Similarly, the image set $T(E_r'')$ is of measure zero. Let us denote by $\bar{e}$ the set of those points z where $\kappa(z, T, Q)=\infty$. Since $\kappa(z, T, Q)$ is summable, $\bar{e}$ is of measure zero. Let us put

$$\overline{E} = \bar{e} + \sum_r T(E_r') + \sum_r T(E_r''), \tag{3}$$

where the summations are extended over all rational numbers r such that $0<r<1$. Obviously

$$|\overline{E}| = 0, \tag{4}$$

where $|\overline{E}|$ denotes the measure of $\overline{E}$. Now take any point $z_0 \notin \overline{E}$. Then $z_0\notin \bar{e}$, and hence $\kappa(z_0, T, Q)=k_0$ is finite. Furthermore, if c_0 is any e.m.m.c. of z_0, then by the definition of $\overline{E}$ it follows that c_0 does not intersect any one of the lines $u=r$ or $v=r$, where r is any rational number such that $0<r<1$. Hence c_0 must reduce to a single point. It follows that $N(z_0, \mathcal{E}^*)=k_0=\kappa(z_0, T, Q)$ (cf. LA IV.1.39). Hence

$$N(z, \mathcal{E}^*) = \kappa(z, T, Q) \qquad \text{for } z \notin \overline{E}. \tag{5}$$

Clearly (4) and (5) imply (v), and thus the proof of our theorem (b) in §3 is complete.

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THE UNIQUENESS OF A CERTAIN LINE INVOLUTION

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In a forthcoming paper [1][1] on the classification of line involutions we pointed out that any line involution in S_3 with the property that a general line and its image do not intersect has associated with it, besides its order, m, and the order, i, of its invariant complex, two other important numerical characteristics which we called k and n. The first of these, k, is the number of lines of an arbitrary pencil which meet their images. Recalling the well known mapping relation between the lines of S_3 and the points of a nonsingular V_4^2 in S_5, the second, n, is the order of the ruled surface formed by the rays which join the points of a general line of V_4^2 to their respective images. The four quantities, m, i, k, n, are not independent but satisfy the following relations [1]

$$\text{(A)} \qquad n - i = k, \qquad n + i = m + 1.$$

It is the purpose of the present note to establish the fact that there is a unique involution associated with the set $m = n = 2$, $i = k = 1$, namely the involution whose point equivalent on V_4^2 is defined by the transversals of a general line, λ, of S_5 and an S_3 contained in the S_4 which is tangent to V_4^2 at one of the two intersections of λ and V_4^2 [2, 3]. In constructing our proof we shall find it convenient to work exclusively in S_5 rather than in the three-dimensional space where the line involution is actually defined.

We shall begin by establishing the fact that the surface which is the image of a general plane of V_4^2 is of the second order. This cannot be inferred as obvious from the fact that the order of the transformation is two, because one may show by example that far from being self-evident, it is not even true that the order of the image of a plane of V_4^2 is always the order of the involution. Moreover it is not true that planes of opposite families on V_4^2 necessarily have images of the same order. For instance in a paper in this Bulletin [4] Clarkson discussed a simple involution of order two in which bundles of lines are transformed into (3, 1) congruences while plane fields of lines are transformed into (1, 1) congruences. In other words, on V_4^2 an ω-plane is transformed into a surface of the fourth order, while a ρ-plane is transformed into a quadric surface.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Suppose, then, that the image of a general plane, π, on V_4^2 is a surface of order r. A general S_4 of S_5 will meet this image surface in a C_r which must be the proper image of some curve in π, say a C_s. Since the order of the involution is two, the total image of a C_s in π is a C_{2s}. Hence $r \leqq 2s$. Now the S_4 which cuts out C_r meets the image conic of any line, l, in π in exactly two points, and these are necessarily the images of the points common to C_s and l. Thus C_s is met by any line in π in two points. Hence its order is $s=2$ and so $r \leqq 4$.[2]

If the order of the image surface is four it clearly must lie properly in at least four dimensions, since the total intersection of V_4^2 and an S_3 is only a quadric surface. Consider first the possibility that it lies in a four-dimensional subspace of S_5. Then any S_3 in the S_4 which contains the image surface intersects the latter in a C_4, which, since the order of involution is two, must be the image of some curve in π of order $s>1$. Now any line in π will meet such a C_s in s points whose images will be points common to the conic into which the line is transformed, and the C_4. If $s>2$, the S_3 which cuts the C_4 will then contain the image conic of every line in π, which is absurd. Hence $s=2$. However, a general conic in π meets the invariant locus in two points, whereas the C_4 cut by a general S_3 can meet it only once. The hyperplane sections of the image surface must therefore be the images of the conics through some invariant point which is simultaneously a singular point possessing a curve of images. Now on the C_4 cut from the image surface of π by a general S_3 consider four coplanar points such that none of them is an image of the singular invariant point. Each of the S_3's of the pencil on the plane of these four points will cut the image surface in a C_4 which is the image of some conic through the singular invariant point. We thus have the impossible situation of ∞^1 conics passing through five fixed points, and so we must reject the possibility that the image of π is a quartic surface in an S_4.

If the image of π is a V_2^4 lying properly in S_5 its ∞^5 hyperplane sections must be the images of the ∞^5 conics in π. Each such curve must therefore have two invariant points, that is, must meet the invariant line in π in two points. But this requires every S_4 in S_5 to pass through the invariant line in π, which is absurd. This, then, overthrows completely the possibility that the image of π is of the fourth order.

Suppose next that the image of π is a cubic surface, of necessity lying properly in some S_4, or more specifically, lying on the V_3^2 in which some S_4 meets V_4^2. Now by a well known theorem of Felix

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[2] This proof may be restated almost word for word for the general involution of order m, leading to the conclusion that the order of the image surface of a general plane of V_4^2 is not greater than m^2.

Klein, any V_{n-2}^{m} on a nonsingular V_{n-1}^{2} in S_n ($n>3$) is necessarily the complete intersection of V_{n-1}^{2} with a single primal. In the present case we have a V_2^3 lying on a V_3^2 in S_4. Since a V_2^3 cannot be the complete intersection of a V_3^2 and any primal, it follows that the V_3^2 must be singular, that is, that it must be cut from V_4^2 by a tangent S_4. The cubic surface itself is cut from the singular V_3^2 common to V_4^2 and the tangent S_4 by a second hypercone having an entire plane in common with the first. All planes on the singular V_3^2 which belong to the same family as this common plane will meet the cubic surface in conics passing through the vertex of the V_3^2, and these conics will be the images of the lines of some pencil in π, since they have no free intersections among themselves, and since individually they have but one intersection with the image-conic of a general line of π. Now consider any line in the plane of one of these conics, C. Its image is a conic, K, which necessarily meets π in two points on the line, l, whose image is C. The plane of the conic K is thus a plane of V_4^2 meeting π in l. But there is a unique plane of V_4^2 meeting a given plane in a given line. Hence the proper image of the plane of C is the plane of K. Similarly all planes on the singular V_3^2 which belong to the same family as the plane of C have for their images planes meeting π in the lines of a certain pencil. In the aggregate these planes constitute the singular V_3^2 cut from V_4^2 by the S_4 which is tangent at the vertex of this pencil. Thus associated with the general plane, π, of at least one of the two families of planes on V_4^2 there are two singular V_3^2's which are images of each other in such a way that the planes of one family on the first correspond respectively to the planes of one of the families on the second. But when all planes, π, are considered it is apparent that there must be ∞^2 such singular V_3^2's, collectively embracing ∞^3 planes which have only planes for their images. In other words if the planes of one family on V_4^2 have cubic surfaces for their images, the planes of the opposite family have planes for their images. But this is self-contradictory, because any plane and its image must meet in a line, since every plane contains a line of invariant points, and they must therefore belong to opposite families on V_4^2. Since the transformation is involutorial this requires that the planes of one family be simultaneously transformed into planes and into cubic surfaces, which is absurd.

Finally suppose that the images of all planes on V_4^2 are planes. Since a general line in a plane, π, is transformed into a conic on the image surface of π, it follows that any plane of V_4^2 meeting π in a line has for its image a plane with a conic in common with the image of π. In other words any plane meeting π in a line has an image which coincides with the image of π, which is impossible. This leaves a quadric surface as the only possibility for the image of a general plane of V_4^2.

Knowing now that the image of a general plane, π, is a quadric surface meeting π in a line, consider a general line, l, in π. Its image is a conic on the image quadric of π, and this conic is met in two points by the image of any other line, l', in π. One of these points is of course the image of the intersection of l and l'. The other point must be in some sense a singular point, that is, it must be the common image of two distinct points in π. If as l' varied over π this singular intersection varied along the image of l it would follow that every point on the image of a general line was the image of more than one point. This contradicts the involutorial character of the transformation, and so necessarily the image conics of all lines in π pass through a fixed point, P, on the quadric which is the image of π. Moreover since each line in π contains a single point having P for its image, the locus of such points in π is a straight line.

Consider next the quadratic regulus formed by the rays joining the points of a general line, l, in π to their respective images. Among the generators of the other family is one, λ, which passes through P and which necessarily is met by every ray of the first family. For a second line, l', in π there is similarly a line λ' which passes through P and which is met by the image ray of every point on l'. The image ray associated with the intersection, Q, of l and l' thus meets both λ and λ'. Finally consider any third line, l'', of π. Its image regulus of course contains P. Moreover it contains the image ray of the intersection, Q', of l and l'', and the image ray of the intersection, Q'', of l' and l''. The S_3 in which the image regulus of l'' lies thus contains both λ and λ'. In addition to Q' and Q'' it therefore contains Q, that is, it contains the entire plane π, which is absurd. The only escape is for λ and λ' to coincide, which leads to the conclusion that there is one and only one line, λ, which is met by the image rays of every point of π. The uniqueness of λ is evident, because if there were a second line (necessarily not passing through P) which was met by every image ray from π, it would determine with λ an S_3 in which the image regulus of every line in π would have to lie, which is clearly impossible.

Now let R be the second intersection of λ and V_4^2. For any line in π some image ray must pass through R. Hence if such an image ray does not lie entirely on V_4^2 the image of every line in π passes through R as well as P, which is impossible. Thus the points of π whose image rays pass through R are the singular points in π, and the totality of the associated singular rays is the plane of V_4^2, σ, through R and the line of singular points in π.

Consider now any other plane, π', on V_4^2 which meets π in a line. As in π, the lines in π' map into conics on the image surface of π', and these conics all pass through a fixed point, P'. Moreover the image ray of every point in π' meets a unique line, λ', through P', and this line intersects in a point, R', the singular ray, s, through the unique singular point, S, on the intersection of π and π'. And just as for π, all image rays of π' passing through R' are singular and in the aggregate form a plane, σ', of V_4^2 through s and the line of singular points in π'.

Any other line in π through S has a unique plane, π'', passing through it, and by the same reasoning as before, the singular rays through its line of singular points form a plane through s. But only two planes of V_4^2 pass through a line such as s. Hence this third plane cannot be distinct from both σ and σ', and in fact must coincide with the plane of singular rays from π'. This means that the singular points in π'' constitute a line of σ', which as π'' is allowed to vary covers all of σ'. In other words the points of all singular rays from π' are themselves singular. Of course the same remark is true for the singular rays in σ.

Since there are at least two planes, σ and σ', of singular points it follows that the linear complex of singular elements is special, that is, is tangent to V_4^2 at some point of s. But the reasoning leading to this conclusion could equally well have applied to any other singular ray, leading to the conclusion that the vertex of the singular complex had to lie upon it. Hence we infer that all singular rays are concurrent in the vertex of the singular complex. In other words R and R', and the corresponding points associated with all the other planes of V_4^2, must coincide. Hence the lines $\lambda, \lambda', \cdots$ must coincide also. Thus we draw the conclusion that *the image rays of all points on V_4^2 meet a fixed line*, λ, *through the vertex of the singular complex.*

Consider again the image regulus of a general line l. The quadric of which it forms one family of generators is met by the S_4 which contains the singular complex in the singular ray, s, associated with l, and in a second line, m, which necessarily belongs to the same regulus as l. The image rays of all points of l thus meet m, the points of m and the image rays being in 1:1 correspondence. Moreover the image rays of all the points on the quadric common to V_4^2 and the S_3 determined by λ and l also meet m. For consider any generator of this quadric, l', belonging to the regulus not containing l. Like any other line on V_4^2 its image rays meet the singular S_4 in the points of some line, m'. But such a generator meets l in a point and the image conic of l in a point, and these points are not images one of the other since their join, namely l' itself, is a line of V_4^2, which is impossible for a general pair of corresponding points. Thus the image rays of these points are distinct, and hence their intersections with S_4 are two distinct points of m. Therefore the line m' in which the image rays of a general generator, l', meet S_4 must coincide with m.

It is now easy to see that the image rays of the points of a general plane, π, meet S_4 in the points of some plane, σ. For if l_1 and l_2 be two general lines of π meeting in a point Q, the image rays of each meet S_4 along a line. The two lines, m_1 and m_2, say, necessarily intersect, for otherwise the image ray of Q, which must meet both, would lie entirely in S_4, which cannot happen. Finally any other line, l_3, in π meets m_1 and m_2, and so the line, m_3, along which its image rays intersect S_4 must meet m_1 and m_2, that is, must lie in the plane of m_1 and m_2.

Now at a general point, P, of V_4^2 consider three lines, r_1, r_2, and r_3 of V_4^2, so chosen that no one lies in the S_3 determined by λ and either of the others. Each of the S_3's determined by λ and one of the lines r_1, r_2, r_3 has the property noted above that the image rays of all the points which it has in common with V_4^2 meet S_4 along a line. Moreover these three lines, m_1, m_2, m_3, say, necessarily concurrent, are independent, that is define a specific S_3 in the singular S_4. Now any plane, π, on V_4^2 has the property that it meets any quadric on V_4^2 in at least one point. Hence the plane, σ, in which the image rays of π meet S_4 must have one point on each of the three lines m_1, m_2, m_3, and these three points will all be distinct since π does not in general intersect the conic cut from V_4^2 by the plane of P and λ. Therefore σ is contained entirely in the S_3 determined in S_4 by m_1, m_2, m_3. Thus in addition to meeting a fixed line, λ, through the vertex of the singular complex, *the image ray of a general point of* V_4^2 *meets a fixed* S_3 *in the* S_4 *which cuts the singular complex from* V_4^2. This completes our proof.

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THE UNIVERSITY OF UTAH

SUPERPOSABILITY PROPERTIES OF NATURALLY METRIZED GROUPS[1]

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Let p and q be elements of an additively written abelian group G. The unordered pair of elements $(p-q, q-p)$ is written $pq = |p-q|$ nd is called the "distance" of p and q. This distance has the properties of symmetry ($pq = qp$) and vanishing ($pq = |0|$ if and only if $p = q$). The group bearing this distance is called naturally metrized.

Two subsets of G are *congruent* provided there is a distance-preserving mapping of one onto the other. A congruence of G with itself is a *motion* of G. Two subsets of G are *superposable* if one is mapped onto the other by a motion of G. G has the property of *free mobility* if any congruence between any two subsets of G can be induced by a motion of G.[2]

It was shown by Menger[3] that any two congruent pairs of elements of G are superposable. In this note it is shown that G has the property of free mobility.

If $t \in G$ the mapping $x \rightarrow x \oplus t$ of G onto itself is the translation induced by t. The reflection in O of G is the mapping $x \rightarrow -x$ of G onto itself. Clearly all translations and the reflection in O are congruences. Denote by Γ the group of motions of G obtained by adjoining the reflection in O to the group of all translations of G.

THEOREM. *Let $S \subset G$ and α be a mapping of S onto $T \subset G$. A necessary and sufficient condition that there be a motion $\mu \in \Gamma$ which induces α over S is that $ab = a\alpha b\alpha$ for all $a, b \in G$, that is, α is a congruence.*

PROOF.[4] The necessity is obvious. Suppose that α is a congruence.

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[1] This paper is part of a dissertation being written for the Ph.D. degree, University of Missouri, under the direction of L. M. Blumenthal.

[2] The concepts of superposability and free mobility *vis-à-vis* congruence have recently assumed new importance in certain of the characterization problems of metric geometry (cf. Garrett Birkhoff, *Metric foundations of geometry*. I, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 465–492). These notions, logically equivalent in euclidean, spherical, and hyperbolic spaces, are quite distinct, for example, in elliptic space where they have been extensively studied in a recent publication (cf. L. M. Blumenthal, *Congruence and superposability in elliptic space*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 431–451).

[3] K. Menger, *Beiträge zur Gruppentheorie* I. *Über einer Abstand im Gruppen*, Math. Zeit. vol. 33 (1931) pp. 396–418.

[4] The author is indebted to the referee for this more concise form of the original proof.

Since Γ contains all translations and the group of translations is transitive over G, we may assume without loss of generality that $0 \in S$ and that $0\alpha = 0$. Denote by S' the subset of S which is invariant under α. We distinguish two cases:

Case 1. $x \in S'$ implies $2x = 0$.

Suppose $w \in S$, $w \notin S'$. Then $w0 = w\alpha 0\alpha = w\alpha 0$. Since $w \notin S'$, $w \neq w\alpha$. However, $(w, -w) = (w\alpha, -w\alpha)$ so that $w\alpha = -w$. Apply the reflection in O over G. S' is invariant under the reflection and for all other elements of S, $-w = w\alpha$ so that the reflection coincides with α over S.

Case 2. There exists $t \in S'$ so that $2t \neq 0$.

Suppose $w \in S$. As in case 1, $w\alpha$ is either w or $-w$. If $w\alpha = w$, $w \in S'$. Suppose then that $w\alpha = -w$. Since α is a congruence $wt = w\alpha t\alpha = -wt$ ($t \in S'$ and hence invariant under α). Hence $w - t = w + t$ or $w - t = -w - t$. The first equation is impossible since $2t \neq 0$. Therefore $w - t = -w - t$ and $w = -w = w\alpha$ so that $w \in S'$. This shows that the identity mapping of G (translation induced by 0) coincides with α over S.

The generalization of superposability theorems to non-abelian groups is not fruitful (although such groups have been studied from the metric viewpoint)[5] since in these studies it is shown that the reflections of such a group are motions if and only if the group is either abelian or is a Hamiltonian 2-group.[6]

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[5] Cf. O. Taussky, Math. Ann. vol. 108 (1933) pp. 615–620 and Quart. J. Math. Oxford Ser. vol. 12 (1941) pp. 64–67.

[6] A Hamiltonian 2-group is a group in which the order (period) of each element is a power of 2.